$$|G_n(x,y)| \leq \epsilon R_n(x,y) + \epsilon^2|\lambda| \int R_n(x,z)R_n(z,y)dz + \cdots , \quad \text{(A11)}$$

and therefore G_n is also relatively uniformly convergent to zero.

In the limit $n \to \infty$ the second and fourth terms of (A9) vanish by (A2) and (A3), and we have

$$f(x) = g(x) + \lambda \int K_\infty(x,y) f(y) dy. \quad \text{(A12)}$$

If we define a sequence of functions $\{f_n(x)\}$ which satisfy

$$f_n(x) = g(x) + \lambda \int K_n(x,y) f_n(y) dy, \quad \text{(A13)}$$

then the limit $f(x) = \lim_{n\to\infty} f_n(x)$ satisfies

$$f(x) = g(x) + \lambda \int K_\infty(x,y) f(y) dy, \quad \text{(A14)}$$

which is identical to (A12).

We have therefore proven the following theorem:

If $\{K_n\}$ is a relatively uniformly convergent sequence of separable approximations to K, $f = g + \lambda K f$ and $f_n = g + K_n f_n$, then $f = \lim_{n\to\infty} f_n$ up to the possible addition of a solution of the homogeneous equation.[15]

When λ is not a characteristic value of K, the solution of (A9) may be written explicitly in terms of g, G_n, and K_n (Ref. 14, p. 45). One can then determine the rate of convergence of $\{f_n\}$ to f once the rate of convergence of $\{G_n\}$ is known.

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Even-Parity Meson Resonances*

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A recent model of the odd-parity baryon resonances is extended to include even-parity meson resonances. The resonances are assumed coupled to S-wave and D-wave MM states (where M denotes a pseudoscalar or vector meson) with interaction constants such that the resonance pole contributions to the collinear $MM \to MM$ scattering amplitudes satisfy $SU(6)_W$ symmetry. The relative coupling of the 35-fold and 1-fold M representations is taken in accordance with a bootstrap model. The set of even-parity resonances predicted in this model are octets of $j^C = 0^+$, 1^+, 2^+, and 1^-, and 0^+, 2^+, and 1^- singlets, where C is the particle-antiparticle conjugation quantum number of the $I_z = Y = O$ particles. The partial widths for MM decays and the principal terms in the mass splitting are computed in the model. The agreement with the present experimental data is good.

I. INTRODUCTION

IN a recent paper (referred to here as R1), an $SU(6)_W$-symmetric bootstrap model of MB and MM states was developed.[1] The B and M considered were the 56-fold baryon supermultiplet, and 35-fold supermultiplet of odd-parity mesons. The $SU(6)_W$ symmetry was applied to the forward and backward one-particle exchange amplitudes T_f and T_b. The linear combinations $T_f \pm T_b$ represent potentials in states of even and odd orbital angular momentum, respectively. It was found in R1 that the M and B can be bootstrapped successfully only if baryons of odd parity and mesons of even parity exist. The simplest exact solution is one in which these "off-parity" baryons and mesons correspond to the $SU(6)_W$ representations 70 and $35 \oplus 1$, respectively.

Two basic steps are necessary to determine the physical consequences of the model. The first is that of R1, the formulation and solution of the self-consistency equations in terms of $SU(6)_W$ states. The second is the interpretation of the $SU(6)_W$ states in terms of spin and other physical quantum numbers. In a recent work, it was shown that the odd-l MM and MB composites may be associated with the basic M and B particles, and that the even-l MB states correspond physically to the representation (70,3) of $SU(6) \otimes O(3)$.[2] The purpose of this paper is to make a similar analysis of the even-parity meson states of the model, and to compare the predicted properties of these mesons with experiment.

Attention was limited to $M_{35}M_{35}$ and $M_{35}B_{36}$ states in R1, where the subscript is the $SU(6)_W$ multiplicity. (This did not imply the assumption that these are the only important states coupled to the composites, because the bootstrap condition of R1 may be applied separately to processes involving different external par-

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[1] R. H. Capps, Phys. Rev. **161**. 1538 (1967). Herein referred to as R1.
[2] R. H. Capps, Phys. Rev. **158**. 1433 (1967). Herein referred to as R2.

ticles). In order that we may discuss thoroughly the MM branching ratios of the meson resonances, we must consider M_1M_{35} and M_1M_1 states as well as $M_{35}M_{35}$ states. Predictions of ω decay modes would be unreliable without this extension, since the zero helicity state of the vector singlet is the M_1 multiplet, while the other two helicity states are in the M_{35} multiplet. (A similar problem does not arise in the baryon resonance analysis, since M_1B_{56} states are not coupled to the resonance multiplet 70.)

In Sec. II of this paper, the model of R1 is extended to include the M_1M_1 and M_1M_{35} states. The classification in this section is in terms of the group $SU(6)_W$. The interpretation of the predicted resonances in terms of physical spin is given in Sec. III, and compared with experiment in Sec. IV.

II. BOOTSTRAP CONDITIONS FOR M_1M_1 AND M_1M_{35} STATES

We consider the general group $SU(n)_W$, where $n>2$, and denote the odd-parity meson multiplet by $M_r\oplus M_1$, where r and 1 refer to the regular and identity representations of the group.[3] The M are taken to be degenerate. In R1 a bootstrap model was developed, in which the hadrons are composites of superpositions of two-hadron states, held together by one-hadron exchange forces. A self-consistency equation was derived and applied to the one-meson-exchange potentials in M_rM_r states of both parities. A solution was found involving the composites M_r, N_r, and N_1, where N_i is an even-parity state corresponding to the $SU(6)_W$ representation i. The physical significance of these $SU(6)_W$ states is not discussed until Sec. III. Here, we extend the arguments to M_1M_r and M_rM_r states.

The possible Yukawa interactions that involve at least two M mesons are

$$M_rM_rM_r,\quad N_rM_rM_r,\quad N_1M_rM_r,\quad N_rM_1M_r,\quad N_1M_1M_1.$$

The coupling constant associated with the first of these interactions is denoted by F_r, and the constant associated with the $N_\alpha M_\beta M_\gamma$ interaction is denoted by $G_{\alpha,\beta\gamma}$. These are total constants, i.e., the squares are sums over MM states. For example, $G_{r,rr}^2=\sum_{jk}G(N_iM_jM_k)^2$, where the sum includes every M_rM_r state once and only once. The constants F_r, $G_{r,rr}$, and $G_{1,rr}$ are equal to F_M, F_D, and F_I of R1.

The M_1 are not coupled to MM states, and so are not bootstrapped in this paper. The M_1 are coupled to MN states. However, it was pointed out in R1 that states of different external particles may be considered separately, so that the neglect of MN states in this paper is not an approximation, but means only that all possible self-consistency conditions are not utilized.

[3] The $SU(n)_W$ group is defined and discussed by H. J. Lipkin and S. Meshkov, Phys. Rev. **143**, 1269 (1966).

Consideration of M_rM_r states in R1 yields the results

$$G_{1,rr}^2=\frac{2(n^2-1)}{n^2-4}G_{r,rr}^2=\kappa_e^{-1}\frac{2(n^2-1)}{n^2}F_r^2,\qquad(1)$$

where κ_e is an undetermined constant that measures the effective ratio of N and M exchange potentials in the even-parity MM states. We must extend the bootstrap argument to the following four processes:

$$\begin{array}{lll}\text{Singlet processes:} & \text{(A)} & M_1M_1\to M_1M_1\\ & \text{(B)} & M_1M_1\to M_rM_r\\ \text{Representation } r \text{ processes:} & \text{(C)} & M_1M_r\to M_1M_r\\ & \text{(D)} & M_1M_r\to M_rM_r.\end{array}\qquad(2)$$

Only the exchange of even-parity mesons (N_1 and N_r) contribute to the potentials for these processes, but the potentials may be present in states of both parities. We consider the even-parity states first. The self-consistency equation for these states is Eqs. (17) of R1, with the constants related by Eq. (20). In the absence of M exchange potentials, this condition for the process $M_aM_b\to M_cM_d$ may be written in the simple form

$$2G_{i,ab}G_{i,cd}=\textstyle\sum_j C_{ij}G_{j,ac}G_{j,bd}+\sum_k C_{ik}G_{k,ad}G_{k,bc},\qquad(3)$$

where the C are crossing matrix elements.

This condition is easily transformed into the $SU(6)_W$ representation. The N_1 exchange contributes to processes A and C of Eq. (2), while N_r exchange contributes to processes B, C, and D. Since all processes involve the singlet, the crossing matrix elements may be determined easily, although one must be careful to take into account the symmetry of the MM states properly. The results of applying the condition to these processes are, respectively,

$$2G_{1,11}^2=2G_{1,11}^2,\qquad(4a)$$

$$2G_{1,11}G_{1,rr}=(n^2-1)^{1/2}G_{r,1r}^2,\qquad(4b)$$

$$2G_{r,1r}^2=G_{r,1r}^2+2(n^2-1)^{-1/2}G_{1,11}G_{1,rr},\qquad(4c)$$

$$2G_{r,r1}G_{r,rr}=2G_{r,r1}G_{r,rr}.\qquad(4d)$$

The first and last of these equations are identities, and the second and third are equivalent.

Self-consistency also requires that the potentials in odd-parity states vanish. There are no odd-parity M_1M_1 states, so this condition applies only to processes C and D. In process D, the odd-parity M_rM_r states of the representation r are of the antisymmetric (f-type) coupling, while the N_r exchange couples to M_rM_r states with symmetric (d-type) coupling. It is easy to show that crossing does not connect f- and d-type $M_1M_r\to M_rM_r$ processes, so the odd-parity potential vanishes for process D. The N_1 and N_r exchange contributions to the odd-parity potential for process C separately do not vanish; however, the N_r potential is associated with an "exchange-type" interaction and thus is opposite in sign from the corresponding even-parity potential. The condition that the total odd-parity potential vanish is

$$2(n^2-1)^{-1/2}G_{1,11}G_{1,rr}-G_{r,1r}{}^2=0.$$

This condition is equivalent to Eq. (4b).

Thus, all the bootstrap conditions are satisfied if Eq. (4b) is satisfied. The G are determined only within a multiplicative constant K^ν, where ν is the number of M_1 particles associated with the vertex. Consideration of the MN states might allow the determination of the constant K. We will follow an alternative procedure, choosing the value of K that leads to a "total d-type" interaction, which possesses a very simple symmetry. One may determine the coupling constants corresponding to such an interaction by associating an n by n matrix with each of the N and M multiplets. The MM state associated with the ik element of the N matrix is then given by the symmetrized product $N_{ik}\sim\sum_j (M_{ij}M_{jk}+M_{jk}M_{ij})$. This construction is a well known method of computing the $G_{r,rr}$ Clebsch-Gordan coefficients.[4] We assume that the value of K is given by this prescription for three reasons: (i) The prescription is simple; (ii) it is consistent with the bootstrap requirements of Eqs. (1) and (4b); (iii) it is supported by the experimental evidence on B meson decay, as discussed in Sec. IV A.

The ratios of the G^2 that follow from this prescription are

$$G_{r,rr}{}^2:G_{r,1r}{}^2:G_{1,rr}{}^2:G_{1,11}{}^2=(\tfrac{1}{2}n^2-2):2:(n^2-1):1. \quad (5)$$

III. PREDICTED PROPERTIES OF THE MESON RESONANCES

We now turn our attention to the physical interpretation of the N resonances. The formalism used is the logical extension of that of R2; we will discuss it here briefly. The even-parity potential T in MM states is the average of the forward and backward, one-particle-exchange amplitudes. The potential is determined from the arguments of Sec. II; in the $SU(6)_W$ representation it may be written

$$T_{\beta fm,\beta' f'm'}=G_{35,\beta}G_{35,\beta'}\delta_{f,35}\delta_{f',35}\delta_{mm'}+G_{1,\beta}G_{1,\beta'}\delta_{f,1}\delta_{f'1}, \quad (6)$$

where f and f' denote irreducible representations of $SU(6)_W$, m and m' are states within these representations, and β and β' indicate whether the state is of the type $M_{35}M_{35}$, M_1M_{35}, or M_1M_1. The G are the constants of Sec. II. We may combine the f and β labels by defining MM states {35a}, {35b}, {1a}, and {1b} in the following fashion:

$$\{35a\}=(8/9)^{1/2}\{35,(35,35)\}+(1/9)^{1/2}\{35,(1,35)\}, \quad (7a)$$

$$\{1a\}=(35/36)^{1/2}\{1,(35,35)\}+(1/36)^{1/2}\{1,(1,1)\}. \quad (7b)$$

The states {35b} and {1b} are orthogonal to the a states. The coefficients have been chosen so that only the a states resonate. With these labels, Eq. (6) may be rewritten in the form

$$T_{fm,f'm'}=G_{35}{}^2\delta_{f,35a}\delta_{f',35a}\delta_{mm'}+G_1{}^2\delta_{f,1a}\delta_{f',1a}, \quad (8)$$

$$G_{35}{}^2=G_{35,(35,35)}{}^2+G_{35,(1,35)}{}^2, \quad (9a)$$

$$G_1{}^2=G_{1,(35,35)}{}^2+G_{1,11}{}^2. \quad (9b)$$

The potentials are attractive. Since we are not interested in the over-all magnitude of T, we set

$$G_{35}{}^2=1,\quad G_1{}^2=2. \quad (10)$$

The relation between these two values follows from Eqs. (5), (9a), and (9b).

The wave functions of the eigenstates of the potentials must be expressed in terms of physical spin and internal quantum numbers. We denote by $(MM)_{\alpha s}{}^M$ an MM state of internal quantum numbers α, total intrinsic physical spin s, and z component of spin m. One computes the potentials T in the αsM representation, using $SU(6)$ Clebsch-Gordan coefficients and the known correspondence of the P and V states with $SU(6)_W$ states.[3,5] The amplitudes X corresponding to the exchange of spin Δ are determined from the T by the formula of R2, i.e.,

$$X_{\alpha s,\alpha's',\Delta}=\left(\frac{2\Delta+1}{2s+1}\right)^{1/2}\sum_m C(s'\Delta s;m0m)T_{\alpha's',\alpha s}{}^m, \quad (11)$$

where the C are angular-momentum Clebsch-Gordan coefficients. In the MB case discussed in R2, the $SU(6)_W$ symmetry restricts Δ to the values 0 and 2, but in the MM case $\Delta=4$ is also possible. (This can be shown to follow from the fact that in the quark construction, there are two antiquarks in MM states and only one in MB states.) However, the $\Delta=4$ terms do vanish, as is shown later. It is assumed that the lowest partial waves dominate, the S-S and S-D amplitudes being the lowest partial waves that correspond to $\Delta=0$ and 2, respectively. The S-S potential connecting the states αs and $\alpha's$ is denoted by $U_{\alpha s,\alpha'}$ and the S-D potential connecting the S state αs with the D state $\alpha's'$ is denoted by $U_{\alpha s,\alpha's'}$. These are given in terms of the X by the equations

$$U_{\alpha s,\alpha'}=(2s+1)^{-1/2}X_{\alpha s,\alpha's,0}, \quad (12a)$$

$$U_{\alpha s,\alpha's'}=(2s+1)^{-1/2}X_{\alpha s,\alpha's'2}. \quad (12b)$$

[4] See, for example, J. J. Sakurai, *Theoretical Physics, Lectures presented at a Seminar, Trieste*, (International Atomic Energy Agency, Vienna, 1963), pp. 227–249.

[5] Conveinent tables of $SU(6)$ Clebsch-Gordan coefficients are given by C. L. Cook and G. Murtaza, Nuovo Cimento **39**, 531 (1965).

We discuss the results for the $SU(3)$ octet states first. These are associated with the $SU(3)\otimes SU(2)_W$ representations $(8,3_W)$ and $(8,1_W)$ of the $SU(6)_W$ multiplet **35**. We denote an octet state of W spin w and $W_z=m$ by $\chi(w)^m$, omitting the $SU(3)$ indices. One may use $SU(6)$ Clebsch-Gordan coefficients and the known W-spin properties of the M to derive the relations[5]

$$\chi(1)^1=(\tfrac{1}{2})^{1/2}\psi_2{}^1+(\tfrac{1}{2})^{1/2}\psi_1{}^1, \tag{13a}$$

$$\chi(0)^0=(\tfrac{2}{3})^{1/2}\psi_2{}^0+(\tfrac{1}{3})^{1/2}\psi_0{}^0, \tag{13b}$$

where

$$\begin{aligned}\psi_2&=(5/9)^{1/2}(VV)_2{}^d+(4/9)^{1/2}(VV_1)_2,\\ \psi_1&=(VP)^f,\\ \psi_0&=(5/12)^{1/2}(PP)^d+(\tfrac{1}{3})^{1/2}(PP_1)+(5/36)^{1/2}(VV)_0{}^d\\&\quad+(1/9)^{1/2}(VV_1)_0.\end{aligned} \tag{14}$$

The symbols P, P_1, V, and V_1 denote pseudoscalar octet and singlet and vector octet and singlet, the subscripts of the ψ and (VV) are the total intrinsic spin values, and d and f denote d- and f-type octet-octet-octet interactions of $SU(3)$. The $I_z=Y=0$ members of these multiplets are all of particle-antiparticle conjugation parity C equal to $+1$. The corresponding equations for $\chi(1)^0$ are

$$\chi(1)^0=\psi(-)_1{}^0, \tag{15}$$

$$\psi(-)_1=(5/18)^{1/2}(PV)^d+(\tfrac{1}{9})^{1/2}(VP_1)+(\tfrac{1}{9})^{1/2}(V_1P)+(\tfrac{1}{2})^{1/2}(VV)_1{}^f, \tag{16}$$

where the $(-)$ indicates that $C=-1$ for the $I_z=Y=0$ states.

The coefficients of Eqs. (13a) and (13b) are such that the $\Delta=4$ term connecting the state ψ_2 to itself vanishes, as may be seen from Eq. (11) and a table of angular-momentum Clebsch-Gordan coefficients. We follow the procedure of R2, first diagonalizing the S-S potential. The eignestate of internal symmetry α and spin j is denoted by $\psi_{\alpha j}$ and the eigenvalue by $U_{\alpha j}{}^S$. The D state connected by the S-D potential to $\psi_{\alpha j}$ is denoted by $\varphi_{\alpha j}{}^D$, and is computed from the formula

$$\varphi_{\alpha j}{}^D=\sum_{\beta s} U_{\alpha j,\beta s}\varphi_{\beta js}/U_{\alpha j}{}^D, \tag{17}$$

where $\varphi_{\beta js}$ is the D state of total angular momentum j formed by coupling the internal symmetry and spin state $\psi_{\beta s}$ with orbital angular momentum 2. The $U_{\alpha j}{}^D$ is the S-D matrix element in the representation of the $\psi_{\alpha j}$ and $\varphi_{\alpha j}{}^D$, and is given by $U_{\alpha j}{}^D=(\sum_{\beta s} U_{\alpha j,\beta s}{}^2)^{1/2}$.

The octet amplitudes all result from the $G_{35}{}^2$ terms of T in Eq. (8). Thus, T is a projection operator for these states. It follows from the theorems of Sec. IV A of R2 that the U have the three following simple properties: (i) All S-S eigenvalues $U_{\alpha j}{}^S$ are of the same sign (positive, corresponding to attraction); (ii) the $\varphi_{\alpha j}{}^D$ corresponding to orthogonal S eigenstates $\psi_{\alpha j}$ are orthogonal; and (iii) the $U_{\alpha j}{}^S$ and $U_{\alpha j}{}^D$ are given in terms of one parameter $\lambda_{\alpha j}$ by the equations

$$U_{\alpha j}{}^S=\tfrac{1}{3}(1+\lambda_{\alpha j}),\quad |U_{\alpha j}{}^D|^2=(2/9)\times(1+\tfrac{1}{2}\lambda_{\alpha j}-\tfrac{1}{2}\lambda_{\alpha j}{}^2). \tag{18}$$

The four octet eigenstates of the S-S potential are the ψ_i of Eqs. (14) and (16). One may show by using Eqs. (11), (12a), (13a), (13b), and (15) that all four λ parameters are zero. The D-state vectors corresponding to the ψ_i are

$$\begin{aligned}\varphi_2{}^D&=(1/5)^{1/2}\varphi_{20}-(9/20)^{1/2}\varphi_{21}-(7/20)^{1/2}\varphi_{22},\\ \varphi_1{}^D&=(\tfrac{1}{4})^{1/2}\varphi_{11}+(\tfrac{3}{4})^{1/2}\varphi_{12},\\ \varphi_0{}^D&=\varphi_{02},\\ \varphi(-)_1{}^D&=\varphi(-)_{11},\end{aligned} \tag{19}$$

where φ_{ij} is the D state of total angular momentum i formed from the spin and internal symmetry state ψ_j.

The $SU(3)$ singlet potentials correspond to the $SU(3)\otimes SU(2)_W$ multiplet $(1,3_W)$ of the $SU(6)_W$ representation **35** and the $(1,1_W)$ multiplet of the representation **1**. The equations corresponding to Eqs. (13a) through (16) are

$$\chi(1)^1=\psi_2{}^1, \tag{20a}$$

$$\chi(0)^0=(\tfrac{2}{3})^{1/2}\psi_2{}^0+(\tfrac{1}{3})^{1/2}\psi_0{}^0, \tag{20b}$$

$$\chi(1)^0=\psi(-)_1{}^0, \tag{20c}$$

$$\begin{aligned}\psi_2&=(8/9)^{1/2}(VV)+(1/9)^{1/2}(V_1V_1),\\ \psi_0&=(2/3)^{1/2}(PP)+(1/12)^{1/2}(P_1P_1)+(2/9)^{1/2}(VV)\\&\quad+(1/36)^{1/2}(V_1V_1),\\ \psi(-)_1&=(8/9)^{1/2}(PV)+(1/9)^{1/2}(P_1V_1).\end{aligned} \tag{21}$$

The $SU(3)$ indices are suppressed. There are two main features of the singlet case that differ from the corresponding octet features. First, there is no $j=1$ state of positive C parity, as is seen from Eq. (20a). Second, the potential T corresponding to $\chi(0)^0$ depends on the constant $G_1{}^2$ of Eq. (8), and is thus twice the other $SU(6)_W$ potentials. These features combine so that the $\Delta=4$ terms vanish here also.

Because of the $SU(6)_W$ singlet term, the $SU(3)$ singlet part of T is not a projection operator; however, the potential U is fairly simple. The eigenvalues of the S-S potentials corresponding to the singlet states ψ_2, ψ_0, and $\psi(-)_1$ are given by

$$U_2{}^S=U_0{}^S=2U(-)_1{}^S=2U(8)^S, \tag{22}$$

where $U(8)^S$ is the common value for the octet states. The equations for φ^D differ from the octet equations [Eq. (19)] only in that there is no $\varphi_1{}^D$(for $C=+1$) and no φ_{21}. The magnitudes of the S-D potentials associated with the φ_{20}, φ_{22}, and φ_{02} terms are each twice the corresponding octet magnitudes.

The predicted set of even-parity mesons is almost the same as that predicted in the quark model. Nonets of

$j^C=0^+$, 2^+, and 1^- are predicted. However, for $j^C=1^+$, only an octet is predicted. (In the quark model, a $j^C=1^+$ singlet exists, but cannot decay into MM states.)

IV. COMPARISON WITH EXPERIMENT

A. B-Meson Decay

We make the usual assumptions that the 1208-MeV B meson is of even parity, with j^C quantum numbers equal to 1^-. This particle is identified with the isovector member of the 1^- octet predicted here. The predicted $\pi\varphi/\pi\omega$ branching ratio R depends not only on the $\omega-\varphi$ mixing angle, but also on the coupling constant ratio $G_{r,1r}/G_{r,rr}$ defined in Sec. II. This follows from the fact that the 1^- states are associated with an $SU(6)_W$ potential corresponding to $W_z=0$ [Eqs. (15) and (20c)], together with the fact that the $W_z=0$ state of the V singlet is the $SU(6)_W$ singlet.

The predicted branching ratio R may be written in the form

$$R=\rho\tan^2(\theta-\beta),\quad \beta=\tan^{-1}(G_{r,rr}/4G_{r,1r}),\qquad (23)$$

where ρ is the phase-space ratio and θ is the $\omega-\varphi$ mixing angle.[6] If the phase-space factors are proportional to the decay momenta, as is expected if the S-wave decay modes dominate, then $\rho=0.37$. Experimentally, the branching ratio is quite small, i.e., $R \lesssim 0.015$.[7] Thus, we assume that $(\theta-\beta)\approx 0$. It is usually assumed that $\theta=\tan^{-1}(2^{-1/2})$, since this value is obtained from the quark model, and is nearly consistent with the V mass spectrum and the assumption that the Gell-Mann–Okubo mass formula is valid.[8] If we accept this value of θ, then Eq. (23) and the condition $(\theta-\beta)=0$ imply $G_{r,1r}/G_{r,rr}=(\frac{1}{8})^{1/2}$. This is the ratio of Eq. (5). Thus, B decay provides supporting evidence for our choice of the arbitrary constant of Sec. II.

The small $\pi\varphi/\pi\omega$ decay ratio also provides evidence in support of the present model over that of a previous treatment, in which only odd-parity meson (M) exchange forces were included.[9] It can be shown that in the M-exchange model, a predicted zero value of R occurs only if the central force may be neglected in comparison with the tensor force. In the present model, the result is independent of the relative importance of the central ($\Delta=0$) and tensor ($\Delta=2$) terms.

B. Branching Ratios

It has been pointed out previously that experimental candidates exist for most of the even-parity meson states of the $j^C=2^+$, 1^+, 0^+, and 1^- octets and singlets.[10] These candidates are listed below[11]:

$$\begin{aligned}&2^+:\ A_2(1306),\quad K_V(1411),\quad f(1254),\quad f'(1514),\\ &1^+:\ A_1(1079),\quad K_c(1215),\quad D(1285),\\ &0^+:\ \pi_V(1003),\quad K_V(1080),\quad \eta_V(1050),\\ &1^-:\ B(1208),\quad K_A(1320),\quad H(975).\end{aligned}\qquad (24)$$

For each multiplet, the first particle is the isovector meson, the second is the strangeness (±1) meson, and the others are isoscalar mesons. These assignments are tentative, since spin-parity measurements have not been made for many of these particles.

The relative partial widths for S-wave MM decays, and the relative widths for D-wave MM decays, may be predicted from the results of Sec. III, provided phase-space factors are assumed and the effects of symmetry breaking on the coupling constants are neglected. The phase-space factors are taken to be k and k^5/M^4 for the S and D decays, where k is the decay momentum and M is the mass of the decaying particle. For those decays that involve isoscalar particles (either in the resonance or M multiplets) the singlet-octet mixing angles are also needed. These angles cannot be predicted theoretically, since they depend sensitively on the masses, and no accurate theory of the mass-splitting exists. However, for the P and V multiplets and the $j^C=2^+$ resonance multiplet, all the nonet members have been identified, so that the assumption of the Gell-Mann–Okubo formula for the squares of the masses allows one to calculate the magnitude of the angle. The mixing angle I is defined by the relations

$$\begin{aligned}Z_a&=(\cos\theta)Z_1-(\sin\theta)Z_8,\\ Z_b&=(\sin\theta)Z_1+(\cos\theta)Z_8,\end{aligned}\qquad (25)$$

where the Z_a, Z_b pair is either the (ω,φ), (X,η), or (f,f') pair. The calculation yields $|\theta(V)|\approx 40°$ for the $\omega-\varphi$ mixing angle; since this is close to the quark-model prediction, we choose the magnitude and sign as given by the quark model,[8] i.e., $\theta(V)=\tan^{-1}(2^{-1/2})$. The calculation yields, for the other two angles, $|\theta(P)|\approx 10.5°$ and $|\theta(2^+)|\approx 32.5°$ .Both signs of $\theta(P)$ are considered, while the sign of $\theta(2^+)$ is chosen so as to lead to the smaller of the two possible $f'\to\pi\pi$ partial widths.

In Table I the predicted (MM) partial widths are compared with experiment for all particles of the N supermultiplet (except the η_V) for which approximate partial widths are listed in Ref. 11. The η_V is not included, because there are insufficient data for an estimate of the mixing angle for the $j^C=0^+$ multiplet. The S and D decay modes are distinguished with subscripts,

[6] One may derive Eq. (23) by extending the technique used in Sec. III B of R. H. Capps, Phys. Rev. **144**, 1182 (1966).

[7] K. I. Hess, Lawrence Radiation Laboratory Report No. UCRL-16832 (unpublished); see also Ref. 11.

[8] F. Gürsey, T. D. Lee, and M. Nauenberg, Phys. Rev. **135**, B467 (1964).

[9] R. H. Capps, Phys. Rev. Letters, **16**, 1066 (1966).

[10] See for example, R. H. Dalitz, in *Proceedings of the Thirteenth Annual International Conference on High-Energy Physics, Berkeley, California, 1966* (University of California Press, Berkeley, Calif., 1967), p. 219. This paper contains a review of the quark model. The list of Eq. (24) above is an updated version of Dalitz's list.

[11] Except where otherwise noted, the symbols and experimental data concerning the resonances are taken from the compilation of A. H. Rosenfeld, A. Barbaro-Galtieri, W. J. Podolsky, L. R. Price, P. Soding, C. G. Wohl, M. Roos, and W. J. Willis, Rev. Mod. Phys. **39**, 1 (1967).

TABLE I. Calculated and experimental partial widths (in MeV) of the even-parity meson resonances. The experimental numbers are taken from Ref. 10.

Particle	Mode	S- or D-State probability	Calculated partial width	Experimental partial width
$j^C=2^+$ particles				
$A_2(1306)$	$\rho\pi$	3/10	39	75
	$K\bar{K}$	1/20	7.1	≈ 3
	$\eta\pi$	0.018(0.051)	7.5(22)	≈ 2
	$X\pi$	0.082(0.041)	1.4(0.7)	<2
$K_V(1411)$	$K^*\pi$	9/80	9.5	33
	$K\rho$	9/80	2.8	8
	$K\pi$	3/40	48(input)	48
	$K\eta$	0.019(0.002)	3.4(0.3)	<5
$f_0(1254)$	$\pi\pi$	0.15	155	110
	$K\bar{K}$	0.058	5.5	≈ 3
	$\eta\eta$	0.004(0.013)	<1	Unseen
$f_0'(1514)$	$(K\bar{K}^*+K^*\bar{K})$	0.32	4	<34
	$\pi\pi$	0.0004	0.5	<12
	$K\bar{K}$	0.092	32	>52
	$\eta\eta$	0.059(0.027)	13(6)	Unseen
$j^C=1^-$ particles				
$B(1208)$	$(\omega\pi)_S$	1/6	44	$\lesssim 119_{(S+D)}$
	$(\omega\pi_D)$	1/6	10	
$K_A(1320)$	$(K^*\pi)_S$	1/8	33	$\lesssim 80(K^*\pi$
	$(K^*\pi)_D$	1/8	5.4	$+K\rho,\ S+D)$
	$(K\rho)_S$	1/8	19	
	$(K\rho)_D$	1/8	0.4	
	$(K\omega)_S$	1/24	5	$<8_{(S+D)}$
	$(K\omega)_D$	1/24	<0.1	
$j^C=1^+$ particle				
$A_1(1079)$	$(\rho\pi)_S$	2/3	127(input)	$\approx 130_{(S+D)}$
	$(\rho\pi)_D$	1/6	3	
$j^C=0^+$ particle				
$\pi_V(1003)$	$(K^\pm K^0)_S$	1/4	15	$\lesssim 70$
	$(\pi\eta)_S$	0.09(0.26)	23(66)	?

except in the case of the 2^+ particles, where no S-wave decays are allowed. The $K_V(1411)\to K\pi$ and $A_1\to\rho\pi$ are taken as the input D and S modes. The first predicted width listed for modes involving the η or X corresponds to $\theta(P)=10.5°$, while the numbers in partentheses correspond to $(-10.5°)$. For all cases except the f and f' decays, the theoretical probability is the square of the appropriate coefficient in the normalized S- or D-state vector; the product of this probability and the phase-space factor is proportional to the predicted partial width. The f and f' probabilities differ only in that account has been taken of the fact that the $SU(3)$ singlet (PP) amplitudes are larger by $\sqrt{2}$ than the corresponding octet amplitudes. (This follows from the results of Sec. III that certain of the singlet potentials, which are quadratic in the coupling constants, are twice the corresponding octet potentials.)

There are no accurate data concerning S-wave decays, so the S-wave input is tentative. However, the prediction that the $A_1\to\rho\pi$ mode is the largest of all S-wave modes is in accord with the present limited data.

The successful prediction of a small $f'\to\pi\pi$ partial width is striking, because the phase-space factor for this mode is the largest of all the D-state factors, and is much larger than most. A successful prediction of this effect has been made previously, on the basis of the A_2 and $K_V(1411)$ data, $SU(3)$ symmetry, and the assumption $\theta(2^+)\approx 30°$.[12]

Recently, Elitzur *et al.* have published a model in which even-parity mesons corresponding to the $SU(3)$ representations **27**, **8**, and **1** resonate and are mixed.[13] The $f\to\pi\pi$ and $K_V(1411)\to\pi K$ partial widths are both fit in this model as well as in the present model, while the $A_2\to K\bar{K}$ width is predicted to be 7 MeV in the present paper and zero in Ref. 13. The experimental data on the A_2 decay do not favor either model over the other. Of course, if even-parity mesons corresponding to the $SU(3)$ representation **27** are discovered in this mass region, the present model would have to be discarded.

The over-all agreement between the predictions and data in Table I is satisfactory. However, most of the data concerns the 2^+ mesons, and for these the only predictions that do not result entirely from $SU(3)$ symmetry and the assumed mixing angles are the over-all PP/PV ratio and the details of the f and f' partial widths. More accurate data concerning the decays of the 1^+, 0^+, and 1^- multiplets are needed. More data con-

[12] S. L. Glashow and R. H. Socolow, Phys. Rev. Letters **15**, 329 (1965).

[13] M. Elitzur, H. R. Rubinstein, H. Stern, and H. J. Lipkin, Phys. Rev. Letters **17**, 420 (1966).

cerning all η decay modes are also needed, since there is little direct evidence concerning the actual magnitude of the $\eta-X$ mixing angle.

C. Mass Spectrum

In R2 is was shown that the differences in the average masses of the $SU(3)\otimes SU(2)$ multiplets of the (70,3) baryon resonance supermultiplet are given approximately by a three-parameter formula, the three parameters being λ [see Eq. (18)] and the root mean squares of the meson and baryon masses in the composite S-state wave functions. We now carry out a similar analysis for the N supermultiplet.

In the meson case, there is no mass-splitting of the λ type. It is pointed out in Sec. III that λ is zero for the four octets, and the $j^G=1^-$ singlet. The eigenvalues U^S of the S-state potentials of the 0^+ and 2^+ singlets are greater than the octet eignevalues, but the experimentally observed approximate degeneracy of the 2^+ singlet and octet indicates that the N masses do not depend appreciably on this potential difference. This fact, like the approximate $\rho-\omega$ degeneracy, is somewhat surprising in the $SU(6)_W$-symmetry bootstrap model.

Because of the approximate singlet-octet degeneracy and the lack of data on the singlets, we consider only the average square masses of the four octets, and attempt to explain the differences in these masses in terms of the root-mean-square masses of the mesons in the S-state wave functions. We list below the S-state, P-meson probability $\mathcal{P}$ corresponding to each of the four octets.

$$\mathcal{P}(0^+)=\tfrac{3}{4},\quad \mathcal{P}(1^+)=\tfrac{1}{2},\quad \mathcal{P}(1^-)=\tfrac{1}{4},\quad \mathcal{P}(2^+)=0. \quad (26)$$

Since the average P mass is small, Eq. (26) implies that the 0^+ octet should be the lightest, and that the 0^+, 1^+, 1^-, 2^+ spacing should be nearly uniform. It is seen from Eq. (24) that this agrees very well with experiment. A simple possible explanation of the equal-spacing rule exists in the quark model as well, i.e., the rule could result from a spin-orbit term in the Hamiltonian.[10] In the present model, one would expect the average 1^+ mass to be slightly lower than that given by the above rule, since the S-state wave function for this particle contains no P singlet terms, and the P singlet is heavier than the P octet. The data are not sufficiently accurate to determine whether or not such an effect exists.

This mechanism, i.e., the effect of the lack of degeneracy of the M mesons in the S-state wave functions, should be responsibe for the splitting within the N octets also. The observed splitting within the P and V octets and all the N octets is such that the isovector member is lightest. It has been shown previously that with this mechanism, the f-type interaction would lead to a light isovector member of the composite octet, while the d-type octet-octet-octet interaction would lead to a heavy isovector composite.[14] However, it can be shown that if the d-type interaction is of the "nonet type," i.e., the singlet terms are as mixed as in the d-type terms of Eqs. (14) and (16), and if the $\omega-\varphi$ and $\eta-X$ mixing amplitudes are equal to the quark-model values, the d-type interaction also leads to light isovector composites. Furthermore, the strength of the effect for such an interaction is the same as for an f-type interaction. Thus the sign of the K^*-isovector meson mass-splitting terms of Eq. (24) fits the model. One would expect the octet mass-splitting (in terms of mass squared) to be smallest in the 0^+ multiplet, because the probability of the $(PP)^d$ states is high in this case, and the $\eta-X$ mixing amplitude probably is not close to the quark-model value. If the identifications of the $\pi_V(1003)$ and $K_V(1080)$ in Eq. (24) are correct, then this effect exists.

In conclusion, the predictions concerning the even-parity mesons that result from the $SU(6)_W$-symmetric bootstrap model are similar to the baryon-resonance predictions of R2 in two respects. First, the predictions are similar in many ways to those of the quark model, and second, the agreement with experiment is good. However, there are very few accurate data on the meson-resonance decay partial widths at present; these data are needed to test the model. Many of the predictions are similar to those of Ref. 9, in which only odd-parity meson exchanges were considered. However, the results of Ref. 9 depend sensitively on the assumption that the tensor ($\Delta=2$) potentials dominate the central ($\Delta=0$) potentials, while the results of this paper do not depend on such an assumption.

[14] R. H. Capps, Phys. Rev. 134, B460 (1964).