Solvable Two-Dimensional Field Theory Based on Currents*

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In this paper we study a solvable 2-dimensional field theory specified by a closed equal-time algebra generated by commutation of the components of the electric current and the energy-momentum tensor $\theta_{\mu\nu}$. As in the Thirring model, from which this model is abstracted, the physical content of the theory turns out to be trivial. The results are nevertheless of interest (a) because they allow one to readily understand why the Thirring model is solvable and why it has trivial physical consequences and (b) because they provide an example of a case where the requirement of a positive energy spectrum places important, and explicit, constraints on a theory written in terms of currents. We emphasize that although the present model was abstracted from the Thirring model we do not maintain that the resulting theory, or its solution as given here, is the same as the Thirring model, if by this term one means the conventional field-theoretic model.

I. INTRODUCTION

We have recently been investigating the possibility that the weak and electromagnetic hadron currents and the hadron energy-momentum tensor, when interrelated through a set of equal-time commutation relations whose structure has been specified in one way or another, might provide a complete formulation of strong-interaction physics.[1,2] A simple illustration of these ideas is provided by a solvable 2-dimensional field theory, abstracted from the Thirring model,[3] which we discuss in this paper.

Our discussion is not directed towards formal mathematical questions regarding the model. The results are of interest, instead, because they provide a tractable model of a theory based on the currents and the energy-momentum tensor, and because they allow one to see very readily (a) why the Thirring model is solvable and (b) why it has trivial physical consequences.

As will be clear from the following analysis, the solvability of this model depends critically on the fact that it is a 2-dimensional model. It is not likely that any of the specific features of this model can be generalized to more realistic cases, or that they will provide a useful guide to the state of affairs in the real world.

II. SPECIFICATION OF THE MODEL

The Thirring model, which provides the starting point from which we abstract the set of relations between the currents and the energy-momentum tensor which define the model to be discussed here, is a field theory in 2-dimensional space-time[4] in which a massless Fermi field interacts with itself through a current-current coupling. Without loss of generality we may suppose that the current involved is simply a conserved vector current.[5] The Lagrangian density for this theory is thus

$$\mathcal{L}(x)=\bar{\psi}i\gamma\cdot\partial\psi+gj_\mu j^\mu, \tag{1}$$

with the current given by $j_\mu(x)=\bar{\psi}(x)\gamma_\mu\psi(x)$. The resulting equation of motion for ψ is

$$i\gamma\cdot\partial\psi+2g\gamma_\mu\psi j^\mu=0. \tag{2}$$

Finally, we can write an expression for the symmetric, conserved energy-momentum tensor in this theory, which is

$$\theta_{\mu\nu}=\tfrac{1}{4}i\left\{\bar{\psi}\gamma_\mu\frac{\partial\psi}{\partial x_\nu}+\bar{\psi}\gamma_\nu\frac{\partial\psi}{\partial x_\mu}-\frac{\partial\bar{\psi}}{\partial x_\nu}\gamma_\mu\psi-\frac{\partial\bar{\psi}}{\partial x_\mu}\gamma_\nu\psi\right\}-g_{\mu\nu}\{\bar{\psi}i\gamma\cdot\partial\psi+gj_\mu j^\mu\}. \tag{3}$$

It is a direct consequence of the equation of motion, Eq. (2), that $\theta_{\mu\nu}$ is traceless.

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[1] R. F. Dashen and D. H. Sharp, second preceding paper, Phys. Rev. **165**, 1857 (1968), henceforth referred to as I.
[2] D. H. Sharp, first preceding paper, Phys. Rev. **165**, 1867 (1968), henceforth referred to as II.
[3] W. Thirring, Ann. Phys. (N. Y.) **3**, 91 (1958).
[4] For extensive discussions of the Thirring model (Ref. 3) and related two-dimensional field theories see A. S. Wightman, in *High Energy Electromagnetic Interactions and Field Theory*, edited by M. Levy (Gordon and Breach Science Publishers, Inc., New York, 1967), Vol. II; F. A. Berezin, *The Method of Second Quantization* (Academic Press Inc., New York, 1966), Chap. IV.
[5] Making the current out of a vector and an axial-vector piece would, as in the case of neutrinos, simply result in the theory breaking up into two independent pieces, one involving "left-handed" fermions, the other "right-handed" fermions. Each piece has the same structure as the theory which we actually discuss.

III. CLOSED EQUAL-TIME ALGEBRA FOR THE CURRENTS AND $\theta_{\mu\nu}$

In this section we shall write down the equal-time commutation relations satisfied by $\theta_{\mu\nu}(x)$ and $j_\mu(x)$.

First, it is easy to conclude that in a relativistically invariant world having one space dimension, the tracelessness of $\theta_{\mu\nu}$ implies that the algebra generated by commuting the various components of $\theta_{\mu\nu}$ among themselves is closed. As we shall see, it is really precisely this circumstance which accounts for the solvability of this model, and which does not readily generalize to more realistic situations.

We shall show that the $\theta_{\mu\nu}$ algebra is closed by calculating explicitly the desired commutators. For this purpose it is useful first to eliminate all time derivatives of the fields appearing in Eq. (3), using the equations of motion for ψ and $\bar{\psi}$. This procedure gives the following expressions for the components of $\theta_{\mu\nu}$:

$$\theta_{00}(x,t)=\theta_{11}(x,t)=\tfrac{1}{2}i\left[\bar{\psi}\gamma_1\frac{\partial\psi}{\partial x}-\frac{\partial\bar{\psi}}{\partial x}\gamma_1\psi\right]-gj_\mu(x,t)j^\mu(x,t)\,, \quad (4)$$

$$\theta_{01}(x,t)=\theta_{10}(x,t)=\tfrac{1}{2}i\left[\bar{\psi}\gamma_0\frac{\partial\psi}{\partial x}-\frac{\partial\bar{\psi}}{\partial x}\gamma_0\psi\right]. \quad (5)$$

Using these expressions, and the anticommutation relations $\{\psi_\alpha(x,t),\psi_\beta^\dagger(y,t)\}=\delta_{\alpha\beta}\delta(x-y)$, we find

$$[\theta_{00}(x,t),\theta_{00}(y,t)]=i(\partial/\partial x-\partial/\partial y)(\delta(x-y)\theta_{01}(x,t))\,, \quad (6a)$$

$$[\theta_{01}(x,t),\theta_{01}(y,t)]=i(\partial/\partial x-\partial/\partial y)(\delta(x-y)\theta_{01}(x,t))\,, \quad (6b)$$

$$[\theta_{00}(x,t),\theta_{01}(y,t)]=i(\partial/\partial x-\partial/\partial y)(\delta(x-y)\theta_{00}(x,t))\,. \quad (6c)$$

Next, commuting $\theta_{\mu\nu}(x)$ with the conserved vector current $j_\mu(x)=\bar{\psi}\gamma_\mu\psi$, we obtain

$$[\theta_{00}(x,t),j_0(y,t)]=-i\partial(j_1(x,t)\delta(x-y))/\partial y\,, \quad (7a)$$

$$[\theta_{00}(x,t),j_1(y,t)]=-i\partial(j_0(x,t)\delta(x-y))/\partial y\,, \quad (7b)$$

$$[\theta_{01}(x,t),j_0(y,t)]=-i\partial(j_0(x,t)\delta(x-y))/\partial y\,, \quad (7c)$$

$$[\theta_{01}(x,t),j_1(y,t)]=-i\partial(j_1(x,t)\delta(x-y))/\partial y\,. \quad (7d)$$

Finally, we need the commutation relations between the components of the vector current. We shall suppose these have the form

$$[j_0(x,t),j_0(y,t)]=0\,, \quad (8a)$$

$$[j_0(x,t),j_1(y,t)]=ic\partial[\delta(x-y)]/\partial x\,, \quad (8b)$$

$$[j_1(x,t),j_1(y,t)]=0\,, \quad (8c)$$

where c is a dimensionless number.

The above algebra is very simple. It is so simple, in fact, that one could almost have guessed its structure *ab initio*, without reference to the underlying field-theory model. The relations (6), incorporating the assumption that $\theta_{\mu\nu}$ is traceless, are the simplest choice that guarantee that $\int\theta_{00}(x,t)dx$ and $\int\theta_{01}(x,t)dx$ are the generators of time and space translations. Equation (7) likewise guarantees that $j_\mu(x,t)$ is a conserved current which transforms like a vector. The simplest choice for the commutator $[j_\mu(x,t),j_\nu(y,t)]$ would be to suppose that it vanishes, for each choice of μ and ν. However, this choice will not work, as is of course to be expected on the basis of the work of Schwinger,[6] Johnson,[7] and others. But it is not necessary to refer to the underlying field theory to find this out. If one ignores the Schwinger term on the right side of Eq. (8b), one is simply unable to find a representation of the current algebra which leads to a positive energy spectrum. So the next simplest choice for the current algebra, Eq. (8), is what we assume.

IV. EQUATIONS OF MOTION AND THEIR SOLUTION

The above algebra leads to trivially solvable equations of motion for both the current $j_\mu(x,t)$ and the energy-momentum tensor $\theta_{\mu\nu}(x,t)$. To derive these equations, we simply note that the time dependence of any operator $\mathcal{O}(x,t)$ is given by

$$\partial\mathcal{O}(x,t)/\partial t=i[P_0,\mathcal{O}(x,t)]\,,$$

with

$$P_0=\int\theta_{00}(x)dx\,,$$

the energy operator.

From Eqs. (6a) and (6c) we find

$$\dot{\theta}_{00}(x,t)=\partial\theta_{01}(x,t)/\partial x\,, \quad (9)$$

$$\dot{\theta}_{01}(x,t)=\partial\theta_{00}(x,t)/\partial x\,, \quad (10)$$

which together imply

$$(\partial^2/\partial t^2-\partial^2/\partial x^2)\theta_{00}(x,t)=0\,, \quad (11)$$

$$(\partial^2/\partial t^2-\partial^2/\partial x^2)\theta_{01}(x,t)=0\,, \quad (12)$$

while Eqs. (7a) and (7b) give

$$\partial j_0(x,t)/\partial t=\partial j_1(x,t)/\partial x\,, \quad (13)$$

$$\partial j_1(x,t)/\partial t=\partial j_0(x,t)/\partial x\,, \quad (14)$$

with the result

$$(\partial^2/\partial t^2-\partial^2/\partial x^2)j_0(x,t)=0\,, \quad (15)$$

$$(\partial^2/\partial t^2-\partial^2/\partial x^2)j_1(x,t)=0\,. \quad (16)$$

We see that each component of the current and the energy-momentum tensor satisfies a homogeneous wave equation, i.e., a "free-field" equation. The consequence of this is that this model has essentially trivial physical

[6] J. Schwinger, Phys. Rev. Letters 3, 296 (1959).
[7] K. Johnson, Nucl. Phys. 25, 431 (1961).

consequences. For example, it is clear from the well-known properties of the solutions of such wave equations that a moving energy packet will simply undergo translation at the speed of light, remaining completely unchanged in other respects. Likewise, two such energy packets, heading towards one another, will simply pass through each other without change.

It is evident that such a situation must be described by a trivial S matrix, i.e., one which is energy-independent and of modulus 1. This result follows directly from the observation that the form of Eqs. (9)–(16) is the same for the "free" theory [$g=0$ in Eqs. (1)–(3)] and the "interacting" theory. Any theory for which this is the case must have a trivial S matrix, essentially by definition.[8]

We can also see this from another point of view. The only place in Eqs. (6)–(16) where a parameter appears whose variation could reflect a transition from a free to an interacting theory[9] is on the right side of the commutator of $j_0(x,t)$ and $j_1(x,t)$, Eq. (8b). But this dimensionless number c does not enter at all into the equations of motion, (9)–(16).

The variation of the parameter c does, however, account for the only discernible effect of the interaction.

A change in the value of c must result in a scale change in the current j_μ (because c is dimensionless). This in turn must appear as a change in scale of the total charge $Q=\int j_0(x)dx$, which can be interpreted as a charge renormalization. Such a charge renormalization is not inconsistent with current conservation, because the particles in this theory are massless.

V. ENERGY SPECTRUM

In this section we show that the energy spectrum in this model is positive, and identical to that of a system of free, massless scalar mesons. For this purpose, we need a representation of the algebra defined by Eqs. (6)–(8).

This representation is easily found. First, we can write $\theta_{\mu\nu}$ as an explicit function of the current j_μ in such a way that Eqs. (6)–(8) are all satisfied, provided that the components of the current commute as in Eq. (8). The appropriate expression is

$$\theta_{\mu\nu}=(1/2c)[j_\mu j_\nu+j_\nu j_\mu-g_{\mu\nu}(j_\alpha j^\alpha)]. \qquad (17)$$

It is interesting that the above expression bears no immediately evident relationship to the form of $\theta_{\mu\nu}$ as given in terms of the fermion field ψ, Eq. (3). It was not obtained by starting from Eq. (3) for $\theta_{\mu\nu}$ and then reexpressing things in terms of the current, but simply by guessing a form for $\theta_{\mu\nu}$ which turned out to satisfy the required equations. It is worth emphasizing that this form for $\theta_{\mu\nu}$ does not work if one tries to disregard the Schwinger term in the commutation of j_0 and j_1, Eq. (8b). For example, using Eq. (17) and Eq. (8b) without the Schwinger term does not give the current-conservation equation correctly.

To complete the analysis, one needs a representation of the current algebra, Eq. (8). The current algebra is solved by writing

$$c^{-1/2}j_0(x)=\Pi(x), \qquad (18)$$

$$c^{-1/2}j_1(x)=\partial\varphi(x)/\partial x, \qquad (19)$$

and requiring $\varphi(x)$ and $\Pi(x)$ to commute like a canonical scalar field, $[\varphi(x,t),\Pi(y,t)]=i\delta(x-y)$, or, equivalently, by supposing that $c^{-1/2}j_\mu(x)$ is the gradient of a scalar field $\varphi(x)$.[10] With this representation for the current, $\theta_{\mu\nu}$, as given in Eq. (17), takes the form

$$\theta_{\mu\nu}=\tfrac{1}{2}[\partial_\mu\varphi\partial_\nu\varphi+\partial_\nu\varphi\partial_\mu\varphi-g_{\mu\nu}(\partial_\alpha\varphi\partial^\alpha\varphi)]. \qquad (20)$$

That is, it is the energy-momentum tensor of a free, massless scalar field. At this point, one could introduce the Fock representation for the scalar field, annihilation and creation operators, etc., and verify in detail that the energy and momentum operators have the expected properties, but there is little to be gained by going over these well-known details.

VI. SUMMARY AND CONCLUSIONS

Having solved the commutation relations connecting $\theta_{\mu\nu}(x,t)$ and $j_\mu(x,t)$, let us see how the present results fit into the pattern developed in papers I and II. In this regard, there are two points that should be discussed.

In papers I and II, we constructed complete dynamical theories but were, of course, unable to solve them. Here we have solved the dynamics but, as it turns out, we may not have completely specified the theory, at least in so far as we define "the theory" to be the conventional field-theoretic Thirring model. The reason for this may be seen as follows. We note that both of the quantities $Q=\int j^0(x,t)dx$ and $Q'=\int j^1(x,t)dx$ commute with all of our basic coordinates $j^0(x,t)$, $j^1(x,t)$, $\theta^{00}(x,t)$, and $\theta^{01}(x,t)$. Thus both Q and Q' would have to be constants in any irreducible representation of the algebra which means either (a) that both Q and Q' lead to superselection rules or (b) that we must work with reducible representations of the algebra. Now the electric charge Q certainly leads to a superselection rule, but one would rather not have to suppose that Q' does also. Therefore, it seems that one must accept alternative (b) and consequently we do not, in the language

[8] The S matrix need not be identically equal to unity, however. It can have an energy-independent phase. But, in contrast to the 3-dimensional case, there is in 1 dimension no natural way to fix the phase of the S matrix. In 1 dimension, such a phase factor represents a convention as to the phase of the in-fields. Since this has no physical effect, clearly we cannot determine it by following the motion of lumps of charge and energy.

[9] If one evaluates the commutator (8b) starting from the underlying field theory, one finds that c is a function of g, the coupling constant appearing in Eq. (1). See, e.g., Wightman, Ref. 4.

[10] The observation that in the Thirring model the current can be written as the gradient of a scalar field is trivial and well known. See Wightman, Ref. 4.

of papers I and II, have a complete set of coordinates.

Fortunately, none of our arguments concerning the dynamical triviality of the theory is affected by this difficulty. In particular, the expression (17) for $\theta^{\mu\nu}$ in terms of the current remains valid. Moreover, if we choose to think of our model as just a model, and not *the* Thirring model, the above comments are, of course, not germane.

The second point has to do with the Schwinger term. In papers I and II we were never forced to introduce such a term into the current algebra. It may well be that this was due to the fact that we never really tried to solve the dynamics. Recall that in the present model the Schwinger term was forced upon us when we found that without it the energy would not be positive. We have already stressed the point that the requirements of Lorentz invariance impose severe, and useful, constraints on the commutation relations between the various operators. Here we see that the requirement of positivity of the energy spectrum likewise imposes important constraints.

One should not jump to the conclusion, however, that the present results nullify the content of papers I and II where Schwinger terms were ignored. As long as we were talking about simple models such as nonrelativistic quantum mechanics or relativistic free fields, our previous results are formally correct. In the case of interacting relativistic theories, one might have to modify the assumed commutation relations in order to obtain a positive energy spectrum. However, it is not obvious how one is to know whether or not this is necessary until at least a partial solution of the dynamics is available.

In conclusion, it seems reasonable to say that, basically, we have learned two things: (i) why the Thirring model is solvable (essentially because the components of $\theta_{\mu\nu}$ form a closed algebra) and (ii) that positivity of the energy spectrum places important and, in the present case, explicit restrictions on a theory written in terms of currents.

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Poles and Resonances in η Production*

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Poles and resonances are used to find χ^2 fits to the data up to several hundred MeV above threshold for the processes $\pi^-+p\rightarrow n+\eta$ and $\gamma+p\rightarrow p+\eta$. Below center-of-mass energy $W=1700$ MeV, a consistent picture of these reactions is achieved with the following ingredients: a nucleon pole and $P_{11}(1400)$, $S_{11}(1570)$, and $D_{13}(1512)$ resonances for production by pions; a nucleon pole, a vector meson pole, and $S_{11}(1570)$ and $D_{13}(1512)$ resonances for photoproduction. Some $F_{15}(1688)$ also improves the fits. The value of the η-nucleon coupling constant, $g_\eta^2/4\pi$ is $\lesssim 0.002$ in both processes. Implications of these results for quark models and $SU(3)$ symmetry are discussed. Above $W=1700$, possible additional resonances are considered in η production by pions.

I. INTRODUCTION

CROSS-SECTION data are available on the process $\pi^-+p\rightarrow\eta+n$ up to several hundred MeV above threshold (pion laboratory kinetic energy $T_\pi=562$ MeV). Bulos *et al.*,[1] who presented results at nine energies up to $T_\pi=1151$ MeV (center-of-mass energy $W=1822$ MeV), found that the total cross section for the process $\pi^-+p\rightarrow\eta+n$ ($\eta\rightarrow 2\gamma$) rises rapidly to a peak of 1 mb at $T_\pi=650$ MeV and then begins a sharp but somewhat more moderate decline, going below 0.5 mb at $T_\pi=1151$ MeV. Bulos *et al.*[1] find little evidence for anisotropy in the angular distribution of the produced η's below $T_\pi=1003$ MeV, but above this energy a term at least linear in the cosine of the production angle is needed.

Richards *et al.*,[2] who studied the same reaction as Bulos *et al.*[1] at seven energies up to $T_\pi=1300$ MeV, confirmed the structure and magnitude of the total cross section for η production but with improved statistics found evidence for anisotropy in the angular distribution beginning at $T_\pi=655$ MeV.

Additional data on η production from the process $\pi^-+p\rightarrow\eta+n$ ($\eta\rightarrow$ all neutrals) are available from the experiments of Jones *et al.*[3] ($T_\pi\leq 589$ MeV) and of

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[1] F. Bulos, R. E. Lanou, A. E. Pifer, A. M. Shapiro, M. Widgoff, R. Panvini, A. E. Brenner, C. A. Bordner, M. E. Law, E. E. Ronat, K. Strauch, J. J. Szymanski, P. Bastien, B. B. Brabson, Y. Eisenberg, B. T. Feld, V. K. Fischer, I. A. Pless, L. Rosenson, R. K. Yamamoto, G. Calvelli, L. Guerriero, G. A. Salandin, A. Tomasin, L. Ventura, C. Voci, and F. Waldner, Phys. Rev. Letters **13**, 486 (1964).

[2] W. Bruce Richards, Charles B. Chiu, R. D. Eandi, A. Carl Helmholz, R. W. Kenney, B. J. Moyer, J. A. Poirier, R. J. Cence, V. Z. Peterson, N. K. Sehgal, and V. J. Stenger, Phys. Rev. Letters **16**, 1221 (1966).

[3] W. G. Jones, D. M. Binnie, A. Duane, J. P. Horsey, D. C. Mason, J. A. Newth, I. U. Rahman, J. Walters, N. Horwitz, and P. Palit, Phys. Letters **23**, 597 (1966).