Pole Approximations in the N/D Equations*

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We propose the use of Padé approximants (which are essentially a prescription for constructing pole approximations) to make the kernels of the N/D integral equations separable. If certain conditions on the function being approximated are satisfied, it can be shown that the sequence of approximate solutions which one obtains will converge to the exact solution. The method is applied to the netural pseudoscalar-pseudoscalar-scalar bootstrap, and the known result is easily reproduced.

I. INTRODUCTION

SINCE their introduction by Chew and Mandelstam[1] the N/D equations have received an enormous amount of attention as a method for constructing unitary scattering amplitudes and for defining self-consistent (bootstrapped) systems of particles.[2] It has recently been proven[3] that the N/D representation exists even for the coupled-channel problem, and we may now say that the principle of the method is on firm ground. In practice, however, it has not been possible to obtain exact solutions in closed form, and a variety of approximation techniques have been proposed.[4–6]

The coupled-integral equations for the N and D functions (matrices in the multichannel case) which one obtains from unitarity and analyticity may be solved by brute force via matrix inversion, but the amount of computation time involved for even the simplest cases forces one to seek alternatives. Unfortunately, all of the approximation schemes that have been proposed suffer from one or more of the following defects: (1) the solutions depend on an arbitrary subtraction point; (2) a symmetric driving force does not produce a symmetric solution, thereby violating time reversal invariance; (3) ambiguities are introduced through the necessity of cutting off various integrals that arise; (4) it is not known whether the sequence of approximate solutions converges (in the mathematical sense) to the exact solution. It is to this last point that we address our attention in this paper, in the context of the methods of Martin[5] and Pagels,[6] which are in a sense complementary.

We propose to replace the kernels of certain integral equations which we shall encounter by Padé approximants.[7] Various useful theorems on the convergence of Padé approximants are known, and in the Appendix we present a theorem on the convergence of solutions of approximate-integral equations to the exact solution of the equation being approximated. Since the Padé approximation is essentially a prescription for the construction of a pole approximation to the function in question, it follows that the sequence of approximate solutions obtained by using pole approximations in the N and D equations converges to the exact solution, provided that the pole parameters are chosen to be those given by the Padé approximants.

In Sec. II we present the N/D equations and discuss Martin's and Pagels's approximations to them. Section III is devoted to a discussion of Padé approximants and series of Stieltjes. Although this material might well have been relegated to an appendix, the remainder of the paper is not very readable without some familiarity with these concepts. Our presentation is necessarily quite sketchy; we have restricted ourselves to defining the terminology and quoting a selection of relevant theorems. The interested reader should consult Wall[7] and the excellent review article by Baker[7] for more details. In Sec. IV we present the parameters of the Padé approximants for some simple examples, and we give some numerical results for the neutral pseudoscalar-pseudoscalar-scalar bootstrap. Finally, Sec. V contains a discussion and summary of our results.

II. N/D EQUATIONS: MARTIN'S AND PAGELS'S METHOD

We assume that each partial-wave amplitude $T(s)$ is analytic in the complex s plane except for the unitarity cuts R and the dynamical cuts L, satisfies elastic unitarity $[\mathrm{Im}T^{-1}(s+i\epsilon)=-\rho(s)$ on $R]$, and behaves sufficiently well at $|s|\to\infty$ to enable us to write an unsubtracted dispersion relation. We will discuss the single-channel problem, but all of our equations apply to the multichannel case if appropriate matrix notation is used. Furthermore, the modifications of the formalism needed to incorporate inelasticity are straightforward.

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[1] G. F. Chew and S. Mandelstam, Phys. Rev. **119**, 467 (1960).

[2] It is, of course, impossible to cite even a small fraction of the work that has been done on these topics. Practically any current journal issue can serve as a starting point for a search of the literature.

[3] R. L. Warnock, Nuovo Cimento **50**, 894 (1967); J. B. Hartle and J. R. Taylor, J. Math. Phys. **8**, 651 (1967).

[4] M. Baker, Ann. Phys. (N. Y.) **4**, 271 (1958); T. Fulton, in *Elementary Particle Physics and Field Theory, 1962 Brandeis Lectures* (W. A. Benjamin, Inc., New York, 1963); G. L. Shaw, Phys. Rev. Letters **12**, 345 (1964); J. S. Ball, Phys. Rev. **137**, B1573 (1965).

[5] A. W. Martin, Phys. Rev. **135**, B967 (1964).

[6] H. Pagels, Phys. Rev. **140**, B1599 (1965).

[7] H. S. Wall, *Analytic Theory of Continued Fractions* (D. Van Nostrand, Inc., Princeton, N. J., 1948). G. A. Baker, Jr., in *Advances in Theoretical Physics*, edited by K. A. Brueckner (Academic Press Inc., New York, 1965).

With these assumptions we have (except for poles)

$$T(s)=B(s)+\frac{1}{\pi}\int_R ds'\frac{\rho(s')}{s'-s}|T(s')|^2, \tag{1}$$

where

$$B(s)=\frac{1}{\pi}\int_L ds'\frac{\mathrm{Im}T(s')}{s'-s}, \tag{2}$$

i.e., $\mathrm{Im}B(s)=\mathrm{Im}T(s)$ on L.

The function $B(s)$, the contribution of the dynamical singularities, is assumed to be regular on R, and $\rho(s)$ is a known kinematical factor. In practice $B(s)$ is assumed to be given by some simple mechanism such as single-particle exchange. This is about the best we can do as long as the problem of fully incorporating crossing symmetry into the partial-wave formalism remains unsolved.

We now write

$$T(s)=N(s)/D(s), \tag{3}$$

where N has cuts on L only and D has cuts on R only. As $|s|\to\infty$ the function N is assumed to vanish and D to be not worse than constant. We should remark at this point that we have nothing new to say about the problem of Castillejo-Dalitz-Dyson (CDD) poles[8] and we assume that they are absent.

With the decomposition (3), Eq. (1) for T becomes a set of coupled linear equations

$$D(s)=1+\frac{s-s_0}{\pi}\int_R ds'\frac{\mathrm{Im}D(s')}{(s'-s_0)(s'-s)}$$
$$=1-\frac{s-s_0}{\pi}\int_R ds'\frac{\rho(s')N(s')}{(s'-s_0)(s'-s)}, \tag{4}$$

$$N(s)=\frac{1}{\pi}\int_L ds'\frac{\mathrm{Im}N(s')}{s'-s}=\frac{1}{\pi}\int_L ds'\frac{\mathrm{Im}B(s')D(s')}{s'-s}, \tag{5}$$

with the normalization $D(s_0)=1$. Equation (4) is, of course, equivalent to an unsubtracted dispersion relation for $D(s)/(s-s_0)$:

$$\frac{D(s)}{s-s_0}=\frac{1}{s-s_0}+\frac{1}{\pi}\int_R\frac{ds'}{s'-s}\frac{\mathrm{Im}D(s')}{s'-s_0}$$
$$=\frac{1}{s-s_0}-\frac{1}{\pi}\int_R\frac{ds'}{s'-s}\frac{\rho(s')N(s')}{s'-s_0}. \tag{6}$$

We prefer this form, since the inhomogeneous term is now square integrable on R.

Martin's method consists of substituting (4) into (5), which leads to

$$N(s)=B(s)+\frac{1}{\pi}\int_R K(s,s')N(s')ds', \tag{7}$$

where

$$K(s,s')=\frac{\rho(s')}{s'-s}\left[B(s')-\frac{s-s_0}{s'-s_0}B(s)\right], \tag{8}$$

and approximating B by a sum of poles. The kernel then becomes separable, and the problem may be solved by algebra.

Pagels, on the other hand, substitutes (5) into (4) [or equivalently into (6)] to obtain

$$\frac{D(s)}{s-s_0}=\frac{1}{s-s_0}+\frac{1}{\pi}\int_L K(s,s',s_0)(s'-s_0)\times\mathrm{Im}B(s')\frac{D(s')}{s'-s_0}ds', \tag{9}$$

where

$$K(s,s',s_0)=\frac{sF(s)}{(s-s')(s-s_0)}+\frac{s'F(s')}{(s'-s)(s'-s_0)}+\frac{s_0F(s_0)}{(s_0-s)(s_0-s')}, \tag{10}$$

$$F(z)=zH(z)=\frac{z}{\pi}\int_R\frac{\rho(s'')ds''}{(s'')^2(s''-z)},$$

and then approximates $F(z)$ or $H(z)$ by a sum of poles. The kernel again becomes separable, and the problem again reduces to algebra. In both methods the result for $T=N/D$ is independent of s_0.

We propose to use Padé approximants for the function B in (8) and for the function F in (10). This will amount to a pole approximation in which the positions of the poles and their residues are fixed once the order of the approximation is specified. If the conditions of the theorem of the Appendix are satisfied, one can be confident that the sequence of approximate solutions converges to the exact solution.

It turns out that the variable s (or even $s-s_t$, where s_t is the start of the right-hand cut) is not convenient to use in the construction of the approximants, since it does not lead to a relatively uniformly convergent sequence of approximate kernels in (7) or (9), and our theorem cannot be applied. For example, if we have a uniformly convergent sequence of approximations to the function B in Eq. (8), i.e., $|B(z)-B_n(z)|<\epsilon$ for $n>n_0$, then the residual kernel $Q_n=K-K_n$ satisfies $|Q_n(s,s')|<\epsilon\rho(s')/(s'-s_0)$, and the convergence of Q_n to 0 is not relatively uniform since $\rho(s')/(s'-s_0)=P(s,s')$ is certainly not square integrable on $s_t\leq s$, $s'\leq\infty$. The same difficulty arises in attempting to solve (9) with a uniformly convergent sequence of approximations to the function H. One must either examine the *rate* of convergence of the approximants to B and H as a function of their arguments, or find a better variable to use in the construction of the approximants. The second alternative is much easier in most cases.

[8] L. Castillejo, R. H. Dalitz, and F. J. Dyson, Phys. Rev. **101**, 453 (1956).

III. PADÉ APPROXIMANTS AND SERIES OF STIELTJES

In this section we define some mathematical concepts which are somewhat unfamiliar in the context of high-energy physics, and will state without proof several theorems given by Baker[7] which are relevant for our purposes. We generally follow Baker's notation.

Definition: The $[N,M]$ Padé approximant to a function $f(z)$ is the ratio of two polynomials, the numerator $P(z)$ being of degree M and the denominator $Q(z)$ being of degree N, such that the first $M+N+1$ terms in the power series for f and $[N,M]$ are identical.

This implies

$$f(z)Q(z)-P(z)\sim z^{M+N+1}+\cdots, \tag{11}$$

and the coefficients in the polynomials P and Q may be determined uniquely by equating the coefficients of z^k, $k\leq M+N$, on the left of this equation to zero, using the normalization $Q(0)=1$.

Definition: A power series $\sum f_n(-z)^n$ is a series of Stieltjes if and only if there is a bounded, nondecreasing function $\phi(u)$ taking on infinitely many values in $0\leq u\leq\infty$ such that

$$f_n=\int_0^\infty u^n d\phi(u).$$

For our purposes this means that we must be able to write

$$f(z)=\sum f_n(-z)^n=\int_0^\infty \frac{d\phi(u)}{1+uz}, \tag{12}$$

with $\phi(u)$ having the properties mentioned above.[9]

THEOREM: If $\sum f_n(-z)^n$ is a series of Stieltjes, then the poles of the $[N,N+j]$, $j\geq -1$, Padé approximants are on the negative real axis, and all the residues are positive. The roots of the numerator interlace those of the denominator, and the poles of successive approximants interlace.

THEOREM: The $[N,N]$ and $[N, N-1]$ sequences of approximants to a series of Stieltjes are, respectively, the best upper and lower bounds obtainable with a given number of coefficients, and use of additional coefficients improves the bounds.

In fact, if $f(z)=\sum f_n(-z)^n$ in a series of Stieltjes then for z real and non-negative we have

$$[N,N]\geq f(z)\geq [N, N-1]$$

and (13)

$$(-1)^{1+j}\{[N+1, N+1+j]-[N, N+j]\}\geq 0,$$

$$(-1)^{1+j}\{[N, N+j]-[N-1, N+j+1]\}\geq 0,\quad j\geq -1.$$

THEOREM: Any sequence of $[N, N+j]$ approximants to a series of Stieltjes $\sum f_n(-z)^n$ converges to an analytic function in the cut $(-\infty\leq z\leq 0)$ complex plane. If the series converges with radius of convergence R, then any $[N, N+j]$ sequence converges in the cut plane $(-\infty\leq z\leq -R)$ to the analytic function defined by the series.

Baker also gives several theorems on the convergence of Padé approximants and on the construction of series of Stieltjes which are quite interesting, although not directly relevant to the present work.

We shall use the following (quite trivial) theorem to construct series of Stieltjes for functions which are analytic except for cuts: If $f(z)$ is analytic in the cut $(-\infty\leq z\leq -L)$ plane, $\mathrm{Im}f(z+i\epsilon)\leq 0$ on the cut, and $f(\infty)=0$, then f is a series of Stieltjes. The proof is straightforward; we write a dispersion relation for f:

$$f(z)=\frac{1}{\pi}\int_{-\infty}^{-L}\frac{\mathrm{Im}f(x+i\epsilon)}{x-z}dx$$

and change variables to $y=-1/x$ to get

$$f(z)=\frac{1}{\pi}\int_0^{1/L}\frac{-\mathrm{Im}f(-y^{-1}+i\epsilon)}{y}\frac{dy}{1+yz},$$

which satisfies the definition if $\mathrm{Im}f\leq 0$ and $f(\infty)=0$. This theorem can be easily modified to apply to functions with cuts on $z>0$ or with asymptotic behavior which requires subtractions; all that is necessary is that f be real analytic and $\mathrm{Im}f(z+i\epsilon)\leq 0$. Furthermore, if $\mathrm{Im}f$ oscillates a finite number of times, we can multiply f by a polynomial chosen in such a way that the imaginary part of the resulting function does not oscillate, and then perform a suitable number of subtractions. It is clear that the class of functions which can be related to a series of Stieltjes in this simple fashion is quite large.

IV. EXAMPLES

We shall first consider the well-known neutral pseudoscalar-pseudoscalar-scalar bootstrap. The scalar particle is self-consistent if its coupling to the pseudoscalars is $g^2=5.3$ and its mass (in units of the pseudoscalar's mass) is $\mu^2=3.5$.[10] The 0^--0^- S-wave scattering amplitude $T_0(s)=\sin\delta\exp(i\delta)/\rho$, $\rho=[(s-4)/s]^{1/2}$, is an analytic function of s and satisfies a dispersion relation.

The contribution of the single 0^+ exchange term to the 0^--0^- S wave is

$$B_0(s)=\frac{2g^2}{s-4}\ln\left(1+\frac{s-4}{\mu^2}\right)=\frac{2g^2}{\mu^2}\int_0^1\frac{dw}{1+(s-4)w/\mu^2} \tag{14}$$

which is a series of Stieltjes in $z=(s-4)/\mu^2$. The Padé approximants to B_0 necessarily converge, but as we have pointed out above this condition is not strong enough to allow the theorem of the Appendix to be applied. However, the function $z^{-1}\ln(1+z)$ belongs to

[9] The integral is to be taken in the sense of Stieltjes, but we shall encounter only Riemann integrals with finite limits.

[10] As quoted by Ball (Ref. 4) and by Pagels (Ref. 6).

TABLE I. The self-consistent values of g^2 and μ^2 obtained for the neutral-scalar-meson bootstrap, using $[N+1, N]$ Padé approximants for the Born term. The approximants are given by $(\mu^2/2g^2)B_0(s)=\sum a_n z^n/\sum b_n z^n$, with $b_0=1$ (normalization) and $a_0=(\mu^2/2g^2)B_0(s=4)=1$.

$N=$	0	1	2
g^2	6.69	5.52	5.31
μ^2	3.43	3.51	3.53
a_1	...	$\frac{1}{2}$	1
a_2	...	...	11/60
b_1	$\frac{1}{2}$	1	$\frac{3}{2}$
b_2	...	$\frac{1}{6}$	$\frac{3}{5}$
b_3	...	...	1/20

a class which has been investigated by Luke,[11] and from his work it follows that the Padé approximants do converge sufficiently rapidly (as functions of z) for large z. We can apply our theorem, but we are restricted to $[N+j, N]$, $j\geq 1$, approximants if all the integrals which arise are to remain finite. From Eq. (13) it is clear that $j=1$ is the best choice. The approximants cannot, of course, reproduce the behavior of B for $s\to\infty$, but we are primarily interested in the region $s\lesssim 16$.

Table I shows the self-consistent values we have obtained for g^2 and μ^2 using the [1,0], [2,1], and [3,2] approximants to $(\mu^2/2g^2)B_0(s)$, together with the parameters needed to form the approximants. One could, of course, decompose the approximants by partial fractions and uniquely determine an equivalent set of poles and residues, but there is no particular advantage to be gained thereby. To illustrate the accuracy of the approximants, we may mention that the [3,2] is in error by less than 0.3% at $s=16$ for $\mu^2=3.5$.

Obviously good results may be obtained in this way using only low-order approximants, but it would be quite tedious to perform an analysis similar to Luke's for every case which might be encountered. We therefore seek a change of variable which will make the residual kernel of (7) square integrable, while the function to be approximated remains (or at least is closely related to) a series of Stieltjes.

TABLE II. The self-consistent values obtained for g^2 and μ^2 in the neutral-scalar-meson bootstrap, using a few low-order $[N,N]$ and $[N+1,N]$ Padé approximants to the function $A(y)$ defined in the text.

$[N,M]$	g^2	μ^2
[0,0]	7.95	3.46
[1,0]	5.75	3.48
[1,1]	5.46	3.52
[2,1]	5.32	3.53
[2,2]	5.27	3.53

To this end we try the linear fractional transformation

$$y=(1-z)/(1+z)\,, \tag{15}$$

and we get

$$B_0(s)=\frac{2g^2}{\mu^2}\frac{2}{1+z}A(y)\,, \tag{16}$$

where

$$A(y)=\int_0^1\frac{dv}{1+v}\frac{1}{1+vy} \tag{17}$$

is a series of Stieltjes in y. The Padé approximants to $A(y)$ necessarily converge, and the factor $(1+z)^{-1}$ in (16) makes the residual kernel of (7) square integrable. Our theorem can be applied directly, and the resulting self-consistent values of g^2 and μ^2 obtained with a few low-order approximants to $A(y)$ are shown in Table II. The results obtained with a given order are not as much better as it might seem, since an N-pole approximation to $A(y)$ results in an $(N+1)$-pole approximation to $B_0(s)$, but we have indeed avoided the necessity of an *a priori* examination of the rate of convergence of the approximants.

The transformation (15) is obviously not the only way to construct a representation for B_0 which has the form square integrable function times series of Stieltjes. Restricting ourselves to linear fractional transformations, the most general possibility we can use is[12]

$$y=(\alpha-z)/(\beta+\gamma z)\,, \tag{18}$$

which leads to

$$B_0(s)=\frac{2g^2}{\mu^2}\frac{\beta+\gamma\alpha}{\beta+\gamma z}A(y)\,, \tag{19}$$

with

$$A(y)=\int_{(1-\beta/\gamma)/(1+\alpha)}^1 dv\frac{1}{\beta/\gamma+\alpha v}\frac{1}{1+\gamma y v}\,. \tag{20}$$

Now an approximation of $A(y)$ will force us to have a pole in $B_0(s)$ at $z=-\beta/\gamma$ which will not be on the cut unless $|\beta/\gamma|>1$ and both have the same sign. On the other hand, $A(y)$ is not a series of Stieltjes unless either $\beta/\gamma<1$ or they have opposite signs, assuming, as will always be the case, that $1+\alpha>0$. (See below.) We are therefore forced to take $\beta=\gamma$, giving

$$B_0(s)=\frac{2g^2}{\mu^2}\frac{1+\alpha}{1+z}A(y)\,, \tag{21}$$

$$A(y)=\int_0^1\frac{dv}{1+\alpha v}\frac{1}{1+\gamma y v}\,,\quad y=\frac{\alpha-z}{\gamma(1+z)}\,. \tag{22}$$

The value of γ has become irrelevant, since $A(y)$ depends only on $\gamma y=(\alpha-z)/(1+z)$, which is independent of γ.

[11] Y. L. Luke, J. Math. and Phys. **37**, 110 (1958); *ibid.* **38**, 279 (1960).

[12] The coefficient of z in the numerator must not vanish, since this would map $z=\infty$ onto $y=0$ and the Taylor series for $A(y)$ would not exist.

We are thus left with only one parameter, α, which specifies which value of z is mapped into $y=0$. Since any Padé approximant to $A(y)$ will be exact at $y=0$, we are able to insist that $B_0(s)$ should be given correctly at any one particular point, i.e., $s=4+\alpha\mu^2$. It is convenient to have this freedom, but it does not seem to be particularly important in the 0^-0^--0^+ bootstrap.

It is also possible to specify the value of $B_0(s)$ at any particular point in other ways. For example,

$$\frac{\mu^2}{2g^2}B_0(s)=\frac{A}{z+A}+\frac{z}{z+A}\int_0^1\frac{(1-Aw)dw}{1+wz} \tag{23}$$

will be exact at $z=0$ ($s=4$) for any Padé approximant to the integral. (We must pick $A=1$, since a pole will be induced at $z=-A$ and the integral will be a series of Stieltjes only if $A\leq 1$.) The transformation (18) may now be performed, and the conditions of the theorem of the Appendix will be satisfied. B_0 will now be given exactly at two points but, again, this freedom does not seem to be important in practice.

Turning to a case where some preliminary manipulation is necessary, we consider the P-wave projection of the single-scalar-exchange amplitude. Such terms would have to be investigated if one wanted to verify that the scalar bootstrap did not produce bound states or low-lying resonances in other angular momenta. Except for a multiplicative constant we have [$z=(s-4)/\mu^2$ as above]

$$B_1(s)=\frac{2}{z}-\frac{z+2}{z^2}\ln(1+z)$$
$$=-\int_{-\infty}^{-1}\frac{dz'}{z'-z}\frac{z'+2}{(z')^2}=\int_0^1\frac{(2w-1)dw}{1+wz}, \tag{24}$$

which is not a series of Stieltjes since $\mathrm{Im}B_1(z)=-\pi(z+2)/z^2$ changes sign at $z=-2$. However, $C(z)=(z+2)B_1(z)/z$ is such that $\mathrm{Im}C(z)$ does not change sign, and has no new singularities since $B_1(0)=0$. Now we have

$$C(z)=-\int_{-\infty}^{-1}\frac{dz'}{z'-z}\frac{1}{z'}\left(\frac{z'+2}{z'}\right)^2=-\int_0^1\frac{(2w-1)^2dw}{1+wz}, \tag{25}$$

so that $B_1(z)=-z/(z+2)$ times a series of Stieltjes. We can now approximate $C(z)$ using exactly the same techniques as were applied to $B_0(z)$ in the first part of this section.

Finally, we consider Pagels's function

$$H(s)=\frac{1}{\pi}\int_4^\infty\frac{\rho(s')ds'}{(s')^2(s'-s)}, \tag{26}$$

which is a series of Stieltjes in $(-s)$, as may be seen by changing the integration variable to $s''=1/s'$. (Note that the function ρ is always positive.) However, convergence of the approximants to H is not sufficient to guarantee that the residual kernel of (9) is square integrable. This function can be analyzed with manipulations similar to those used on $B_0(s)$ above. In particular, with $z=-(s+4)/(s-4)$ we have

$$H(s)=-\frac{1}{4\pi}\frac{1}{s-4}I(z), \tag{27}$$

where

$$I(z)=\int_0^1\rho\left(s'=4\frac{1+w}{1-w}\right)\frac{1-w}{(1+w)^2}\frac{dw}{1+wz} \tag{28}$$

is a series of Stieltjes in z. We are now free to approximate the integral (28), and the residual kernel of (9) will be square integrable if $\mathrm{Im}B(s)$ is.

V. CONCLUSIONS

We have shown how to construct a sequence of what are essentially pole approximations to the kernels of the N/D equations in such a way that one can be sure that the sequence of solutions to the approximate equations converges to the solution of the exact equations. The amount of effort required is not increased thereby, since some preliminary computation is required to determine the parameters of any approximation. Probably less preliminary work is required in our method, since it provides a systematic technique for constructing many-pole approximations. The resulting kernels are separable, and the N/D equations may, as in any pole approximation, be solved algebraically.

It was necessary to construct series of Stieltjes related to the functions that we wished to approximate, since the limits of the class of functions for which the Padé approximants converge are not known. The class of functions which may be related to series of Stieltjes is, however, quite large, so that this is not a real handicap. If the Padé conjecture[7] is valid, some of the preliminary manipulations may be dispensed with, although they might still be a desirable way to improve the rate of convergence.

When the Born term comes from the exchange of particles with high spin, integrals arise which must be cut off if they are to be finite. The Padé approximants can serve as a kind of natural cutoff in these cases; at least one has the advantage of working with a denumerable infinity of cutoff functions instead of the continuum of cutoff parameters usually employed. One can determine which order approximant to use with physical arguments; elastic unitarity is, of course, incorrect above the inelastic threshold, so that the function chosen should be a good approximation to the Born term below this point and should deviate toward zero fairly rapidly above it. This appears to be a reasonably restrictive criterion, leaving only a few approximants to be considered.

Forming Padé approximants in the coupling constant is known to be a useful approach toward the solution of integral equations,[13] but the idea of using them to construct a sequence of separable approximate kernels also seems quite fruitful.

APPENDIX: THEOREM ON THE APPROXIMATION OF SOLUTIONS OF INTEGRAL EQUATIONS

Before we can discuss the conditions under which a sequence of approximate solutions to an integral equation converges to the exact solution, we must first introduce the notion of relative uniform convergence and present a theorem on the reduction of an integral equation to a system of linear algebraic equations. Every text on integral equations discusses this material; we have used Smithies's book[14] as a source.

Definition: A sequence $\{K_n(x,y)\}$ of L^2 kernels is said to be relatively uniformly convergent to $K(x,y)$ if there exists an L^2 kernel $H(x,y)$ such that, given $\epsilon>0$, there is an integer $N_0(\epsilon)$ for which

$$|K_n(x,y)-K(x,y)|\leq\epsilon H(x,y) \tag{A1}$$

for $n>N_0$. The limit $K(x,y)$ is again L^2. An infinite series $\sum K_n(x,y)$ of L^2 kernels is said to be relatively uniformly convergent if its partial sums form a relatively uniformly convergent sequence. The series is relatively uniformly absolutely convergent if $\sum|K_n(x,y)|$ is relatively uniformly convergent.

Completely analogous statements hold for functions of a single variable.

Among other things, the following can be shown:

(i) If $K_n(x,y)\rightarrow K(x,y)$ relatively uniformly and $f(y)$ is L^2, then

$$\int K_n(x,y)f(y)dy=\int K(x,y)f(y)dy \tag{A2}$$

relatively uniformly.

(ii) If $K_n(x,y)\rightarrow K(x,y)$ relatively uniformly and $L(x,y)$ is L^2, then

$$\int K_n(x,y)L(y,z)dy=\int K(x,y)L(y,z)dy$$

and (A3)

$$\int L(x,y)K_n(y,z)dy=\int L(x,y)K(y,z)dy$$

elatively uniformly.

Theorem (Ref. 14, p. 42):

Given an L^2 kernel $K(x,y)=P(x,y)+Q(x,y)$ where $P(x,y)=\sum a_n(x)b_n(y)$ is separable and

$$\|Q\|=\left(\int|Q(s,y)|^2dxdy\right)^{1/2}<\epsilon.$$

Let C be the resolvent of Q for $|\lambda|<1/\epsilon$. Then the integral equation

$$f(x)=g(x)+\lambda\int K(x,y)f(y)dy \tag{A4}$$

is equivalent to the equation

$$f(x)=h(x)+\lambda\int H(x,y)f(y)dy, \tag{A5}$$

where

$$h(x)=g(x)+\lambda\int G(x,y)g(y)dy \tag{A6}$$

and

$$H(x,y)=P(x,y)+\lambda\int G(x,z)P(z,y)dz. \tag{A7}$$

The new kernel H is obviously separable since P is, and in the usual way the solution of Eq. (A5) can be found by solving a system of linear algebraic equations.

Now suppose $K_n(x,y)$ is a relatively uniformly convergent sequence of separable approximations to $K(x,y)$. We wish to solve the equation

$$\begin{aligned}f(x)&=g(x)+\lambda\int K(x,y)f(y)dy\\&=g(x)+\lambda\int K_n(x,y)f(y)dy\\&\quad+\lambda\int[K(x,y)-K_n(x,y)]f(y)dy.\end{aligned} \tag{A8}$$

The solution of (A8) may be found by solving the equation

$$\begin{aligned}f(x)=g(x)&+\lambda\int G_n(x,y)g(y)dy+\lambda\int K_n(x,y)f(y)dy\\&+\lambda^2\int\left[\int G_n(x,z)K_n(z,y)dz\right]f(y)dy,\end{aligned} \tag{A9}$$

where G_n is the resolvent of $Q_n=K-K_n$.

By assumption, Q_n is relatively uniformly convergent to zero, i.e., $|Q_n|\leq\epsilon R_n$ for some L^2 kernel R_n.

The Neumann series for G_n is relatively uniformly absolutely convergent for $\epsilon<1/|\lambda|$;

$$G_n(x,y)=Q_n(x,y)+\lambda\int Q_n(x,z)Q_n(z,y)dz+\cdots, \tag{A10}$$

[13] J. S. R. Chisholm, J. Math. Phys. 4, 1506 (1963); D. Masson, *ibid.* 8, 512 (1967).

[14] F. Smithies, *Integral Equations* (Cambridge University Press, Cambridge, England, 1958).

[15] This ambiguity does not affect the solution of the N/D equations; the characteristic values of our kernels manifestly depend on the subtraction point s_0, while the solution $T=N/D$ does not. We can always adjust s_0 to avoid allowing the self-consistent value of g^2 to be a characteristic value.

$$|G_n(x,y)|\leq \epsilon R_n(x,y)$$
$$+\epsilon^2|\lambda|\int R_n(x,z)R_n(z,y)dz+\cdots, \quad \text{(A11)}$$

and therefore G_n is also relatively uniformly convergent to zero.

In the limit $n\to\infty$ the second and fourth terms of (A9) vanish by (A2) and (A3), and we have

$$f(x)=g(x)+\lambda\int K_\infty(x,y)f(y)dy. \quad \text{(A12)}$$

If we define a sequence of functions $\{f_n(x)\}$ which satisfy

$$f_n(x)=g(x)+\lambda\int K_n(x,y)f_n(y)dy, \quad \text{(A13)}$$

then the limit $f(x)=\lim_{n\to\infty}f_n(x)$ satisfies

$$f(x)=g(x)+\lambda\int K_\infty(x,y)f(y)dy, \quad \text{(A14)}$$

which is identical to (A12).

We have therefore proven the following theorem:

If $\{K_n\}$ is a relatively uniformly convergent sequence of separable approximations to K, $f=g+\lambda Kf$ and $f_n=g+K_nf_n$, then $f=\lim_{n\to\infty}f_n$ up to the possible addition of a solution of the homogeneous equation.[15]

When λ is not a characteristic value of K, the solution of (A9) may be written explicitly in terms of g, G_n, and K_n (Ref. 14, p. 45). One can then determine the rate of convergence of $\{f_n\}$ to f once the rate of convergence of $\{G_n\}$ is known.

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Even-Parity Meson Resonances*

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A recent model of the odd-parity baryon resonances is extended to include even-parity meson resonances. The resonances are assumed coupled to S-wave and D-wave MM states (where M denotes a pseudoscalar or vector meson) with interaction constants such that the resonance pole contributions to the collinear $MM\to MM$ scattering amplitudes satisfy $SU(6)_W$ symmetry. The relative coupling of the 35-fold and 1-fold M representations is taken in accordance with a bootstrap model. The set of even-parity resonances predicted in this model are octets of $j^C=0^+$, 1^+, 2^+, and 1^-, and 0^+, 2^+, and 1^- singlets, where C is the particle-antiparticle conjugation quantum number of the $I_z=Y=O$ particles. The partial widths for MM decays and the principal terms in the mass splitting are computed in the model. The agreement with the present experimental data is good.

I. INTRODUCTION

IN a recent paper (referred to here as R1), an $SU(6)_W$-symmetric bootstrap model of MB and MM states was developed.[1] The B and M considered were the 56-fold baryon supermultiplet, and 35-fold supermultiplet of odd-parity mesons. The $SU(6)_W$ symmetry was applied to the forward and backward one-particle exchange amplitudes T_f and T_b. The linear combinations $T_f\pm T_b$ represent potentials in states of even and odd orbital angular momentum, respectively. It was found in R1 that the M and B can be bootstrapped successfully only if baryons of odd parity and mesons of even parity exist. The simplest exact solution is one in which these "off-parity" baryons and mesons correspond to the $SU(6)_W$ representations 70 and $35\oplus1$, respectively.

Two basic steps are necessary to determine the physical consequences of the model. The first is that of R1, the formulation and solution of the self-consistency equations in terms of $SU(6)_W$ states. The second is the interpretation of the $SU(6)_W$ states in terms of spin and other physical quantum numbers. In a recent work, it was shown that the odd-l MM and MB composites may be associated with the basic M and B particles, and that the even-l MB states correspond physically to the representation (70,3) of $SU(6)\otimes O(3)$.[2] The purpose of this paper is to make a similar analysis of the even-parity meson states of the model, and to compare the predicted properties of these mesons with experiment.

Attention was limited to $M_{35}M_{35}$ and $M_{35}B_{36}$ states in R1, where the subscript is the $SU(6)_W$ multiplicity. (This did not imply the assumption that these are the only important states coupled to the composites, because the bootstrap condition of R1 may be applied separately to processes involving different external par-

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[1] R. H. Capps, Phys. Rev. **161**. 1538 (1967). Herein referred to as R1.

[2] R. H. Capps, Phys. Rev. **158**. 1433 (1967). Herein referred to as R2.