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Algebraic Classification of Regge Poles*

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Starting from the Lorentz invariance and usual on-mass-shell analyticity properties of scattering amplitudes, we prove that: (a) massless "particles," transforming according to infinite-spin representations of the two-dimensional Euclidean group, are necessarily "elementary," corresponding to Kronecker-δ singularities in the j plane; (b) the classification algebra of Regge poles, at vanishing invariant mass, is necessarily the Lie algebra of the homogeneous Lorentz group $SL(2,C)$. We calculate the contributions of Regge poles to scattering amplitudes of particles with arbitary finite mass and spin at vanishing momentum transfer, taking into account the "conspiracy" of Regge poles arising from their classification according to $SL(2,C)$. The Regge contributions are indeed found to have the required analyticity properties and, therefore, a uniform asymptotic behavior for large energies.

I. INTRODUCTION

THERE has been much interest lately in the so-called particle "conspiracy." New restrictions on physical theories, imposed by Lorentz invariance, were first realized by Domokos and Suranyi[1] in 1963, when they studied the bound states in a Bethe-Salpeter model; these were investigated in considerable detail in a series of elegant papers by Toller and his collaborators.[2–8] More recently the phenomenon was encountered by Freedman and Wang,[9] in connection with the validity of the Regge representation for backward elastic scattering.

Toller's main point is that the symmetry group of a scattering amplitude is the full Lorentz group at zero four-momentum transfer. This leads to a natural classification of Regge bound states according to representations of the homogeneous Lorentz group. However, his method applies only in the case of the scattering of particles with pairwise equal masses, since only in that case can one assert that one has a zero momentum-transfer *four-vector*.

In this paper, we show that the generally accepted S-matrix analyticity properties and Lorentz invariance together imply that, in *any* scattering process, Regge bound states should be classified according to the representations of the homogeneous Lorentz group, at zero momentum transfer. The result is achieved by clearly distinguishing the *invariance group* of a scattering amplitude from the *classification group* of the bound states, the spectrum of the amplitude. The former depends on "external" properties of the amplitude, in particular the masses of the incoming and outgoing particles, whereas the latter—in view of the unitarity of the S matrix—obviously does not. Therefore, the classification group of the spectrum need not coincide with the invariance group. This fact has been already observed in a Bethe-Salpeter model.[10]

In the next section, we review the group structure of the scattering amplitude for particles of arbitrary finite mass and spin. We then observe that the contribution of a single Regge pole at zero momentum transfer, develops a logarithmic singularity, which is not associated with the kinematic factors which are usually removed. The appearance of this singularity is simply a consequence of the change in group structure of the scattering amplitude at $s=0$. The phenomenon is investigated in detail in the third section. We recall that one associates with Regge poles a Lie algebra, the classification algebra of the spectrum, and show that, provided the structure of the algebra does not change as the "mass" of the pole is varied, the unwanted singularity does not appear. Further, we use a theorem on Lie algebras to prove that the only physically admissible algebra is the Lie algebra $\mathrm{SL}(2,C)$ of the homogeneous Lorentz group.

In the fourth section we calculate the contribution of Regge poles to the scattering amplitude at vanishing invariant momentum transfer. The form of the expression obtained coincides, in the equal-mass case, with that of Toller.[6] In the case of arbitrary-mass particles the form of the expression is the same, provided we make the necessary changes in the kinematics. In particular, it follows that Toller's results on the inclusion of reflections can be taken over to the general mass case without modification.

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In the Appendix we compute the transformation matrix between irreducible representations of the Poincaré group and the homogeneous Lorentz group.

II. GROUP STRUCTURE OF THE SCATTERING AMPLITUDE

Consider a process $(1)+(2) \to (3)+(4)$, where the symbols (i) may denote particle groups or individual particles. Let the squares of the c.m. energies of this process and the two others obtained from it by crossing be denoted by s, t, and u, respectively. Now according as s, the square of the energy-momentum four-vector given by $s=(p_1+p_2)^2$, is positive or negative, the little groups of the energy-momentum vector are $SU(2)$ and $SL(2,R)$, respectively. The scattering amplitude may be considered a function, in fact a matrix function, on the little group and may be decomposed in terms of irreducible representations.[2–8]

We may treat the scattering amplitude as a function, rather than a matrix, on the little group because: (a) the total four-momentum is conserved, (b) by construction the generators of the little group,

$$W_\lambda = \frac{1}{2\sqrt{P^2}} \epsilon_{\lambda\mu\rho\sigma} P_\mu M_{\rho\sigma},$$

commute with both the momentum operators P_μ and the scattering operator. Let us denote the total four-momentum, spin, and helicity of the particle group (i) by p_i, s_i, and λ_i, respectively, with $p_i^2=m_i^2$. Then the scattering amplitude of the process $(1)+(2) \to (3)+(4)$ on the momentum shell with $p_1+p_2=p_3+p_4$ has been written by Trueman and Wick[11] in the following form:

$$\begin{aligned}\langle p_3s_3\lambda_3,p_4s_4\lambda_4|T|p_1s_1\lambda_1,p_2s_2\lambda_2\rangle &= D_{\lambda_3\nu_3}{}^{s_3}(R_3)D_{\lambda_4\nu_4}{}^{s_4}(R_4)D_{\nu_1\lambda_1}{}^{s_1*}(R_1)D_{\nu_2\lambda_2}{}^{s_2*}(R_2)\\ &\quad\times\delta_{\nu_1-\nu_2,\nu}\delta_{\nu_3-\nu_4,\nu'}\langle\nu'|\mathfrak{F}(s,t)|\nu\rangle, \quad (2.1)\end{aligned}$$

where, as is usual, $t=(p_3-p_1)^2$.

The Wigner rotation elements of the little group, $R_1 \cdots R_4$ are given by the expression

$$R_i = L^{-1}(p_i)lL(l^{-1}p_i). \quad (2.2)$$

Here, $L(p)$ is a "boost" which produces the state $|\mathbf{p}s\lambda\rangle$ from the corresponding rest state,

$$|\mathbf{p},s,\lambda\rangle = U(L(p))|0,s,m\rangle,$$

and the Lorentz transformation l is uniquely determined, up to a rotation around the 3 axis, by the requirement that l^{-1} transforms the momentum p_1+p_2 to rest and rotates the vector $\mathbf{p}_1$ into the direction of the positive 3 axis. The amplitude $\langle\nu'|\mathfrak{F}(s,t)|\nu\rangle$ then admits the partial-wave decomposition

$$\langle\nu'|\mathfrak{F}(s,t)|\nu\rangle = \langle\nu'|\mathfrak{F}_+(s,t)|\nu\rangle + \langle\nu'|\mathfrak{F}_-(s,t)|\nu\rangle,$$

where

$$\langle\nu'|\mathfrak{F}_\pm(s,t)|\nu\rangle = \frac{s^{1/2}}{\pi|\mathbf{P}(s)|}\int_c \frac{(2j+1)dj}{\sin j\pi}\langle\nu'|\mathfrak{F}_\pm(s,j)|\nu\rangle \times\left[D_{\nu'\nu}{}^j(R(\phi,\Theta_s,-\phi)) \pm \begin{pmatrix}\cos\Theta_s \to -\cos\Theta_s\\ \nu,\nu' \to -\nu,-\nu'\end{pmatrix}\right]. \quad (2.3)$$

In these formulas, $D_{ab}{}^j(R(\alpha,\beta,\gamma))$ denotes a matrix element of the representation (j) of $SU(2)$ or $SL(2,R)$, depending on whether or not the Euler angle β is real or pure imaginary. These matrix elements can be continued into each other, and the continuation procedure is described in detail in the work of Andrews and Gunson.[12] In Eq. (2.3) $|\mathbf{P}|$ stands for the magnitude of the three-momentum in the center-of-mass system (c.m.s.)

$$|\mathbf{P}| = (4s)^{-1/2}\Delta(s,m_1^2,m_2^2),$$

where

$$\Delta(x,y,z) = [x^2+y^2+z^2-2xy-2xz-2yz]^{1/2}.$$

The variable $\cos\Theta_s$ is the scattering angle in the s-channel c.m.s., which, as is well known,[13] may be expressed in terms of the Mandelstam invariants:

$$\cos\Theta_s = \frac{s(t-u)+(m_1^2-m_2^2)(m_3^2-m_4^2)}{\Delta(s,m_1^2,m_2^2)\Delta(s,m_3^2,m_4^2)}. \quad (2.4)$$

The contour of integration in Eq. (2.3) depends on the nature of the little group. With $s>0$, when the little group is $SU(2)$, c may be taken as a "hairpin" contour around the positive real axis; however with $s<0$, when the little group is $SL(2,R)$, after a suitable deformation of the contour, one obtains an integral over the principal series of unitary representations of $SL(2,R)$,

$$(j=-\tfrac{1}{2}+i\tau,\ -\infty<\tau<\infty),$$

plus a sum over the Regge poles.

Equation (2.1) gives the scattering amplitude in an arbitrary reference frame. In the s-channel c.m.s. we have $\mathbf{p}_1+\mathbf{p}_2=0$ and the right-hand side reduces to $\langle\nu'|\mathfrak{F}|\nu\rangle$, whereas the helicity amplitudes in the t-channel c.m.s., with $\mathbf{p}_1+\mathbf{p}_3=0$, are given by the Trueman-Wick transformation[11] derivable from Eq. (2.1):

$$\begin{aligned}\langle p_3s_3\lambda_3,p_4s_4\lambda_4|\mathfrak{F}(s,t)|p_1s_1\lambda_1,p_2s_2\lambda_2\rangle &= \sum_{\nu_1\nu_2\nu_3\nu_4} d_{\nu_1\lambda_1}{}^{s_1}(\chi_1)d_{\nu_2\lambda_2}{}^{s_2}(\chi_2)d_{\nu_3\lambda_3}{}^{s_3}(\chi_3)d_{\nu_4\lambda_4}{}^{s_4}(\chi_4)\\ &\quad\times\langle p_3s_3\nu_3,\bar{p}_1\bar{s}_1\bar{\nu}_1|\mathfrak{F}(t,s)|\bar{p}_4\bar{s}_4\bar{\nu}_4,p_2s_2\nu_2\rangle. \quad (2.5)\end{aligned}$$

The bar over a particle eigenvalue symbol means one deals with the corresponding antiparticle and, as usual, $d_{ab}{}^s(\chi)$ denotes the matrix representation of a rotation through a real or imaginary angle χ about the 2-axis.

[11] T. L. Trueman and G. C. Wick, Ann. Phys. (N. Y.) **26**, 322 (1964).

[12] M. Andrews and J. Gunson, J. Math. Phys. **5**, 1391 (1964).

[13] T. W. B. Kibble, Phys. Rev. **117**, 1159 (1960).

A partial-wave decomposition analogous to that of Eq. (2.3) holds for $\langle\nu|\mathfrak{F}(t,s)|\nu'\rangle$, with s and t interchanged in Eq. (2.4) and the azimuthal angles interpreted correspondingly.

The angles $\chi_i(i=1,2,3,4)$ are given by the expression

$$\sin\chi_1=\frac{2m_1[\phi(s,t)]^{1/2}}{\Delta(t,m_1^2,m_2^2)\Delta(t,m_3^2,m_4^2)}\text{ (cycle)}, \qquad (2.6)$$

where $\phi(s,t)$ is the Kibble function,

$$\begin{aligned}\phi(s,t)=stu-s(m_2^2-m_4^2)(m_1^2-m_3^2)\\ -t(m_1^2-m_2^2)(m_3^2-m_4^2)\\ -(m_1^2m_4^2-m_2^2m_3^2)(m_1^2+m_4^2-m_2^2-m_3^2).\end{aligned}$$

The representation (2.3) should exist for $s\gtrless 0$. Moreover, we expect that a Regge-pole contribution to $\langle|\mathfrak{F}(s,t)|\rangle$ should be regular in s at $s=0$ with t in its physical region. However, a glance at a term in the representation (2.3) shows that the contribution seems to develop singularities at $s=0$. In fact let us recall the well-known expression for the rotation matrix

$$D_{mm'}{}^j(\alpha\beta\gamma)=e^{-im\alpha}d_{m,m'}{}^j(z)e^{-im'\gamma},$$

where

$$d_{mm'}{}^j(z)=\left[\frac{\Gamma(j+m+1)\Gamma(j-m'+1)}{\Gamma(j+m'+1)\Gamma(j-m+1)}\right]\times\left(\frac{1+z}{2}\right)^{(m+m')/2}\left(\frac{1-z}{2}\right)^{(m-m')/2}\times\frac{F(-j+m,\,j+m+1;\,1+m-m';\,(1-z)/2)}{\Gamma(1+m-m')}$$

and $z=\cos\beta$.

Barring accidents we have $\lim_{s\to0}\cos\Theta_s=1$, and both terms in the square bracket of Eq. (2.3) become singular. The singular contributions of the factor

$$\left(\frac{1\pm z}{2}\right)^{(m-m')/2}\left(\frac{1\mp z}{2}\right)^{(m-m')/2}$$

are well known[14] and have been separated out as kinematical singularities. However, in general, the hypergeometric function also has a singularity when the argument of the exchange term tends to 1. Here we have a logarithmic branch point which has been dealt with essentially by using a Laurent series expansion.[9] This unwanted singularity arises because of the sudden change in the structure of the little group at $s=0$. It is known, from Wigner's classical work,[15] that the little group corresponding to a lightlike momentum is $E(2)$, the group of rigid motions in the Euclidean plane. The effect of contraction of the groups $SU(2)$ or $SL(2,R)$ on our formulas can be followed easily.

[14] L. L. Wang, Phys. Rev. **153**, 1664 (1967).

[15] E. P. Wigner, Ann. Math. **40**, 149 (1939).

Let us recall that the rotation entering (2.3) is defined by the relation $R(\phi,\Theta_s,-\phi)=L^{-1}(l^{-1}p)l^{-1}L(p)$ so it clearly depends on the mass $\sqrt{s}$. In fact, without loss of generality, we can choose both $\mathbf{p}$ and $\mathbf{p}_1$ to lie along the positive 3-axis when we have

$$l^{-1}=R(\phi,\Theta_s,-\phi)z^{-1}(p).$$

Here, $L(p)=z(p)$, is a pure Lorentz transformation along the 3-axis and Θ_s, ϕ are the polar angles of p_3 *in the rest system of* p. The dependence on the "total mass," implicit in this definition, is conveniently exhibited in Eq. (2.4). Indeed, Eq. (2.4) can be looked upon as introducing an invariant parametrization of the little group, involving $(t-u)$ instead of the frame-dependent Euler angles.[16]

As $s\to0$ and the velocity of the c.m.s. tends to the light velocity, the degree of freedom corresponding to the parameter $(t-u)$ "freezes in" so that we have the following relation:

$$\lim_{p^2\to0}D_{\lambda\lambda'}{}^j(L(l^{-1}p)l^{-1}L(p))=e^{-i(\lambda-\lambda')\phi}, \qquad (2.7)$$

where j is an integer or half integer.

The right-hand side of Eq. (2.7) is clearly a matrix element of a finite spin representation of $E(2)$, with a vanishing Casimir operator. Thus we see that the "unwanted singularities" in the scattering amplitude arise entirely as a consequence of the sudden change in the structure of the little group, when the momentum of the c.m.s. becomes lightlike. It is quite easy to show that the converse statement is also true. If we insist on maintaining the analyticity of the scattering amplitude represented by Eq. (2.3), it will have incorrect transformation properties under homogeneous Lorentz transformations.[17] A careful study of the little group will indicate a method of reconciling Lorentz invariance with the analyticity properties of the amplitude.

III. STRUCTURE OF THE LITTLE GROUP AND THE ELIMINATION OF THE UNWANTED SINGULARITIES

To be definite, let us start from a state with timelike momentum. We wish to follow the change in the structure of the little group, as the "mass," or square of the total four-momentum of the state, varies continuously through zero. This study will provide us with a clue, enabling us to avoid the unwanted singularities of the amplitude. In the following argument it proves essential to consider states with a nonvanishing three-momentum. Without loss of generality, we may assume that $p_3\neq0$ and $p_1=p_2=0$. That this is important is borne out by

[16] We apologize to the reader for repeating material which is so well known, but a full understanding of these properties of the little group is essential for later development of the theory.

[17] To show this it is sufficient to take a pure Lorentz transformation along one coordinate axis. The details of the proof are left to the reader.

the following simple argument. For $P^2>0$, $=0$, and <0, the canonical forms of the three-momentum can be chosen as $(0,0,0,m)$, $(0,0,p,p)$, and $(0,0,p,0)$, respectively. Evidently these forms do not go into each other under a continuous variation of P^2. However, if we choose the following representations of the three Wigner classes, $(0,0,p,\sqrt{(p^2+s)})$, $(0,0,p,p)$, and $(0,0,p,\sqrt{(p^2+s)})$, we can effect the "timelike → lightlike → spacelike" transition smoothly, by varying s with $\mathbf{P}$ fixed.

Let us take $P^2=s>0$ and consider a state with angular momentum j and magnetic quantum number m. If $|\mathbf{P}|=0$, the set of states $|\mathbf{P}=0, j, m\rangle$ forms a basis of an irreducible representation of the algebra $SU(2)$, with matrix elements defined in the usual way:

$$\begin{aligned}\langle\mathbf{P}=0, j', m\pm1|S_\pm|\mathbf{P}=0, j, m\rangle &= \delta_{jj'}[(j\mp m)(j\pm m+1)]^{1/2},\\ \langle\mathbf{P}=0, j', m'|S_3|\mathbf{P}=0, j, m\rangle &= m\delta_{jj'}\delta_{mm'},\\ \langle\mathbf{P}=0, j', m'|\mathbf{S}^2|\mathbf{P}=0, j, m\rangle &= j(j+1)\delta_{jj'}\delta_{mm'}.\end{aligned} \tag{3.1}$$

We now construct operators which "step" between states with different values of the total spin j. The simplest possibility, suggested by field theory, is to take a nondegenerate tower of states and enlarge the algebra (3.1) to $\mathbf{SL}(2,C)$. We shall see that this "minimal" enlargement of the spin algebra (3.1) is sufficient to get rid of the unwanted singularities of the scattering amplitudes. Let the minimal spin in the tower be j_0; then we *define* the new operators $T_\pm$, T_3 by giving the nonvanishing matrix elements in the following way.[18]

$$\begin{aligned}\langle\mathbf{P}=0, j+1, m\pm1|T_\pm|\mathbf{P}=0, j, m\rangle &= \mp\frac{1}{j+1}\left[\frac{((j+1)^2-j_0^2)((j+1)^2-(1+\sigma)^2)(j\pm m+2)(j\pm m+1)}{(2j+1)(2j+3)}\right]^{1/2},\\ \langle\mathbf{P}=0, j, m\pm1|T_\pm|\mathbf{P}=0, j, m\rangle &= ij_0(1+\sigma)\left[\frac{(j\pm m+1)(j\mp m)}{j(j+1)}\right]^{1/2},\\ \langle\mathbf{P}=0, j-1, m\pm1|T_\pm|\mathbf{P}=0, j, m\rangle &= \pm\frac{1}{j}\left[\frac{(j^2-j_0^2)(j^2-(1+\sigma)^2)(j\mp m+1)(j\mp m)}{(2j-1)(2j+1)}\right]^{1/2},\\ \langle\mathbf{P}=0, j+1, m|T_3|\mathbf{P}=0, j, m\rangle &= \frac{1}{j+1}\left[\frac{((j+1)^2-j_0^2)((j+1)^2-(1+\sigma)^2)(j-m+1)(j+m+1)}{(2j+1)(2j+3)}\right]^{1/2},\\ \langle\mathbf{P}=0, j, m|T_3|\mathbf{P}=0, j, m\rangle &= ij_0\frac{(1+\sigma)}{j(j+1)},\\ \langle\mathbf{P}=0, j-1, m|T_3|\mathbf{P}=0, j, m\rangle &= \frac{1}{j}\left[\frac{(j^2-j_0^2)(j^2-(1+\sigma)^2)(j-m)(j+m)}{(2j-1)(2j+1)}\right]^{1/2}.\end{aligned} \tag{3.2a}$$

The relations of the operators $S_\pm$, $T_\pm$ to the Cartesian components is the usual one,

$$S_\pm=S_1\pm iS_2, \qquad T_\pm=T_1\pm iT_2.$$

By construction the operators $\mathbf{S}$ and $\mathbf{T}$ satisfy the commutation relations of $\mathbf{SL}(2,C)$:

$$\begin{aligned}[S_i,S_j]&=i\epsilon_{ijk}S_k,\\ [S_i,T_j]&=i\epsilon_{ijk}T_k,\\ [T_i,T_j]&=-i\epsilon_{ijk}S_k.\end{aligned} \tag{3.2b}$$

The labels j_0, and σ characterize the "tower" of states on which the operators $\mathbf{S}$ and $\mathbf{T}$ act. The number $2j_0$ is by definition always an integer, whereas σ is an arbitrary complex number. The Casimir operators of the algebra (3.2b) are expressed in the usual way in terms of σ and j_0:

$$\begin{aligned}F&=\tfrac{1}{2}(\mathbf{S}^2-\mathbf{T}^2)=\tfrac{1}{2}(\sigma(\sigma+2)+j_0^2)\\ G&=\mathbf{S}\cdot\mathbf{T}=ij_0(\sigma+1).\end{aligned} \tag{3.3}$$

We now list some of the types of representations of the algebra (3.3):

$\sigma=j_0=0$ trivial,

$\sigma=-1-i\tau$ $(-\infty<\tau<\infty)$ principal series,

$\sigma=\xi$ $(-1\leq\xi\leq0)$ supplementary series,

$\sigma=n+j_0$, $n\geq0$. Finite dimensional.

$j_0=0, \tfrac{1}{2}, 1\cdots$.

The reader should notice that the representations which we are considering here are "classical" in the sense that $2j$ is an integer. "Reggeization" will lead us to consider other representations of the algebra with "unphysical" values of j. The latter cannot in general be used to induce representations of the group generated by the algebra (3.2b); nevertheless, the prescription to obtain them by analytic continuation is well known,[1,8,10] and we shall not discuss it here.

[18] H. Joos, Fortschr. Physik **10**, 65 (1962).

Under a Lorentz transformation which boosts the state $|\mathbf{P}=0, j, m\rangle$ into $|\mathbf{P},j,m\rangle$, the algebra spanned by **S**, **T** undergoes a similarity transformation. Evidently we have $|\mathbf{P},j,\lambda\rangle = U(L(\mathbf{P}))|\mathbf{P}=0, j, m\rangle$, so the operators

$$S_i(\mathbf{P}) = U(L(\mathbf{P}))S_iU^{-1}(L(\mathbf{P}))$$

and

$$T_i(\mathbf{P}) = U(L(\mathbf{P}))T_iU^{-1}(L(\mathbf{P})) \qquad (3.4)$$

have the same matrix elements between the boosted states as S_i and T_i had between rest states. Although Eqs. (3.1) to (3.4) are sufficient in principle to define a "tower" with arbitrary momentum **P**, it proves convenient to give explicit expressions for the generators $S_i(\mathbf{P})$ and $T_i(\mathbf{P})$ by stating index transformation properties under the boost $L(\mathbf{P})$. We define the spin pseudovector S_μ and vector T_μ in such a way that in the rest system they have components $(\mathbf{S},0)$ and $(\mathbf{T},0)$, respectively. Then, after noticing that the components of the boost $L(\mathbf{P})$ in the defining (vector) representation are

$$L_{00}(\mathbf{P}) = \omega/\sqrt{s},$$

$$L_{0i}(\mathbf{P}) = L_{i0}(\mathbf{P}) = P_i/\sqrt{s},$$

$$L_{ij}(\mathbf{P}) = \delta_{ij} + \frac{P_iP_j}{(\sqrt{s})(\omega+\sqrt{s})},$$

with $\omega = (s+\mathbf{P}^2)^{1/2}$, we obtain the relations

$$\mathbf{S}(\mathbf{P}) = \mathbf{S} + \frac{(\mathbf{P}\cdot\mathbf{S})\mathbf{P}}{(\sqrt{s})(\omega+\sqrt{s})}, \qquad (3.5a)$$

$$S_0(\mathbf{P}) = (1/\sqrt{s})\mathbf{P}\cdot\mathbf{S},$$

$$\mathbf{T}(\mathbf{P}) = \mathbf{T} + \frac{(\mathbf{P}\cdot\mathbf{T})\mathbf{P}}{(\sqrt{s})(\omega+\sqrt{s})}, \qquad (3.5b)$$

$$T_0(\mathbf{P}) = (1/\sqrt{s})\mathbf{P}\cdot\mathbf{T}.$$

The four-vectors S_μ and T_μ satisfy the relations

$$P_\mu S_\mu(\mathbf{P}) = P_\mu T_\mu(\mathbf{P}) = 0$$

and

$$[S_\mu, P_\nu] = 0. \qquad (3.6)$$

Evidently the operators (3.5a) coincide with the generators of the little group, the Pauli Lubanski spin operators,

$$W_\lambda = (4s)^{-1/2}\epsilon_{\lambda\mu\nu\rho}P_\mu M_{\nu\rho}(\mathbf{P}). \qquad (3.7)$$

To see this it is sufficient to recall an explicit expression[18] for the generators of Lorentz transformations $M_{\mu\nu}(\mathbf{P})$ operating on a state $|\mathbf{P},j,m\rangle$,

$$M_{jk}(\mathbf{P}) = \epsilon_{ijk}M_i(\mathbf{P}), \quad M_{0i}(\mathbf{P}) = N_i(\mathbf{P}),$$

where

$$\mathbf{M}(\mathbf{P}) = i\mathbf{P}\times\nabla_p + \mathbf{S},$$

$$\mathbf{N}(\mathbf{P}) = i\omega\nabla_p + \frac{\mathbf{P}\times\mathbf{S}}{\sqrt{s}+\omega}, \qquad (3.8)$$

and insert relations (3.8) into Eq. (3.7).

It is worth noticing that had we chosen to describe our tower of states by a multicomponent field, the algebra which we have constructed would coincide with the generators of index transformations on the Fourier transform of the field. Using a spinor basis we should simply obtain

$$S_\lambda(\mathbf{P}) = P_\mu\tilde{S}_{\mu\lambda}, \qquad T_\lambda(\mathbf{P}) = P_\mu S_{\mu\lambda},$$

where $S_{ik} = \epsilon_{ikr}S_r$, $S_{0k} = T_k$ and $\tilde{S}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}S_{\rho\sigma}$.

It is clear that the assignment of the above transformation properties to the operators S_i and T_i is consistent with the following set of commutation relations between S_i, T_i and the generators of rotations and pure Lorentz transformations or boosts M_i and N_i, respectively:

$$[M_i, S_j] = i\epsilon_{ijk}S_k,$$
$$[M_i, T_j] = i\epsilon_{ijk}T_k,$$
$$[S_i, N_j] = i\epsilon_{ijk}T_k, \qquad (3.9)$$
$$[T_i, N_j] = -i\epsilon_{ijk}S_k.$$

It follows that as far as general algebraic properties, in particular the commutation relations, are concerned, we need not distinguish between the generators of Lorentz transformations and the operators defined by Eqs. (3.1) to (3.4); in other words we can speak about *one* **SL**$(2,C)$ algebra instead of one generating homogeneous Lorentz transformations and the other acting on spin states in the rest frame. So for the purpose of the following considerations we identify the operators **S**, **T** according to the relations

$$S_i \to M_i, \qquad T_i \to N_i.$$

The states $|\mathbf{P}j\lambda\rangle$ behave in the usual way under a Lorentz transformation, Λ.

$$\begin{aligned}U(\Lambda)|\mathbf{P}j\lambda\rangle &= U(\Lambda)U(L(\mathbf{P}))|\mathbf{P}=0, jm\rangle \\ &= U(L(\Lambda\mathbf{P}))U(L^{-1}(\Lambda\mathbf{P})\Lambda L(\mathbf{P}))|\mathbf{P}=0, jm\rangle \\ &= \sum_{j'm'} U(L(\Lambda\mathbf{P}))|\mathbf{P}=0, j'm'\rangle\langle\mathbf{P}=0, j'm'| \\ &\qquad \times U(L^{-1}(\Lambda\mathbf{P})\Lambda L(\mathbf{P}))|\mathbf{P}=0jm\rangle.\end{aligned}$$

The operator $L^{-1}(\Lambda\mathbf{P})\Lambda L(\mathbf{P})$ is evidently an element of the little group generated by the operators (3.5a). The reader can verify this by taking for example, an infinitesimal transformation Λ in the $(\frac{1}{2},0)$ representation of the Lorentz group.

We now come to the crucial point. *Keeping* **P** *fixed*, as discussed at the beginning of this section, we let s tend to zero. A glance at the Eqs. (3.5a,b) indicates that both the operators $\mathbf{S}(\mathbf{P})$ and $\mathbf{T}(\mathbf{P})$ develop singularities. In particular, the coefficient of the leading term in $\mathbf{S}(\mathbf{P})$ becomes

$$s^{1/2}\mathbf{S}(\mathbf{P}) \to (\mathbf{P}\cdot\mathbf{S})\mathbf{P}/|\mathbf{P}|, \quad (s\to 0), \qquad (3.10)$$

and the little group *contracts* to the group of rotations around the direction of **P**. This we have already explicitly observed with the help of representations in the previous section.

Let us pause here for a moment and point out an important physical consequence of the previous considerations. Evidently the integrand in Eq. (2.3) forms a basis of the type $|\mathbf{P}=0, jm\rangle$ for the algebras defined here. In particular, if we isolate one Regge pole, its contribution transforms according to an irreducible—in general nonunitary representation of the algebra (3.5b). As we follow the trajectory of the Regge pole, $j=\alpha(s)$, to $s=0$, the family of representations contracts to a representation of the smaller algebra (3.10).

To be specific, let us choose $\mathbf{P}=(0,0,p)$. Then, apart from the irrelevant factor p, (3.10) reduces to M_3. On the other hand we know from Wigner's work that the generators of the little group of the four-momentum $P=(0,0,p,p)$, where $s=0$, are given by the operators M_1+N_2, M_2-N_1 and M_3. In other words, after performing the limit operation $s\to 0$ we obtain not the full Lie algebra of $E(2)$ but only the subalgebra spanned by the single generator M_3. The representations of the latter coincide with those of $E(2)$ corresponding to the zero eigenvalue of the Casimir operator, Wigner's "finite spin" representations. We have thus proved the following:

1. *Theorem:* In a theory where the spin of the states is a function of the mass, $j=\alpha(s)$, and $\lim_{s\to 0}(\alpha(s))$ exists, only the finite-spin-type representations of the lightlike little group are realized.

An immediate consequence of this theorem is that if for some reason one needs a "tower" of particles with zero mass, then either those particles are all elementary, in which case they can belong to irreducible representations of $E(2)$, or they lie on Regge trajectories and necessarily individually span reducible representations of the lightlike little group. The significance of this remark should be obvious to the reader who is familiar with the elementary theory of the so-called "conspiracy." We shall return to the "conspiracy" problem later. Note that the elementary particles correspond to Kronecker-δ singularities in j and can be subtracted out of the amplitude. In this paper we are principally concerned with the singularities of the analytic amplitude corresponding to Regge poles or cuts. The result of the previous theorem is more easily understandable if one recalls the Wigner-Inonu procedure of obtaining $E(2)$ by contraction from $SU(2)$. In fact if the contraction is performed on the representations in order to obtain the full group $E(2)$ one has to let the mass tend to zero and the spin j tend to infinity with $j=0(1/s)$. Evidently this would mean that a Regge trajectory has a first-order pole at $s=0$. Such a possibility is excluded by the unitarity of the scattering operator.

Let us now continue our group-theoretical investigations.

At the point of contraction, $s=0$, the singularity discovered in Eq. (3.5a) must be reflected in at least one of the matrix elements in the representations of the algebra. In fact one can prove that *if, at the point $s=0$, none of the representation matrices of the algebra* (3.5) *have a singularity in s, then the point $s=0$ cannot be a point of contraction.* The proof is straightforward and is left to the reader.

Conversely, if we want to make sure that none of the transition amplitudes (2.3) have an unwanted singularity at $s=0$, *we have to arrange the irreducible amplitude contributions to form the basis of a representation of an algebra which is preserved at the contraction point.* The question as to which subalgebras "survive" contraction is answered by a theorem[19] on Lie algebras. Provided certain conditions are satisfied, which they are in our applications, we have the following:

2. *Theorem:* A subalgebra $\mathbf{g}$ of an algebra $\mathbf{G}$ survives the contraction if and only if it is spanned by the root vectors of $\mathbf{G}$.

For the reader's convenience, we briefly recall some of the definitions involved, which can be found for example in Hermann's book on Lie groups.[20]

If $\mathbf{G}$ is a Lie algebra and $\mathbf{M}$ its maximal compact subalgebra, then the decomposition $\mathbf{G}=\mathbf{M}\oplus\mathbf{N}$ with $[\mathbf{M},\mathbf{N}]\subset\mathbf{N}$ and $[\mathbf{N},\mathbf{N}]\subset\mathbf{M}$ is a Cartan decomposition. A Cartan subalgebra, $\mathbf{A}$ is a maximal Abelian subalgebra of $\mathbf{N}$. A *root vector, W,* of $\mathbf{G}$ is an eigenvector of the elements of $\mathbf{A}$ in the sense that for $N\in\mathbf{A}$ we have $[N,W]=\lambda(N)W$, where the root $\lambda(N)$ is a linear form on $\mathbf{A}$.

The elements $g=\exp tX$, where $X\in\mathbf{G}$, form a one-parameter subgroup of G, the Lie group generated by $\mathbf{G}$. The symbol Ad $g(Y)$ denotes the set of elements of $\mathbf{G}$ of the form gYg^{-1} where $Y\in\mathbf{G}$. We say that if

$$\mathbf{g}(\infty)=\lim_{t\to\infty}\mathrm{Ad}\,\exp tX(\mathbf{g})$$

exists it is a *contraction* of $\mathbf{g}$, and $\mathbf{g}$ *survives* contraction if $\mathbf{g}(\infty)$ is isomorphic to $\mathbf{g}$.

We now apply theorem 2 to the Lorentz group contractions. We work with Hermitean generators and the transformation in question, the boost along the 3-axis, for example, is represented by $\exp(i\beta N_3)$, where $\cosh\beta=\omega/\sqrt{s}$. Evidently we have $\beta\to\infty$ as $s\to 0$ with $|\mathbf{P}|$ fixed. It is an easy task to find the root vectors X such that $[X,N_3]=\lambda N_3$:

$$\left.\begin{array}{l} N_3 \\ M_3 \end{array}\right\}\lambda=0$$
$$\left.\begin{array}{l} W_1=M_1+M_2+N_2-N_1 \\ W_2=M_1-M_2+N_2+N_1 \end{array}\right\}\lambda=i \qquad (3.7)$$
$$\left.\begin{array}{l} W_3=M_1+M_2+N_1-N_2 \\ W_4=M_2-M_1+N_1+N_2 \end{array}\right\}\lambda=-i.$$

Some algebras spanned by the root vectors are:

(1) the Abelian algebra, (M_3,N_3)

(2) the algebra $E(2)$ where we have "two copies" with generators M_1+N_2, M_2-N_1, M_3 and M_1-N_2, M_2+N_1, M_3, respectively, and last but not least

(3) the algebra $SL(2,C)$ itself.

According to theorem 2, these are the only algebras

[19] G. Domokos and G. L. Tindle, Commun. Math. Phys. **7**, 160 (1968).

[20] R. Hermann, *Lie Groups for Physicists* (W. A. Benjamin, Inc., New York, 1966).

which survive the contraction $s\to 0$, so at first glance we could choose any one of them to classify the singularities at $s=0$. Physical considerations, however, restrict the choice. If s is nonzero, say $s>0$, the Regge poles are classified by $SU(2)$ and going to $s=0$ cannot change their number. Now we know that $SU(2)$ does not survive the contraction so we must classify the Regge poles according to a *reducible* representation of $SU(2)$, and the "surviving algebra" must contain $SU(2)$ as a proper subalgebra. This restricts the choice to $SL(2,C)$. The argument could be formalized by invoking the continuity theorem on the singularities of a function of two complex variables but we do not want to dwell on this here. We now summarize these results.

3. *Theorem: In order to make the analyticity properties and Lorentz invariance of the scattering amplitude compatible, it is necessary and sufficient for the analytic part of the partial-wave amplitudes at $s=0$ to "conspire" to form the basis of a representation of $SL(2,C)$. In particular, the analytic singularities, Regge poles or cuts, of the amplitude must form the basis of a reducible representation of $SU(2)$ which for $s=0$ goes over to one of $SL(2,C)$.*

IV. REGGEIZATION OF SCATTERING AMPLITUDES WITH ARBITRARY MASSES AND SPINS IN THE REGION OF VANISHING MOMENTUM TRANSFER

Using Eq. (2.3), we can easily exhibit the contribution of Regge poles to the scattering amplitude. Let us, first of all, recall that the partial-wave expansion (2.3) can be obtained by decomposing the amplitude,

$$\langle p_3s_3\lambda_3,p_4s_4\lambda_4|T|p_1s_1\lambda_1,p_2s_2\lambda_2\rangle,$$

with respect to irreducible representations of the Poincaré group. Writing formally $\sum_j$, instead of the contour integral, we have, in an arbitrary reference frame,

$$\langle p_3s_3\lambda_3,p_4s_4\lambda_4|T|p_1s_1\lambda_1,p_2s_2\lambda_2\rangle$$
$$=\int d^4Pd^4P'\sum_{jj'mm',\nu_1\nu_2\nu_3\nu_4}\langle p_3s_3\lambda_3;\,p_4s_4\lambda_4|Pjm;\,\nu_3\nu_4\rangle$$
$$\times\langle Pjm;\,\nu_3\nu_4|T|P'j'm';\,\nu_1\nu_2\rangle$$
$$\times\langle P'j'm';\,\nu_1\nu_2|p_1s_1\lambda_1,p_2s_2\lambda_2\rangle+(\text{crossed term}).$$

Now

$$\langle Pjm;\,\nu_3\nu_4|T|P'j'm';\,\nu_1\nu_2\rangle=\delta(P-P')\delta_{jj'}\delta_{mm'}\langle\nu_3\nu_4\|T(s,j)\|\nu_1\nu_2\rangle,$$

where $P^2=s$ and $\langle\nu_3\nu_4\|T(s,j)\|\nu_1\nu_2\rangle$ is the reduced matrix element with respect to the Poincaré group. Further, it is convenient to introduce the eigenamplitudes $T_\gamma(s,j)$ by a unitary transformation:

$$\langle\nu_3\nu_4\|T(s,j)\|\nu_1\nu_2\rangle=\sum_\gamma\langle\nu_3\nu_4|\gamma\rangle T_\gamma(s,j)\langle\gamma|\nu_1\nu_2\rangle.$$

Let us denote the Clebsch-Gordan coefficient of the Poincaré group,[21] in the representation where the scattering amplitude is diagonal, by $\langle p_1s_1\lambda_1,p_2s_2\lambda_2|Pjm\gamma\rangle$, where

$$\langle p_1s_1\lambda_1,p_2s_2\lambda_2|Pjm\gamma\rangle=\sum_{\nu_1\nu_2}\langle p_1s_1\lambda_1,p_2s_2\lambda_2|Pjm;\,\nu_1\nu_2\rangle\langle\nu_1\nu_2|\gamma\rangle.$$

(The set of quantum numbers γ contains parity if we assume, as is usual, that it is conserved by strong interactions; in some cases, such as πN scattering, no other quantum numbers are needed.) In this representation the contribution of Regge poles to the amplitude can be written as follows[22]:

$$\mathcal{T}=\langle p_3s_3\lambda_3,p_4s_4\lambda_4|T|p_1s_1\lambda_1,p_2s_2\lambda_2\rangle_{(\text{pole})}$$
$$\sim\sum_{mm'\gamma,\alpha_{i\gamma}(s)}\int d^4P\langle p_3s_3\lambda_3,p_4s_4\lambda_4\|P\alpha_{i\gamma}(s),m,\gamma\rangle\delta_{mm'}$$
$$\times\beta_{i\gamma}^*(s,\alpha_{i\gamma}(s))\frac{1\pm e^{-i\pi\alpha_{i\gamma}(s)}}{\sin\pi\alpha_{i\gamma}(s)}\beta_{i\gamma}(s,\alpha_{i\gamma}(s))$$
$$\times\langle P,\alpha_{i\gamma}(s),m'\gamma\|p_1s_1\lambda_1,p_2s_2\lambda_2\rangle$$
$$\times\delta(P-p_1-p_2)\delta(p_3+p_4-p_1-p_2),\quad(4.1)$$

where we have omitted some unimportant normalization factors. Here, $\alpha_{i\gamma}(s)$ is the ith Regge pole in the eigenchannel γ and $\beta_{i\gamma}(s,\alpha_{i\gamma}(s))$ is the corresponding form factor. The coefficient $\langle p_1s_1\lambda_1p_2s_2\lambda_2\|P,\alpha,m\gamma\rangle$ is the continuation of the "reduced" Clebsch-Gordan coefficient of the Poincaré group (the momentum-conserving δ function has been factored out) to arbitrary values of angular momentum (cf. Andrews and Gunson).[12]

Theorem 3 of the previous section now states that:

(a) in every eigenchannel γ, there is at least one subset of Regge trajectories, say $\alpha_{i_\kappa\gamma}(s)$, such that

$$\lim_{s\to0}\alpha_{i_\kappa\gamma}(s)=\alpha_{i_0\gamma}(0)-\kappa,\quad(\kappa=0,1,2\cdots);\quad(4.2)$$

(b) the residues $\beta_{i_\kappa\gamma}$ become correlated in such a way that

$$\lim_{s\to0}\beta_{i_\kappa\gamma}(s,\alpha_{i_\kappa\gamma}(s))\beta_{i_\kappa\gamma}^*(s,\alpha_{i_\kappa\gamma}(s))\delta_{m,m'}$$
$$=\lim_{s\to0}\Gamma_\gamma(s,\sigma_{i\gamma}(s))\Gamma_\gamma^*(s,\sigma_{i\gamma}(s))\delta_{mm'}$$
$$\times\sum_{j''m''}\langle\alpha_{i_\kappa\gamma}(s),m|\sigma_{i\gamma}(s)j_0;\,j''m''\rangle$$
$$\times\langle\sigma_{i\gamma}(s)j_0;\,j''m''|\alpha_{i_\kappa\gamma}(s),m\rangle.\quad(4.3)$$

The coefficients $\langle\alpha m|\sigma j_0;\,jm\rangle$ are essentially generalized Clebsch-Gordan coefficients, decomposing a basis state

[21] G. C. Wick, Ann. Phys. (N. Y.) 18, 65 (1962).

[22] We recall that the residue factorization theorem holds for the eigenamplitudes cf., e.g., G. Domokos, thesis, Dubna, 1963 (unpublished).

$|\sigma j_0,jm\rangle$ of a representation (σj_0) of $SL(2,C)$ into representations of $SL(2,R)$. Explicit expressions for these, together with some of the important properties, were given by Sciarrino and Toller.[6] The coefficients Γ_γ are the form factors of a Lorentz pole. [Let us note at this point that, strictly speaking, the "states" $|\alpha,m\rangle$, $|\sigma j_0,jm\rangle$ form bases for the representations of the "boosted" algebras (2.5), so that we should write them in the form $|\sigma j_0,jm;P\rangle$, $|\alpha,m;P\rangle$. However, the dependence of the coupling coefficients on the four-momentum P is trivial.]

The position of the Lorentz pole $\sigma_{i\gamma}(0)$ is simply given by the relation

$$\sigma_{i\gamma}(0)=\alpha_{i_0\gamma}(0).$$

After inserting (4.2) and (4.3) into (4.1), we obtain the pole contribution to the amplitude $\mathcal{T}$ arranged according to Lorentz poles. The evaluation of the resulting expression in the general case is quite complicated. However, if we make the customary assumption that in some region (for example when the invariant energies of the crossed channels t and u tend to infinity) the amplitude is dominated by the contributions of the Regge poles, the expression (4.1), together with (4.2) and (4.3), can be evaluated quite easily.

In this case we can assume that the set of improper states $|\alpha m\rangle$ is complete so that operators of the type

$$\sum_{\alpha m}\int\frac{d^3P}{2\omega_P}|\alpha mP\rangle\langle\alpha mP|$$

can be replaced by the unit operator (a "Reggeized" version of the familiar closure approximation). In this way after some straightforward manipulations we arrive at the expression

$$\mathcal{T}\sim\sum_{\sigma_{i\gamma}(s),\gamma}\int d^4P\Gamma_\gamma(s,\sigma_{i\gamma}(s))\frac{1\pm e^{-i\pi\sigma_{i\gamma}(s)}}{\sin\pi\sigma_{i\gamma}(s)}\Gamma_\gamma^*(s,\sigma_{i\gamma}(s))$$
$$\times\sum_{jm}\langle p_3s_3\lambda_3p_4s_4\lambda_4\|P\sigma_{i\gamma}(s)j_0;\,jm\rangle$$
$$\times\langle P\sigma_{i\gamma}(s)j_0;\,jm\|p_1s_1\lambda_1p_2s_2\lambda_2\rangle$$
$$\times\delta(p_1+p_2-P)\delta(p_1+p_2-p_3-p_4)$$
$$(t\to\infty,\,s\to0). \quad (4.4)$$

[The integration over P is again trivial and $\alpha_{i_0\gamma}(s)$ has been replaced by $\sigma_{i\gamma}(s)$ according to (4.2).]

We now define the c.m. amplitude $\mathfrak{F}$:

$$\lim_{s\to0}\mathfrak{F}=\lim_{s\to0}\sum_{\sigma_{i\gamma}(s),\gamma}\Gamma_\gamma(s,\sigma_{i\gamma}(s))\frac{1\pm e^{-i\pi\sigma_{i\gamma}(s)}}{\sin\pi\sigma_{i\gamma}(s)}\Gamma_\gamma^*(s,\sigma_{i\gamma}(s))$$
$$\times\sum_{jm}\langle\mathbf{p}_3s_3\lambda_3,\,-\mathbf{p}_3s_4\lambda_4|P_0\sigma_{i\gamma}(s)j_0;\,m\rangle$$
$$\times\langle P_0\sigma_{i\gamma}(s)j_0;\,jm|\mathbf{p}_1s_1\lambda_1,\,-\mathbf{p}_1s_2\lambda_2\rangle, \quad (4.5)$$

with $P_0=\sqrt{s}$. In the Appendix we calculate the coefficients $\langle ps\lambda,\,-ps'\lambda'|P_0\sigma j_0jm\rangle$ and find that they are proportional to the matrix element $D_{j'\lambda,jm}{}^{(\sigma j_0)}(L^{-1}(p))$ of the inverse of the Lorentz transformation $L(p)$ which produces the relative momentum p.

Thus, after multiplying together the Lorentz transformations and taking into account the results of the Appendix, we find

$$\lim_{s\to0,t\to\infty}\mathfrak{F}=\lim_{s\to0,t\to\infty}\sum_{\sigma_{i\gamma}(s),\gamma}\Gamma_\gamma(s,\sigma_{i\gamma}(s))\frac{1\pm e^{-i\pi\sigma_{i\gamma}(s)}}{\sin\pi\sigma_{i\gamma}(s)}$$
$$\times\Gamma_\gamma^*(s,\sigma_{i\gamma}(s))\sum_{j'j''}\langle s_3\lambda_3s_4-\lambda_4|j'\lambda'\rangle$$
$$\times D_{j',+\lambda';j'',+\lambda''}{}^{(\sigma_{i\gamma}(s),j_0)}(L^{-1}(p_3)L(p_1))$$
$$\times\langle j''\lambda''|s_1\lambda_1,\,s_2-\lambda_2\rangle, \quad (4.6)$$

where j' and j'' are the total spins of the initial and final states.

We note that the unwanted singularities occurred in the amplitude (2.3) when considering $D^\alpha(L(p))$ with complex α after j Reggeization. However, we now Reggeize in σ, and $D^j(L(p))$ appears where j, the total spin, is integral or half integral and $D^j(L(p))$ is simply a polynomial in the cosine of the scattering angle.

With trivial kinematical modifications, Eq. (4.6) is identical to the expression derived for the scattering amplitude by Toller,[6] for the equal-mass case, and generalizes what was called a "broken-symmetry expansion" in Ref. 10 to the case of arbitrary spins.

However, we must emphasize that we obtained Eq. (4.6) on the basis of theorem 3 and the customary assumption that the Regge poles dominate the amplitude as the energy tends to infinity and have not used the invariance of the amplitude under $SL(2,C)$ at zero four-momentum transfer.

V. CONCLUSION

The results of the present work show clearly that the so-called "conspiracy" is essentially a kinematic phenomenon and has nothing to do with some mysterious dynamical interplay of various Regge poles, like the π-A_1 conspiracy.

We were able to resolve the problem, in full generality, because we recognized the essential difference between the invariance group of a scattering amplitude and the classification group of the spectrum. The role of the homogeneous Lorentz group is to classify the singularities in the angular momentum plane, at vanishing invariant energy, and this role is universal in the sense that the properties of the spectrum are independent of any particular scattering process. The circumstance that a scattering amplitude, with all the external particle masses equal, actually becomes invariant under the homogeneous Lorentz group, at vanishing momentum transfer, thus appears somewhat accidental.

Evidently our procedure is not restricted to two particle amplitudes or to Regge poles. A generalization to Regge cuts is straightforward, but the extension to many-body amplitudes will require a careful analysis of the analytic properties of the group structure of these amplitudes.

There has been some controversy in the literature whether spin is or is not an essential complication in S-matrix theory. We do not want to take sides in this controversy but want to emphasize the point that Lorentz invariance is an essential complication.

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APPENDIX

In order to calculate the coupling coefficient, $\langle P_0(\sigma j_0)jm|\mathbf{p}s_1\lambda_1-\mathbf{p}s_2\lambda_2\rangle$, used in Sec. 4, we first of all note that, following Jacob and Wick,[23] we can write the helicity states $|\mathbf{p}s_1\lambda_1,-\mathbf{p}_1s_2\lambda_2\rangle$ as follows:

$$|\mathbf{p}s_1\lambda_1;-\mathbf{p}s_2\lambda_2\rangle = U(R(\phi,\theta,-\phi))|p_3s_1\lambda_1;-p_3s_2\lambda_2\rangle, \quad \text{(A1)}$$

where on the right-hand side we have a helicity state with momentum along the 3-axis.

After defining the one-particle momenta in a general Lorentz frame by

$$p_1=\tfrac{1}{2}P+p\,,\qquad p_2=\tfrac{1}{2}P-p \quad \text{(A2)}$$

($2p$ being the relative momentum, P the total four-momentum of the two-particle system), we can obtain the given values of the relative momentum p by a Lorentz transformation acting on p but not on P.

[23] M. Jacob and G. C. Wick, Ann. Phys. (N. Y.) **7**, 404 (1959).

In particular, if the relative momentum p points along the 3-axis, we have

$$\begin{aligned} p_3&=z_{30}(v)\tilde{p}^0=(p^2)^{1/2}v/(1-v^2)^{1/2}\,,\\ p_0&=z_{00}(v)\tilde{p}^0=(p^2)^{1/2}/(1-v^2)^{1/2}\,,\\ p_1&=p_2=0\,, \end{aligned} \quad \text{(A3)}$$

where $\tilde{p}^0=(p^2)^{1/2}=\tfrac{1}{2}[2(m_1^2+m_2^2)-s]^{1/2}$, obtained directly by adding the squares of Eqs. (A2). As usual we have $p_1^2=m_1^2$, $p_2^2=m_2^2$ and $P^2=s$. [This procedure corresponds to a partial recoupling of the product state (A1).] After combining these results with (A1) we obtain the equation

$$|\mathbf{p}_1s_1\lambda_1,-\mathbf{p}s_2\lambda_2\rangle=U(L(p))|\mathbf{p}=0,s_1\lambda_1;\mathbf{p}=0,s_2\lambda_2\rangle\,, \quad \text{(A4)}$$

where $L(p)$ is a Lorentz transformation acting on the relative momentum alone:

$$L(p)=R(\phi,\theta,-\phi)z(v)\,. \quad \text{(A5)}$$

The state on the right-hand side of Eq. (A4) can be reduced out with respect to the rotation group

$$|\mathbf{p}=0,s_1\lambda_1;\mathbf{p}=0,s_2\lambda_2\rangle=\sum_{j'=|s_1-s_2|}^{s_1+s_2}|j'\lambda\rangle\langle j'\lambda|s_1\lambda_1,s_2-\lambda_2\rangle\,,$$

with $\lambda=\lambda_1-\lambda_2$. Thus for the coupling coefficient we have

$$\langle P_0,\sigma j_0jm|\mathbf{p}s_1\lambda_1-\mathbf{p}s_2\lambda_2\rangle=\sum_{j'}\langle P_0\sigma j_0jm|U^{-1}(L^{-1}(p)) \times|j'\lambda\rangle\langle j'\lambda|s_1\lambda_1s_2-\lambda_2\rangle. \quad \text{(A6)}$$

(We omitted the labels $\mathbf{p}=0$, P_0 from the right-hand side.) The bra-vector $\langle P_0\sigma j_0jm|$ in Eq. (A6) transforms according to an irreducible representation of the Lorentz group so that Eq. (A6) becomes

$$\langle P_0\sigma j_0jm|\mathbf{p}s_1\lambda_1,-\mathbf{p}s_2\lambda_2\rangle=\sum_{j'm'j'}D_{jmj''m''}{}^{(\sigma j_0)}(L^{-1}(p)) \times\langle P_0\sigma j_0j''m''|j'\lambda\rangle\langle j'\lambda|s_1\lambda_1s_2-\lambda_2\rangle\,,$$

and finally, noticing that we have the relation

$$\langle P_0\sigma j_0j''m''|j'\lambda\rangle=\delta_{j''j'}\delta_{m''\lambda}\,,$$

we obtain the desired result

$$\langle P_0\sigma j_0jm|\mathbf{p}s_1\lambda_1-\mathbf{p}s_2\lambda_2\rangle=\sum_{j'}D_{j,j;j',\lambda}{}^{(\sigma,j_0)}(L^{-1}(p)) \times\langle j'\lambda|s_1\lambda_1,s_2-\lambda_2\rangle.$$