PHYSICAL REVIEW VOLUME 165, NUMBER 5 25 JANUARY 1968

Currents and Hadron Dynamics*

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(Received 25 August 1967)

In this paper we continue the investigation of the preceding paper into the possibility that complete, dynamical theories can be formulated in terms of current densities and related operators. Here, these ideas are illustrated by showing how a model of self-interacting charged scalar mesons can be formulated in conformity with them. Because of the relative simplicity of this model as compared to other, more realistic, relativistic models, it is possible to take up in some detail topics like the irreducibility of the coordinates, the Heisenberg equations of motion for the current densities, the determination of the energy-momentum tensor by its equal-time commutation relations, and the use of functional representations of current algebras. It is also shown how electromagnetism can be incorporated into strong-interaction theories based on currents. The results of these two papers are reviewed and an attempt is made to abstract from them a tentative set of working hypotheses.

I. INTRODUCTION

HERE and in an accompanying paper[1] we explore the idea that a complete and possibly useful formulation of strong-interaction physics can be given in terms of the weak and electromagnetic hadron current densities, the hadron stress–energy-momentum tensor $\theta_{\mu\nu}(x)$, and the algebraic relations between these quantities generated by their commutation at equal times.

In I we have tried to bring out some of the essential physical points involved by considering a number of different models from this point of view, concentrating on the quark model. Here, we treat a relativistic model of charged scalar mesons using similar methods. While such a model is primarily of academic interest, it still repays study because its simpler spin structure as compared to more realistic models allows us to give a more systematic and detailed discussion of some of the points brought up in I, and to take up other points relating to dynamics not pursued there.

In order to make the present paper self-contained we wish, before plunging into detail on the charged scalar model, to review and expand on some of the ideas which might motivate an approach to hadron physics in terms of currents and related quantities. One of the leading questions behind this is the following: What is an appropriate set of "coordinates" or "building blocks" in terms of which to describe hadrons?

While one can not look directly to the Bohr correspondence principle for guidance on this question, as one to some extent could in seeking an answer to the analogous question in the case of atomic structure, one can hope to make at least a plausible guess as to the answer by asking what strong-interaction physics itself tells us about coordinates. The following points seem clear at the moment.

First, it seems reasonable, at the present stage of development, to try to choose the building blocks from among quantities which are closely related to experiment. There are at least two reasons for this: (a) When one looks at other theories of matter (solid state, liquid helium, theory of nuclear matter), and indeed at recent work in strong-interaction physics, one sees that to make practical progress it is usually vital to be able to use phenomenology or intuition directly to arrive at a good starting point for approximate calculations. This is most easily done if the formulation of the theory is close to experiment. (b) One has the feeling that quantities whose matrix elements are directly measurable ought to make some sort of mathematical sense, while this need not be true of any posited "underlying" field.

Secondly, judging from the complexity of the experimental hadron spectrum on the one hand, and from the insights into hadron dynamics furnished by the bootstrap program of S-matrix theory on the other, it is difficult to believe that any one of the observed hadron states is more "elementary" than another, or that the particle states themselves are simple.

This state of affairs suggests that if one is going to base the theory on observables it is necessary either to include all the hadron states from the beginning, or none. There are many perfectly cogent reasons for trying to exploit the first possibility, as is done in S-matrix theory. However, it does not seem likely that a relatively simple theory of particle structure, as opposed to a theory of particle reactions, will result if we choose "particle" coordinates, i.e., if we formulate the theory in such a way that there is an essentially 1-to-1 correspondence between physical particle states and the basic quantities in the theory.[2,3] As an alternative we aim, here and in I, towards a theory of hadrons formulated in terms of local observables, in which it is not

* Supported in part by U. S. Atomic Energy Commission under Contract AT(30-1)-2171.

[1] R. F. Dashen and D. H. Sharp, preceding paper, Phys. Rev. **165**, 1857 (1968), henceforth referred to as I.

[2] This would be the case, for example, if we were to try to describe hadrons either in terms of local fields of a type whose quanta are *directly* related to the physical particle states, or in terms of S-matrix elements.

[3] To draw on a widely used analogy, the hydrogen atom is "simple." However, the simplicity of the hydrogen atom does not reside directly in the rather complicated properties of its energy spectrum, but in the fact that the spectrum can be obtained from the Heisenberg equations of motion for the three position coordinates of the electron, or from the Schrödinger equation.

necessary to specify at the outset what the particle states are. This is done without prejudice to the question of whether or not a complete S-matrix theory also exists, or even of whether a theory of hadrons based on an underlying field (presumably a fundamental quark field) exists.

Which local observables should one pick? We do not know, but the following fact has emerged: The hypothesis that the equal-time commutation relations between the weak and electromagnetic hadron currents have a simple algebraic structure has proven very useful in strong-interaction physics[4] (in contrast to the sterility that has marked attempts to exploit canonical commutation relations directly). We shall be guided by this fact, and suppose that the hadron currents will continue to be useful as coordinates in a dynamical theory of hadronic matter.

The electromagnetic and weak hadron current densities are local observables in that all their matrix elements can presumably be measured through their coupling to external photons and lepton pairs.[5]

Beside the currents, one can hardly avoid introducing the hadron stress–energy-momentum tensor $\theta_{\mu\nu}(x)$ because we know (a) that the structure of its equal-time commutation relations is intimately connected with ensuring the Lorentz invariance of the theory,[6] (b) it contains the dynamics, and (c) it is on a symmetrical footing with the currents as far as measurement is concerned; the components of $\theta_{\mu\nu}(x)$ [in particular $\theta_{00}(x)$, the energy density] being measurable in principle through their coupling to external gravitons.

We shall pick our coordinates, then, from among these local observables. The next question which must be answered is: Which of these form a complete or irreducible set of coordinates at a given time? We find that it is always possible, in the models we have studied, to pick a complete set of local operators which form a closed Lie algebra under commutation at equal times. The particular set of operators involved depends on the theory, but always includes the hadron currents themselves and never the energy density $\theta_{00}(x)$.

As a result of the completeness of these operators, one knows that $\theta_{\mu\nu}(x)$, and in particular $\theta_{00}(x)$, can in principle be expressed in terms of them. A closed dynamical theory will then result, in which the dynamics is expressed either in the form of Heisenberg equations of motion for the current operators, or by the Heisenberg algebra generated by commuting the components $\theta_{\mu\nu}(x)$ of the energy-momentum tensor with the currents and themselves.

In illustrating these ideas, our order of presentation is as follows. In Sec. II we discuss in detail a model of self-interacting charged scalar mesons. We first obtain the closed equal-time algebra defining the observable coordinates, abstracting from an "underlying" field theory. We give an argument (the details of which are included in Appendix A) to show that the elements of this algebra form an irreducible set. The energy-momentum tensor is next expressed in terms of these, and the Heisenberg equations of motion for the "coordinates" are derived.

As an alternative formulation of the dynamics, we next show how $\theta_{\mu\nu}(x)$ can be introduced directly into the theory, defining it not by giving a formula for $\theta_{\mu\nu}$ in terms of the currents, but rather by specifying its equal-time commutation relations. The resulting system of algebraic relations contains sufficient information to determine the dynamics and can replace the Heisenberg equations of motion, as discussed in Appendix B.

Next, a functional representation of the current algebra is introduced, mainly to illustrate one way in which one might go about solving a theory expressed entirely in terms of currents from scratch, in the event that one had no idea whether or not it came from an underlying theory. In the final part of this section, we briefly discuss renormalization.

In Sec. III, we show that electromagnetic interactions can be incorporated into strong-interaction theories based on local currents in a natural way, by adding as a further set of observable quantities the electromagnetic field strengths $F_{\mu\nu}(x)$. The resulting electrodynamics is gauge-invariant and path-independent. In Sec. IV, the conclusions of this paper are summarized.

In a very important sense, the formulation of strong-interaction physics discussed here is not new. The operators we use are, of course, local fields in the sense of abstract field theory. Moreover, even the detailed point of view under discussion is implied by the formulation of field theory in terms of rings of local observables.[7] We simply make specific choices for these local observables, concentrating on finding a complete set at a fixed time. While we can in this way go further in the direction of seeing what the explicit structure of theories of this kind might be, we stress that no claim to mathematical rigor can be made for the methods or results which follow.

[4] This is not the place for an extensive review of the literature on this subject. As is well known, the application of current algebra to strong interaction physics was initiated by M. Gell-Mann [Phys. Rev. **125**, 1067 (1962)] in connection with the formulation of strong-interaction symmetries. The next phase was introduced by S. L. Adler [Phys. Rev. Letters **14**, 1051 (1965)] and W. Weisberger [*ibid.* **14**, 1047 (1965)], who showed how to derive useful sum rules starting with current commutators. For reviews of the subject see R. F. Dashen, in *Proceedings of the Thirteenth Annual International Conference on High-Energy Physics, Berkeley, 1966* (University of California Press, Berkeley, 1967), and the forthcoming book by S. L. Adler and R. F. Dashen, *Current Algebras* (W. A. Benjamin, Inc., New York, to be published).

[5] Our statement that the hadron currents are directly observable assumes that the electromagnetic, weak, and gravitational couplings of the currents and $\theta_{\mu\nu}$ can be treated in first-order perturbation theory.

[6] J. Schwinger, in *Theoretical Physics* (International Atomic Energy Agency, Vienna, 1963), pp. 89–134; Phys. Rev. **130**, 406 (1963); **130**, 800 (1963).

[7] One of the basic papers in this field is R. Haag and D. Kastler, J. Math. Phys. **5**, 848 (1964). For a good review of the subject, and further references to the literature, see the lectures by D. W. Robinson, in *Particle Symmetries and Axiomatic Field Theory*, edited by M. Chrétien and S. Deser (Gordon and Breach Science Publishers, Inc., New York, 1966), Vol. I, p. 389.

II. MODEL OF SELF-INTERACTING CHARGED SCALAR MESONS

A. Introduction

In the present section we will illustrate some of the ideas outlined in Sec. I by showing how a theory of self-interacting scalar mesons can be formulated in conformity with them. While we know, of course, that such a theory can not describe the real world, the study of this model is instructive because most of the essential points come up here, yet without all the complexities present in more realistic models.

We start with the canonical form of a theory of charged scalar mesons. This is specified by giving the Lagrangian density:

$$\mathcal{L}(x)=\frac{\partial\varphi^*}{\partial x_\mu}\frac{\partial\varphi}{\partial x^\mu}-\mu^2\varphi^*\varphi; \tag{2.1}$$

defining momenta conjugate to the complex fields $\varphi(x)$ and $\varphi^*(x)$ by

$$\pi(x)=\frac{\delta\mathcal{L}}{\delta\dot\varphi}=\dot\varphi^*(x);\quad \pi^*(x)=\frac{\delta\mathcal{L}}{\delta\dot\varphi^*}=\dot\varphi(x), \tag{2.2}$$

and supposing that the coordinates and their conjugate momenta commute at equal times as follows:

$$[\varphi(\mathbf{x},t),\pi(\mathbf{y},t)]=i\delta(\mathbf{x}-\mathbf{y})=[\varphi^*(\mathbf{x},t),\pi^*(\mathbf{y},t)], \tag{2.3}$$

with the equal-time commutators of all other pairs of coordinates and momenta vanishing.

In addition, one specifies a Hamiltonian density

$$\mathcal{H}(x)=\pi^*(x)\pi(x)+\nabla\varphi^*(x)\cdot\nabla\varphi(x)+\mu^2\varphi^*(x)\varphi(x), \tag{2.4}$$

from which one finds the Heisenberg equations of motion for the fields φ and φ^* to be

$$\begin{aligned}(\Box+\mu^2)\varphi(x)&=0,\\ (\Box+\mu^2)\varphi^*(x)&=0.\end{aligned} \tag{2.5}$$

We shall deal with self-interacting scalar mesons. When we find it necessary to specify the form of the interaction, we shall suppose $\mathcal{L}_I(x)=-\frac{1}{2}\lambda(\varphi^*\varphi)^2$, where λ is a coupling constant.

The quantities of fundamental importance to us in the following are the conserved electromagnetic current operator $j_\mu(x)$ and the symmetric, conserved energy-momentum tensor[8] $\theta_{\mu\nu}(x)$. These can be expressed in terms of the fields $\varphi^*(x)$ and $\varphi(x)$ as

$$j_\mu(x)=ie_0[\varphi^*(x)\partial_\mu\varphi(x)-\partial_\mu\varphi^*(x)\varphi(x)] \tag{2.6}$$

and

$$\theta_{\mu\nu}(x)=\partial_\mu\varphi^*(x)\partial_\nu\varphi(x)+\partial_\nu\varphi^*(x)\partial_\mu\varphi(x)-g_{\mu\nu}\mathcal{L}(x). \tag{2.7}$$

So defined, the above Heisenberg operators are all Hermitian. For simplicity, we have not normal-ordered these operators.

The quantity e_0 appearing in Eq. (2.6) is the bare electric charge. Since we assume in what follows that one can work to first order in the electric coupling,[5] e_0 can be taken to coincide with the physical charge e. We shall henceforth pick $e_0=1$, which procedure can be regarded either as the effect of a choice of units in which $e_0=1$, or as the result of introducing a current $j_\mu'=j_\mu/e_0$ from which the electric coupling has been removed.

B. Equal-Time Current Algebra

We now want to formulate this model in terms of the local observables $j_\mu(x)$ and $\theta_{\mu\nu}(x)$, instead of the fields $\varphi(x)$, $\varphi^*(x)$, $\pi(x)$, and $\pi^*(x)$.

The natural place to start is with the equal-time commutation relations between the components of the electromagnetic current $j_\mu(x)$. From the definition of the current, Eq. (2.6), and the equal-time commutation relations for the fields, Eq. (2.3), one finds[9]

$$[\rho(\mathbf{x}),\rho(\mathbf{y})]=0, \tag{2.8a}$$

$$[\rho(\mathbf{x}),j_k(\mathbf{y})]=2i\frac{\partial}{\partial x_k}[S(\mathbf{x})\delta(\mathbf{x}-\mathbf{y})], \tag{2.8b}$$

$$[j_k(\mathbf{x}),j_l(\mathbf{y})]=0, \tag{2.8c}$$

where

$$S(x)=\varphi^*(x)\varphi(x), \tag{2.8d}$$

and we have written $\rho(x)=j_0(x)$. We see that the equal-time algebra does not yet close, so that one must add at least one more operator, $S(x)$, to the algebra. Its commutation relations are simply

$$[S(\mathbf{x}),S(\mathbf{y})]=[S(\mathbf{x}),\rho(\mathbf{y})]=[S(\mathbf{x}),j_k(\mathbf{y})]=0. \tag{2.8e}$$

The equal-time algebra now closes. However, if we refer to the underlying field theory, we can see that at this point we still need to add one more operator to the algebra. The reason for this is the following. From Eqs. (2.6) and (2.8d), it is clear that $j_k(x)$ and $S(x)$ are both specified by giving two numbers per space-time point, namely, $\varphi^*(x)$ and $\varphi(x)$. The set of coordinates $\rho(x)$, $S(x)$, and $j_k(x)$ therefore represent three dynamical degrees of freedom per space-time point, whereas the charged scalar theory itself has four, namely, $\varphi(x)$, $\varphi^*(x)$, $\pi(x)$, and $\pi^*(x)$.

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[8] The symmetric, conserved energy-momentum tensor $\theta_{\mu\nu}(x)$ need not coincide with the "canonical" energy-momentum tensor $T_{\mu\nu}(x)$. In particular, the energy density $\theta_{00}(x)$ is not necessarily equal to the Hamiltonian density $\mathcal{H}(x)$, although the expressions for the energy, $\int\theta_{00}(x)d^3x$ and $\int\mathcal{H}(x)d^3x$, always agree. It is the components of $\theta_{\mu\nu}(x)$ which have a direct physical significance. In the present case, $\theta_{\mu\nu}(x)$ and $T_{\mu\nu}(x)$ are the same. For a discussion of these points, see G. Wentzel, *Quantum Theory of Fields* (Interscience Publishers, New York, 1949). Appendix I.

[9] In writing equal-time commutators, we will not indicate explicitly the time dependence of the operators. For example, we write $\rho(x)=\rho(\mathbf{x},t)=\rho(\mathbf{x})$ when it appears in an equal-time commutator. All commutators in the paper are equal-time commutators unless the contrary is explicitly stated.

As our last operator, it is natural to pick $\dot{S}(x)$, defined by the equation

$$i[P_0,S(x)]=\dot{S}(x). \tag{2.9}$$

Here $P_0=\int\theta_{00}(\mathbf{y})d^3y$ is the energy operator, which generates time displacements.

The commutation relations of $\dot{S}(x)$ with the other coordinates turn out to be

$$[\dot{S}(\mathbf{x}),\dot{S}(\mathbf{y})]=[\dot{S}(\mathbf{x}),\rho(\mathbf{y})]=0, \tag{2.10a}$$

$$[S(\mathbf{x}),\dot{S}(\mathbf{y})]=2iS(\mathbf{x})\delta(\mathbf{x}-\mathbf{y}), \tag{2.10b}$$

$$[j_k(\mathbf{x}),\dot{S}(\mathbf{y})]=2ij_k(\mathbf{x})\delta(\mathbf{x}-\mathbf{y}). \tag{2.10c}$$

The equal-time algebra, specified by Eqs. (2.8) and (2.10), thus closes and is a formal Lie algebra.

The current $j_\mu(x)$ is the electromagnetic current and is measurable in principle through its coupling to external photons. The operator $S(x)$ may be regarded as defined by Eq. (2.8b); its matrix elements can be given a direct physical meaning at least to the extent that the commutator of the components of the electric current itself describes physical processes.[10]

The matrix elements of $\dot{S}(x)$ are related to those of $S(x)$ by Eq. (2.9). In a representation in which P_0 is diagonal, the matrix elements of $\dot{S}(x)$ are just those of $S(x)$ times a power of the energy, so they can be given a physical meaning whenever the corresponding matrix elements of $S(x)$ can.

All of the above commutators have been computed in a formal algebraic way from Eqs. (2.3) and (2.6). There arises the question of "Schwinger terms."[11] This expression seems to be used in two distinct ways. On the one hand, one can refer to any term on the right side of a commutation relation which involves the gradient of a δ function as a Schwinger term. In this sense, Eq. (2.8b) contains a Schwinger term. Alternatively, one can mean by this expression specifically those terms which arise when the commutators are calculated starting with a definition of the currents as limits of some nonlocal product of operators, but which are not present otherwise. Schwinger terms which might arise in this way have not been included here. In the following paper,[12] we see an example of how these extra terms may be necessary to insure properties like positivity of the energy spectrum. In the present model, however, there is no obvious necessity to include such terms.

Our point of view on Schwinger terms is, then, the following. We write a current algebra and insist that any solution of it, together with the equations of motion, satisfy the conditions of Lorentz invariance and positivity of the energy spectrum. If it does, we are satisfied. If it does not, one must modify the commutation relations until one does get a theory consistent with the above constraints. In either case, it may turn out that the currents and other operators can not be expressed as products of canonical fields in a simple or even meaningful way. We would not find this disturbing. Instead, we insist that the currents and other local observables have physically reasonable properties, and base the theory on these quantities, being content to let the chips fall where they may as regards the possible existence of an underlying theory.

C. Irreducibility of the Equal-Time Current Algebra

At this point we would like to discuss briefly the question of whether the operators $j_\mu(x)$, $S(x)$, and $\dot{S}(x)$ form a complete or irreducible set at a given time. This subject is taken up in more detail in Appendix A and in I; here we shall make one or two introductory comments.

The question of irreducibility is partly one of physics and partly one of mathematics and is not a question that can be settled on the basis of the equal-time current algebra alone.

As an example, recall the situation in nonrelativistic quantum mechanics in one dimension. Here one proves that x and p are irreducible or complete in the sense that any operator that commutes with both of them is a constant multiple of the identity. This result does not follow, however, just from the equal-time algebra:

$$[x,x]=[p,p]=0, \quad [x,p]=i. \tag{2.11}$$

(If it did, one would have proved that there could be no spin in the world.) One must add an additional hypothesis, one form of which is to suppose that the spectrum of x is "simple," i.e., each eigenvalue of x occurs only once.

Evidently, one must make comparable assumptions when discussing irreducibility in the present case. What we shall assume for this purpose is that the fields $\varphi(x)$, $\varphi^*(x)$, $\pi(x)$, and $\pi^*(x)$ form an irreducible set at a given time, that the operators $j_\mu(x)$, $S(x)$, and $\dot{S}(x)$ are defined in terms of the fields by Eqs. (2.6), (2.8d), and (2.9), and that they satisfy the equal-time current algebra, Eqs. (2.8) and (2.10).

In Appendix A we show, on the basis of these assumptions, that the elements of the equal-time current algebra are irreducible in any sector of the charged scalar theory having fixed total charge. The argument, for which no claim to mathematical rigor is made, proceeds by assuming that there is an operator $A(x)$ which commutes at equal times with each of the operators $j_\mu(x)$, $S(x)$, and $\dot{S}(x)$, and then showing that any such operator must be a function of the total charge operator. But the charge operator is a constant multiple of the identity acting on the states in a given charge sector, so the above result means that the elements of the equal-time current algebra are irreducible, in the usual sense, in any particular charge sector of the theory.

[10] In the present case, $S(x)$ also appears as a piece of the energy density.

[11] J. Schwinger, Phys. Rev. Letters 3, 296 (1959).

[12] C. G. Callan, R. F. Dashen, and D. H. Sharp, following paper, Phys. Rev. 165, 1883 (1968).

Completely aside from mathematical arguments, the above result is very plausible on physical grounds. It will turn out that the energy-momentum tensor for this theory can be written explicitly in terms of the above operators. Since these quantities, or functions of them, are the only local quantities that one can think of that one can measure, they ought to be irreducible acting on the states which they can create.

D. Heisenberg Equations of Motion for Currents

As a result of the completeness of the operators $j_\mu(x)$, $S(x)$, and $\dot{S}(x)$, we know that it should be possible to express any other quantity in the theory in terms of them. The most important of these is the energy-momentum tensor $\theta_{\mu\nu}(x)$, and we will now show how it can be written in terms of the above operators.

From Eqs. (2.6) and (2.8d), we have

$$\begin{aligned}\mathcal{J}_\mu(x) &\equiv \partial_\mu S(x) - i j_\mu(x) = 2\varphi^*(x)\partial_\mu\varphi(x)\,,\\ \mathcal{J}_\mu{}^\dagger(x) &\equiv \partial_\mu S(x) + i j_\mu(x) = 2\partial_\mu\varphi^*(x)\varphi(x)\,.\end{aligned} \tag{2.12}$$

Recalling the formal identity $S^{-1}=(\varphi^*\varphi)^{-1}=\varphi^{-1}\varphi^{*-1}$, we see that

$$\partial_\mu\varphi^*\partial^\mu\varphi = \tfrac{1}{4}\mathcal{J}_\mu{}^\dagger S^{-1}\mathcal{J}^\mu. \tag{2.13}$$

Thus we may rewrite $\theta_{\mu\nu}(x)$, Eq. (2.7), in the form

$$\theta_{\mu\nu}(x) = \tfrac{1}{4}\mathcal{J}_\mu{}^\dagger\frac{1}{S}\mathcal{J}_\nu + \tfrac{1}{4}\mathcal{J}_\nu{}^\dagger\frac{1}{S}\mathcal{J}_\mu - g_{\mu\nu}\left[\tfrac{1}{4}\mathcal{J}_\alpha{}^\dagger\frac{1}{S}\mathcal{J}^\alpha - \mu^2 S - \tfrac{1}{2}\lambda S^2\right], \tag{2.14}$$

where for definiteness we have picked an interaction of the form $\mathcal{L}_I(x) = -\frac{1}{2}\lambda(\varphi^*\varphi)^2$. In particular, the energy density is

$$\theta_{00}(x) = \tfrac{1}{4}\mathcal{J}_0{}^\dagger\frac{1}{S}\mathcal{J}_0 + \tfrac{1}{4}\mathcal{J}_k{}^\dagger\frac{1}{S}\mathcal{J}_k + \mu^2 S + \tfrac{1}{2}\lambda S^2. \tag{2.15}$$

Note that it is formally Hermitian in spite of the appearance of non-Hermitian factors like $\mathcal{J}_\mu(x)$.

Finally, the dynamical laws can be expressed through the Heisenberg equations of motion for the currents. These are calculated, of course, using the fact that the space integral of $\theta_{00}(x)$ is the generator of time translations,

$$i\left[\int \theta_{00}(\mathbf{y})d^3y, O(\mathbf{x},t)\right] = \dot{O}(\mathbf{x},t)\,, \tag{2.16}$$

and they turn out to be

$$\dot{\rho}(x) + \nabla\cdot\mathbf{j}(x) = 0\,, \tag{2.17}$$

$$\Box S(x) + 2\mu^2 S(x) + 2\lambda S^2(x) = \tfrac{1}{2}\mathcal{J}_\mu{}^\dagger(x) S^{-1}(x)\mathcal{J}^\mu(x)\,, \tag{2.18}$$

$$\begin{aligned}\partial\mathbf{j}_k(x)/\partial t = \tfrac{1}{2}i\{&[\mathcal{J}_0{}^\dagger(x)S^{-1}(x)\mathcal{J}_k(x)]\\ &-[\mathcal{J}_0{}^\dagger(x)S^{-1}(x)\mathcal{J}_k(x)]^\dagger\} + \partial\rho(x)/\partial x_k\,,\end{aligned} \tag{2.19}$$

with $\dot{S}(x)$ given by Eq. (2.9).

Introducing the definitions of $j_\mu(x)$, $S(x)$, and $\dot{S}(x)$ in terms of the fields, we see that the field equations, Eqs. (2.5), can be recovered from Eqs. (2.17) and (2.18), while Eqs. (2.19) and (2.9), only two of which are independent in the underlying theory, provide the definitions of the field momenta $\pi(x)$ and $\pi^*(x)$.

Looking at the above Heisenberg equations of motion, we see that they involve products of operators at a point, not to mention an inverse operator. Although we take a fairly relaxed view of this fact, the presence of such expressions does not inspire confidence in the idea that any of the mathematical difficulties besetting ordinary field theories have been overcome here. For this reason, we will not pause at this point to analyze these equations further, but will instead proceed in the next two sections to rewrite the equations in such a way that the above difficulties, if not eliminated, are at least not present in so glaring a form.

E. Energy-Momentum Tensor Defined by its Equal-Time Commutation Relations

We have heretofore considered the energy-momentum tensor $\theta_{\mu\nu}(x)$ as a secondary quantity in the theory, defining it in terms of the basic operators $j_\mu(x)$, $S(x)$, and $\dot{S}(x)$, Eq. (2.14). From many points of view, however, it is natural to consider $\theta_{\mu\nu}(x)$ as on the same footing as the currents. In this section, and in Appendix B, we discuss how $\theta_{\mu\nu}(x)$ may be introduced directly into the theory, defining it not by Eq. (2.14), but rather by its equal-time commutation relations with the currents and with itself. The resulting system of algebraic relations contains sufficient information to determine the dynamics, in principle, and can replace the Heisenberg equations of motion, Eqs. (2.17)–(2.19) and (2.9).

The full set of equal-time commutation relations involving $\theta_{\mu\nu}(x)$ is written in Appendix B. Here we wish to summarize a number of important general features of this system of equations.

(i) Referring to Eqs. (B1)–(B4), one will note that no inverse operators appear.

(ii) The only product of operators which appears comes in as a result of the interaction term λS^2. This remaining product may be formally eliminated by introducing $\lambda O(x) = \lambda S^2(x)$ in these equations and defining $O(x)$ by its equal-time commutation relations with a complete set of operators, namely,

$$\begin{aligned}&[O(\mathbf{x}),\rho(\mathbf{y})] = [O(\mathbf{x}),j_k(\mathbf{y})] = [O(\mathbf{x}),S(\mathbf{y})] = 0\,,\\ &[O(\mathbf{x}),\dot{S}(\mathbf{y})] = 4iO(\mathbf{x})\delta(\mathbf{x}-\mathbf{y})\,.\end{aligned} \tag{2.20}$$

(iii) The requirement that the theory be Lorentz invariant is essentially contained in the statement that the equal-time commutation relations between the components of $\theta_{\mu\nu}$ have a particular form. For example, the form of Eqs. (B4a) and (B4b) is to a large extent determined by the requirement that the space integrals of $\theta_{0k}(x)$ and $\theta_{00}(x)$ be the generators of infinitesimal displacements in space and time, together with a condition of "least singularity."[13] Schwinger, for example, shows[6] how one is led to the form of the commutator of the energy density with itself from the requirement of energy-momentum conservation.

(iv) New terms ($\partial \mathbf{j}/\partial t$ and $\dot{\theta}_{kl}$), which we can not express as linear combinations of elements in the algebra, are introduced in Eqs. (B16) and (B4c). The algebra generated by successive commutation of $\theta_{\mu\nu}$ with itself and with the currents is therefore not a closed Lie algebra, in general.[14]

In spite of this fact, it is possible to show that a finite number of commutation relations contain enough information to determine the dynamics,[15] in the following sense.

We first show (in Appendix B) that, given any representation of the equal-time current algebra, the finite set of commutators listed in Eqs. (B1)–(B4) uniquely determine the functional dependence[16] of $\theta_{\mu\nu}(x)$ upon $j_\mu(x)$, $S(x)$, and $\dot{S}(x)$. For this purpose we argue that the completeness of the operators j_μ, S, and $\dot{S}$ means that an expression for $\theta_{\mu\nu}$ as a function of these operators exists. We then suppose that there are two expressions for $\theta_{\mu\nu}$ as a function of the currents which are compatible with its equal-time commutation relations, and show that Eqs. (B1)–(B4) imply that the two expressions can differ only by a multiple of the identity. Finally, we note that the same commutation relations which in their "homogeneous" form[17] imply the uniqueness of the solution must in their "inhomogeneous" form actually determine the solution.

Supposing that we have $\theta_{\mu\nu}(x)$ as a function of the currents, the dynamics of the theory is determined in the usual way by first diagonalizing P_0. The time development of any Heisenberg operator is then determined by the equation

$$O(\mathbf{x},t)=e^{iP_0t}O(\mathbf{x},0)e^{-iP_0t}, \tag{2.21}$$

with

$$P_0=\int \theta_{00}(\mathbf{x})d^3x.$$

In this sense, the dynamics is contained in those equal-time commutation relations which fix $\theta_{\mu\nu}(x)$ once a representation of the equal-time algebra of $j_\mu(x)$, $S(x)$, and $\dot{S}(x)$ is specified.[18]

F. Functional Representations of Current Algebras

By a functional representation of a current algebra, we mean the following. One introduces a basis in Hilbert space consisting of the complete set of eigenvectors associated with a maximal commuting set of current operators. An arbitrary state $|\Psi\rangle$ is then represented by giving its components along each of the basis vectors; that is by a wave functional.[19] When the states are represented in this way, the effect of applying any current operator to a state $|\Psi\rangle$ can be represented either as multiplication of the associated wave functional by a c-number function, or by c-number functional differentiation applied to the wave functional. With the current operators realized in this way, the current algebra is automatically satisfied[19] and the dynamics of the problem is contained in the solutions to a functional Schrödinger equation.

In the present section we shall write some functional representations of the current algebra specified by Eqs. (2.8) and (2.10). To what purpose? First, one gains some idea of how one might go about trying to solve a theory based just on currents from scratch, i.e., if one had no idea as to whether or not it came from some underlying theory. This does not imply that introducing

[13] In this connection, see Ref. 1.

[14] It is interesting to note that in certain very special cases, such as the model which we study in Ref. 12, it can happen that the whole algebra, including the commutation relations of $\theta_{\mu\nu}$, is closed. It is also worth noting that the question of the closure of the algebra of the currents and $\theta_{\mu\nu}$ seems to be one of the kinematics rather than dynamics; in the case of the 2-dimensional model of Ref. 12, the algebra closes whether or not a Fermi interaction is included, while in the present case the algebra does not close, whether or not a λS^2 interaction is included.

[15] The point that Lorentz invariance, plus a sufficient amount of information at equal times, constitutes a complete specification of the dynamics of a system was made long ago by E. P. Wigner, Ann. Math. **40**, 149 (1939).

[16] There is no guarantee, of course, that if one employs an arbitrary representation of the current algebra the resulting expression for the energy will turn out to be positive.

[17] By the "homogeneous" form of the commutation relations, we simply mean the set of commutation relations satisfied by the difference between two proposed solutions for $\theta_{\mu\nu}$, corresponding to a single, specific choice for the currents. The "inhomogeneous" form of the commutation relations is just the set written in Appendix B.

[18] Another way of looking at the problem, which in some ways is closer to the spirit of this section than what we have just described, is to try to work with the full set of commutation relations directly, extending to field theory the algebraic methods originally employed in nonrelativistic quantum mechanics by Born, Heisenberg and Jordan. (M. Born and P. Jordan, *Elementare Quantenmechanik*, Berlin, 1930.) One could try to do this numerically, putting all the commutation relations on a grid, or by searching for some analytic techniques. This has the advantage that one does not have to express $\theta_{\mu\nu}$ as a function of other operators at any stage; on the other hand, this approach has never been well adapted to deal with continuum states, and looks very cumbersome.

[19] This is the analog of introducing a wave function in nonrelativistic quantum mechanics. There one represents an abstract state $|\Psi\rangle$ by giving its components along a basis defined, for example, by the complete set of eigenvectors $|x\rangle$ of the operator $\hat{x}$. Thus a position wave function is defined by $\Psi(x)=\langle x|\Psi\rangle$. The operators $\hat{x}$ and $\hat{p}$ are then represented by

$$\hat{x}|\Psi\rangle \leftrightarrow x\Psi(x);\quad \hat{p}|\Psi\rangle \leftrightarrow \frac{1}{i}\frac{d}{dx}\Psi(x).$$

In this representation, the equal-time algebra $[\hat{x},\hat{x}]=[\hat{p},\hat{p}]=0$; $[\hat{x},\hat{p}]=i$, is automatically satisfied. Such representations are also well known in quantum statistical mechanics and quantum field theory. Extensive discussion of these topics is contained in the following representative list of books and review articles: K. O. Friedrichs, *Mathematical Aspects of the Quantum Theory of Fields* (Interscience Publishers, Inc., New York, 1953); W. T. Martin and I. Segal, *Analysis in Function Space* (MIT Press, Cambridge, Mass., 1964); R. P. Feynman and A. R. Hibbs, *Quantum Mechanics and Path Integrals* (McGraw-Hill Book Co., Inc., New York, 1965); F. A. Berezin, *The Method of Second Quantization* (Academic Press Inc., New York, 1966); I. M. Gel'Fand and A. M. Yaglom, J. Math. Phys. **1**, 48 (1960); S. G. Brush, Rev. Mod. Phys. **33**, 79 (1961).

the functional representations will necessarily make this task any easier, or circumvent any problems, but just that it does provide a way to work directly with a theory formulated in terms of currents. Secondly, some people believe that there may be certain technical mathematical advantages to working with functional representations.[20] A list of some of these possible advantages can be found in the preface of Berezin.[19]

Now let us discuss some of these functional representations. Since $\hat{\rho}(\mathbf{x})$ and $\hat{S}(\mathbf{x})$ form one maximal commuting set of operators[21] at a given time, we may consider a complete set of states[22] which are simultaneous eigenstates of $\hat{\rho}$ and $\hat{S}$;

$$\begin{aligned}\hat{\rho}(\mathbf{x})\,|\,\rho(\mathbf{x}),S(\mathbf{x})\rangle&=\rho(\mathbf{x})\,|\,\rho(\mathbf{x}),S(\mathbf{x})\rangle\,,\\ \hat{S}(\mathbf{x})\,|\,\rho(\mathbf{x}),S(\mathbf{x})\rangle&=S(\mathbf{x})\,|\,\rho(\mathbf{x}),S(\mathbf{x})\rangle\,.\end{aligned}\tag{2.22}$$

We next introduce the associated wave functionals as follows:[19]

$$\Psi\{\rho,S\}=\langle\rho(\mathbf{x}),S(\mathbf{x})\,|\,\Psi\rangle\,.\tag{2.23}$$

Evidently, the action of the operators $\hat{\rho}(x)$ and $\hat{S}(x)$ on the state $|\Psi\rangle$ can be represented as multiplication of the wave functional $\Psi\{\rho,S\}$ by the eigenvalues of $\hat{\rho}(x)$ and $\hat{S}(x)$, the functions $\rho(x)$ and $S(x)$, respectively.

Finally, we define functional derivatives[19] $\delta/\delta S(\mathbf{x})$, $\delta/\delta\rho(\mathbf{x})$, and $\delta^k/\delta^k\rho(\mathbf{x})$ by the equations

$$\begin{aligned}\int\frac{\delta\Psi\{\rho(\mathbf{x})\}}{\delta\rho(\mathbf{x})}\gamma(\mathbf{x})d^3x&=\lim_{\epsilon\to0}\epsilon^{-1}[\Psi\{\rho(\mathbf{x})+\epsilon\gamma(\mathbf{x})\}\\&\qquad-\Psi\{\rho(\mathbf{x})\}]\,,\\ \int\frac{\delta^k\Psi\{\rho(\mathbf{x})\}}{\delta^k\rho(\mathbf{x})}\gamma(\mathbf{x})d^3x&=\lim_{\epsilon\to0}\epsilon^{-1}[\Psi\{\rho(\mathbf{x})-\epsilon\partial_k\gamma(\mathbf{x})\}\\&\qquad-\Psi\{\rho(\mathbf{x})\}]\,,\end{aligned}\tag{2.24}$$

etc., and note that, applied to wave functionals $\Psi\{\rho,S\}$, one has

$$\begin{aligned}&[\rho(\mathbf{x}),\rho(\mathbf{y})]=0\,,\\&[\rho(\mathbf{x}),(1/i)\delta/\delta\rho(\mathbf{y})]=i\delta(\mathbf{x}-\mathbf{y})\,,\\&[\rho(\mathbf{x}),(1/i)\delta^k/\delta^k\rho(\mathbf{y})]=i\frac{\partial}{\partial y_k}\delta(\mathbf{x}-\mathbf{y})\,,\end{aligned}\tag{2.25}$$

and similarly for $S(\mathbf{x})$. Now, if we represent our current operators, acting on wave functionals $\Psi\{\rho,S\}$, as follows:

$$\begin{aligned}&\hat{S}(\mathbf{x})\leftrightarrow S(\mathbf{x})\,,\\&\hat{\rho}(\mathbf{x})\leftrightarrow\rho(\mathbf{x})\,,\\&\partial\hat{S}(\mathbf{x})/\partial t\leftrightarrow S(\mathbf{x})\frac{1}{i}\frac{\delta}{\delta S(\mathbf{x})}+\frac{1}{i}\frac{\delta}{\delta S(\mathbf{x})}S(\mathbf{x})\,,\\&\hat{\jmath}_k(\mathbf{x})\leftrightarrow 2S(\mathbf{x})i\delta^k/\delta^k\rho(\mathbf{x})\,;\end{aligned}\tag{2.26}$$

one can easily verify that the current algebra, Eqs. (2.8) and (2.10), is satisfied.[23]

Since $\hat{S}(\mathbf{x})$ and $\hat{\jmath}_k(\mathbf{x})$ also commute at a given time, one expects that a functional representation also exists in which $\hat{S}(\mathbf{x})$ and $\hat{\jmath}_k(\mathbf{x})$ act as multiplication on wave functionals. A representation of this kind is obtained if one represents the current operators in the following way:

$$\begin{aligned}&\hat{S}(\mathbf{x})\leftrightarrow f(\mathbf{x})\,,\\&\hat{\jmath}_k(\mathbf{x})\leftrightarrow 2f(\mathbf{x})\partial g(\mathbf{x})/\partial x_k\,,\\&\partial\hat{S}(\mathbf{x})/\partial t\leftrightarrow f(\mathbf{x})\frac{1}{i}\frac{\delta}{\delta f(\mathbf{x})}+\frac{1}{i}\frac{\delta}{\delta f(\mathbf{x})}f(\mathbf{x})\,,\\&\hat{\rho}(\mathbf{x})\leftrightarrow(1/i)\delta/\delta g(\mathbf{x})\,,\end{aligned}\tag{2.27}$$

in which $f(\mathbf{x})$, $\delta/\delta f(\mathbf{x})$, $g(\mathbf{x})$, and $\delta/\delta g(\mathbf{x})$ commute like two independent coordinates and momenta in the functional representation [see Eq. (2.25)], and we label the wave functional by $\Psi\{f,g\}$.

The energy spectrum and stationary states of the system are determined by the equation

$$\hat{P}_0|\Psi\rangle=P_0|\Psi\rangle\,,\tag{2.28}$$

where as usual $\hat{P}_0=\int\hat{\theta}_{00}(\mathbf{x})d^3x$, and $P_0\equiv\int\epsilon(\mathbf{x})d^3x$ is the associated energy eigenvalue. This equation can be converted into a concrete functional differential equation by making use of Eq. (2.14) which expresses $\theta_{\mu\nu}$ as a function of the currents, together with a specific functional representation of the current algebra. For example, if we employ the representation (2.27), we can write $\hat{\theta}_{00}(x)$ as

$$\begin{aligned}\hat{\theta}_{00}(\mathbf{x})\leftrightarrow&-\frac{1}{4}\left[f(\mathbf{x})\frac{\delta}{\delta f(\mathbf{x})}+\frac{\delta}{\delta f(\mathbf{x})}f(\mathbf{x})+i\frac{\delta}{\delta g(\mathbf{x})}\right]\frac{1}{f(\mathbf{x})}\\&\times\left[f(\mathbf{x})\frac{\delta}{\delta f(\mathbf{x})}+\frac{\delta}{\delta f(\mathbf{x})}f(\mathbf{x})-i\frac{\delta}{\delta g(\mathbf{x})}\right]\\&+\frac{1}{4}\frac{1}{f(\mathbf{x})}[(\partial_kf(\mathbf{x}))^2+4f^2(\mathbf{x})(\partial_kg(\mathbf{x}))^2]\\&+\mu^2f(\mathbf{x})+\tfrac{1}{2}\lambda f^2(\mathbf{x})\,.\end{aligned}\tag{2.29}$$

[20] This is probably a good place to reiterate, however, that we have no reason to believe that the results of this or the preceding section have contributed anything to the problem of finding a mathematically consistent formulation of an interacting field theory.

[21] In this section will distinguish operators from their eigenvalues by putting a caret over the operators. For typographical reasons, we will write the operator for $\dot{\hat{S}}(\mathbf{y})$ as $\partial\hat{S}(\mathbf{y})/\partial t$ in this section. When we write $\partial\hat{S}(\mathbf{y})/\partial t$ in an equal-time commutator, this means

$$[\hat{O}(\mathbf{x},t),\,\partial\hat{S}(\mathbf{y},t')/\partial t']_{t=t'}.$$

[22] We stress that these eigenstates are labelled by giving the values of the functions $\rho(\mathbf{x})$ and $S(\mathbf{x})$ over the whole spacelike 3-surface $t=$constant.

[23] Note that Eq. (2.8b) can be written

$$[\rho(\mathbf{x}),j_k(\mathbf{y})]=-2iS(\mathbf{y})\partial\delta(\mathbf{x}-\mathbf{y})/\partial y_k.$$

The first term in the above expression simplifies somewhat, and we can write Eq. (2.28) in the form

$$\left\{-\frac{1}{4}\frac{1}{f(\mathbf{x})}\left[2f(\mathbf{x})\frac{\delta}{\delta f(\mathbf{x})}-\delta^{(3)}(0)\right]\left[2f(\mathbf{x})\frac{\delta}{\delta f(\mathbf{x})}+\delta^{(3)}(0)\right]\right.$$
$$-\frac{1}{4}\frac{1}{f(\mathbf{x})}\left[2i\delta^{(3)}(0)\frac{\delta}{\delta g(\mathbf{x})}+\frac{\delta^2}{\delta^2 g(\mathbf{x})}\right]$$
$$+\frac{1}{4}\frac{1}{f(\mathbf{x})}[(\partial_k f(\mathbf{x}))^2+4f^2(\mathbf{x})(\partial_k g(\mathbf{x}))^2]+\mu^2 f(\mathbf{x})$$
$$\left.+\tfrac{1}{2}\lambda f^2(\mathbf{x})\right\}\Psi\{f,g\}=\epsilon(\mathbf{x})\Psi\{f,g\}. \quad (2.30)$$

We shall not go on here to discuss Eq. (2.30) in any detail except for one point; the presence of $\delta^{(3)}(0)$ as a coefficient of $\delta/\delta g(\mathbf{x})$. Such terms are common in functional differential equations, being analogs of factor-ordering terms present in operator expressions, and do not necessarily reflect themselves as ill-behaved solutions. For example,[24] the well-behaved functional

$$F\{\varphi\}=\exp\left[-\frac{1}{2}\int\varphi^2(\mathbf{x})d^3x\right] \quad (2.31)$$

satisfies the singular looking equation

$$[\delta^2/\delta^2\varphi(\mathbf{x})-\varphi^2(\mathbf{x})]F\{\varphi\}=-\delta^{(3)}(0)F\{\varphi\}. \quad (2.32)$$

We do not know that this is the kind of situation which occurs in Eq. (2.30), but we can explore the matter a little further in one special case. Suppose we drop the terms involving $\partial_k f(\mathbf{x})$ and $\partial_k g(\mathbf{x})$ from Eq. (2.30) completely, as might be reasonable if we had a dense, essentially uniform distribution of "matter." The resulting equation separates upon introducing $\Psi\{f,g\}=\chi\{f\}\Phi\{g\}$. We shall only consider the equation for $\Phi\{g\}$, which is

$$\left[\frac{\delta^2}{\delta^2 g(\mathbf{x})}+(2i/V_L)\frac{\delta}{\delta g(\mathbf{x})}+\kappa^2(\mathbf{x})-(1/V_L)^2\right]\Phi\{g\}=0. \quad (2.33)$$

Here we have temporarily replaced $\delta^{(3)}(0)$ by $1/V_L$, the finite number to which it corresponds if we introduce a space lattice in which the volume of an individual lattice cell is V_L. Also, we have chosen the separation constant to be $\kappa^2(\mathbf{x})-(1/V_L)^2$. We shall first solve the equation for fixed, finite V_L and later find that all physical results are independent of V_L. One solution of this equation is

$$\Phi\{g\}=A\exp\left\{i\int[\kappa(\mathbf{x}')-(1/V_L)]g(\mathbf{x}')d^3x'\right\}, \quad (2.34)$$

where A is a normalization factor. This expression satisfies Eq. (2.33) for any value of $\kappa(\mathbf{x})$ and $1/V_L$.

We can get a check on the reasonability of this solution, and at the same time learn something about $\kappa^2(\mathbf{x})$, by requiring that the charge operator $\hat{Q}$ have integral eigenvalues (units $e=1$) applied to a wave functional. Using Eq. (2.27) for the charge density operator, we require

$$\hat{Q}\Psi\{f,g\}=\left[\int_V\frac{1}{i}\frac{\delta}{\delta g(\mathbf{x}')}d^3x'\right]\chi\{f\}\Phi\{g\}=N\chi\{f\}\Phi\{g\}, \quad (2.35)$$

where N is an integer.

If $\Phi\{g\}$ is given Eq. (2.34), we must therefore have

$$\left[\int_V[\kappa(\mathbf{x}')-(1/V_L)]d^3x'\right]\Phi\{g\}=N\Phi\{g\}. \quad (2.36)$$

This equation, which must hold for any finite volume V, can be satisfied by picking $\kappa(\mathbf{x})=(1/V_L)+f(\mathbf{x})$, and then suitably choosing $f(\mathbf{x})$. One choice for $f(\mathbf{x})$ would be

$$f(\mathbf{x})=n_1\delta^{(3)}(\mathbf{x}-\mathbf{a}_1)+\cdots+n_p\delta^{(3)}(\mathbf{x}-\mathbf{a}_p), \quad (2.37)$$

with

$$\sum_{i=1}^{p}n_i=N.$$

This choice for $\kappa(\mathbf{x})$ also determines the spectrum of $\hat{\rho}(\mathbf{x})$, according to the equation

$$\hat{\rho}(\mathbf{x})\Phi\{g\}=\frac{1}{i}\frac{\delta}{\delta g(\mathbf{x})}\Phi\{g\}=[\kappa(\mathbf{x})-(1/V_L)]\Phi\{g\}. \quad (2.38)$$

If $f(\mathbf{x})$ is given by Eq. (2.37), we see that $\hat{\rho}(\mathbf{x})$ applied to $\Phi\{g\}$ is $-\infty$, 0, $+\infty$ at a point $\mathbf{x}=\mathbf{a}_i$ where there is located a "charge," according as n_i is $+$, 0, or $-$. The interpretation of the spectrum of $\hat{\rho}(\mathbf{x})$ when it is smeared over some volume of space is also clear.

The expression for $\Phi\{g\}$ corresponding to the above choice for $\kappa(\mathbf{x})$ is

$$\Phi\{g\}=A\exp\{i\textstyle\sum_i n_i g(\mathbf{a}_i)\}. \quad (2.39)$$

It is interesting to note that Φ, $\hat{\rho}(\mathbf{x})$, and $\hat{Q}$ are all independent of V_L, which can now be set equal to zero. It appears, therefore, that Eq. (2.33) has well-behaved solutions [for a suitable choice of separation parameter $\kappa^2(\mathbf{x})$] in spite of the singular way in which it was written. It remains to be seen, of course, whether the equation for $\chi\{f\}$ also has reasonable solutions with this choice of separation parameter.

One can write a number of other functional representations for the current algebra, besides those given above, including the one which comes from a functional representation of an underlying field theory. The question of the relationship of all these representations to each other, and in particular the question of whether any or all of them are equivalent to an underlying field

[24] This example has recently been mentioned by H. Leutwyler, Phys. Rev. 134, B1155 (1964).

theory, remains to be investigated, as does the problem of finding solutions to such equations.

G. Renormalization

In this section we speculate that strong-interaction theories, when formulated as a set of relationships between currents and other observable quantities, do not require renormalization in the usual sense. We think the situation is something like the following.

First, recall that we are working to first order in the electric charge. The charge is not renormalized, to this order, and so the electric current receives no renormalization from this source.

Further renormalizations are usually introduced because one is comparing a physical theory to a "bare" theory. This is the origin of wave-function renormalization as well as mass- and coupling-constant renormalization. However, no such comparison is made here. The theory written in terms of currents is, directly, the physical theory. There is no reason why the parameters μ^2 and λ, carryovers from the underlying theory, should be interpreted as the "bare mass" or "bare coupling" of anything. That identification presupposes the existence and employment of a representation of the algebra that permits the existence of an underlying "field" of the usual variety. Instead, we can regard μ^2 simply as a parameter fixing a scale of length in the theory, while λ is a parameter determining the strength of interaction. The masses and effective couplings of particles must be determined as part of the problem of finding the particle states in the theory, which are determined by the spectrum of the energy operator. Thus the usual reasons for renormalization are lacking.

Another point is that the equal-time algebra determines the scale of all operators, except for a change of scale arising from changing the scale of length.[25] This can be seen as follows. We require that the integral of $\rho(\mathbf{x})$ over any finite volume equal the charge in that volume. This condition requires $\rho(\mathbf{x})$ to scale as λ^3 when $\mathbf{x}$ scales as $1/\lambda$.

From current conservation (or relativity) we must also have $j_k(\mathbf{x}')\to\lambda^3 j_k(\mathbf{x})$. Next, from the commutator

$$[j_k(\mathbf{x}),\dot{S}(\mathbf{y})]=2ij_k(\mathbf{x})\delta(\mathbf{x}-\mathbf{y})\,, \tag{2.10c}$$

we see that $S(\mathbf{x}')\to\lambda^2 S(\mathbf{x})$, since $\delta^{(3)}(\mathbf{x}'-\mathbf{y}')\to\lambda^3\delta^{(3)}(\mathbf{x}-\mathbf{y})$ and, with an appropriate choice for the units of the speed of light, times scale as lengths. This choice automatically satisfies the remaining commutation relations, and one also finds that the energy density scales as λ^4.

Finally, we wish to stress the following point: The possibility that a theory of matter formulated in terms of currents does not need to be renormalized in the usual sense does not necessarily mean that the theory is free of divergences. The possible existence and the character of such divergences are points which remain to be investigated.

[25] Once the units in which one measured the electric charge are fixed.

III. CHARGED SCALAR THEORY. ELECTRODYNAMICS

A. Introduction

In this section, we briefly indicate how the considerations of Sec. II can be extended to include the interaction of charged scalar mesons with a quantized electromagnetic field. The fundamental quantities appearing here are, beside the currents and energy momentum tensor $\theta_{\mu\nu}(x)$, the electromagnetic field strengths $F_{\mu\nu}(x)$. The latter, according to the analysis of Bohr and Rosenfeld,[26] are supposed to be quantities which are measurable in principle, at least when averaged over a small space-time volume. The resulting formulation of the electrodynamics of scalar mesons is gauge-invariant and path-independent.

B. Conventional Formulation

Our starting point is the conventional formulation[27] of charged scalar electrodynamics, in which the Lagrangian density is taken to be[28]

$$\mathcal{L}(x)=\left[\left(\frac{\partial}{\partial x_\nu}-ie_0A^\nu\right)\varphi^*\right]\left[\left(\frac{\partial}{\partial x^\nu}+ie_0A_\nu\right)\varphi\right]-\mu^2\varphi^*\varphi-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}\,, \tag{3.1}$$

with the electromagnetic field strengths $F_{\mu\nu}$ given in terms of the 4-vector potential by

$$F_{\mu\nu}=\partial A_\mu/\partial x_\nu-\partial A_\nu/\partial x_\mu\,. \tag{3.2}$$

The theory is quantized by imposing suitable equal-time commutation relations between the fields φ, φ^*, and $A_\mu(x)$. The commutation relations for the matter fields φ and φ^* are as before, Eq. (2.3), except that the canonical momenta are now given by

$$\begin{aligned}\pi&=\delta\mathcal{L}/\delta\dot{\varphi}=\dot{\varphi}^*-ie_0A_0\varphi^*\,,\\ \pi^*&=\delta\mathcal{L}/\delta\dot{\varphi}^*=\dot{\varphi}+ie_0A_0\varphi\,.\end{aligned} \tag{3.3}$$

Thus we now have, for example,

$$[\varphi(\mathbf{x}),\pi(\mathbf{y})]=[\varphi(\mathbf{x}),\,\dot{\varphi}^*(\mathbf{y})-ie_0A_0(\mathbf{y})\varphi^*(\mathbf{y})]=i\delta(\mathbf{x}-\mathbf{y})\,. \tag{3.4}$$

The form of the equal-time commutation relations imposed on the vector potential depends on the gauge.

[26] N. Bohr and L. Rosenfeld, Kgl. Danske Videnskab. Selskab, Mat.-Fys. Medd. **12**, No. 8 (1933); Phys. Rev. **78**, 794 (1950).
[27] See, for example, G. Wentzel, Ref. 8, or J. Bjorken and S. Drell, *Relativistic Quantum Fields* (McGraw-Hill Book Co., Inc., New York, 1965).
[28] We have not included strong interactions here. These can, of course, be included if desired withing changing any basic results.

In the Lorentz gauge one requires[29–31]

$$[A_\mu(\mathbf{x}),A_\nu(\mathbf{y})]=[\dot{A}_\mu(\mathbf{x}),\dot{A}_\nu(\mathbf{y})]=0,\quad [A_\mu(\mathbf{x}),\dot{A}_\nu(\mathbf{y})]=i\delta_{\mu\nu}\delta(\mathbf{x}-\mathbf{y}). \tag{3.5}$$

Finally, the dynamics is contained in the equations of motion

$$[(\partial/\partial x_\mu+ie_0A_\mu)^2+\mu^2]\varphi=0,\quad [(\partial/\partial x_\mu-ie_0A_\mu)^2+\mu^2]\varphi^*=0, \tag{3.6}$$

together with a set of equations of motion for the vector potential $A_\mu(x)$. Since the latter are gauge-dependent, and will later be replaced by the gauge-invariant Maxwell equations, we will not write them out here. The full set of equations of motion can be obtained, one way or another, from the symmetric, conserved energy-momentum tensor[32,33]

$$\theta_{\mu\nu}=[(\partial\varphi^*/\partial x_\mu-ie_0A_\mu\varphi^*)(\partial\varphi/\partial x_\nu+ie_0A_\nu\varphi)+(\mu\leftrightarrow\nu)-\tfrac{1}{2}(F_\alpha{}^\mu F_\alpha{}^\nu+F_\alpha{}^\nu F^{\mu\alpha})-g_{\mu\nu}\mathfrak{L}]. \tag{3.7}$$

C. Gauge- and Path-Independent Electrodynamics

To pass from the conventional formalism to a gauge-independent formulation of electrodynamics is a question of introducing a set of gauge- and path-independent variables describing the matter, the quantities $F_{\mu\nu}(x)$ describing the electromagnetic field, of course, already being gauge- and path-independent.

It turns out that the same quantities used to describe the matter in terms of currents are also suitable for our present purpose, so that we may now deal with the coordinates $j_\mu(x)$, $S(x)$, $\dot{S}(x)$, and $F_{\mu\nu}(x)$.

The definition of the currents in terms of the underlying fields is different in that the gauge-invariant currents are obtained by replacing $\partial/\partial x_\mu$ by $[\partial/\partial x_\mu+ie_0A_\mu(x)]$. Thus we have[34]

$$j_\mu(x)=ie_0[\varphi^*(x)(\partial\varphi(x)/\partial x_\mu+ie_0A_\mu(x)\varphi(x))-(\partial\varphi^*/\partial x_\mu-ie_0A_\mu(x)\varphi^*(x))\varphi(x)] \tag{3.8}$$

and

$$\dot{S}(x)=[\varphi^*(x)(\partial\varphi(x)/\partial x_\mu+ie_0A_\mu(x)\varphi(x))+(\partial\varphi^*(x)/\partial x_\mu-ie_0A_\mu(x)\varphi^*(x))\varphi(x)], \tag{3.9}$$

while, as before, $S(x)=\varphi^*(x)\varphi(x)$.

The equal-time algebra involving $j_\mu(x)$, $S(x)$, and $\dot{S}(x)$, Eqs. (2.8) and (2.10), is, however, unchanged, as can be verified by explicit computation.[29–31] We must add to this algebra the equal-time commutation relations of the components of the electromagnetic field with themselves, and with the currents. These are

$$[F_{ij}(\mathbf{x}),F_{kl}(\mathbf{y})]=[F_{0i}(\mathbf{x}),F_{0j}(\mathbf{y})]=0, \tag{3.10a}$$

$$[F_{0i}(\mathbf{x}),F_{jk}(\mathbf{y})]=-i\{\delta_{ij}\partial/\partial y_k-\delta_{ik}\partial/\partial y_j\}\delta(\mathbf{x}-\mathbf{y}), \tag{3.10b}$$

and

$$[S(\mathbf{x}),F_{ij}(\mathbf{y})]=[\dot{S}(\mathbf{x}),F_{ij}(\mathbf{y})]=[\rho(\mathbf{x}),F_{ij}(\mathbf{y})]=[j_k(\mathbf{x}),F_{ij}(\mathbf{y})]=0, \tag{3.11a}$$

$$[S(\mathbf{x}),F_{0i}(\mathbf{y})]=[\dot{S}(\mathbf{x}),F_{0i}(\mathbf{y})]=[\rho(\mathbf{x}),F_{0i}(\mathbf{y})]=0, \tag{3.11b}$$

$$[j_k(\mathbf{x}),F_{0i}(\mathbf{y})]=-2ie_0S(\mathbf{x})\delta_{ki}\delta(\mathbf{x}-\mathbf{y}). \tag{3.11c}$$

We have retained e_0 in the right side of Eq. (3.11c) to remind ourselves that this term would not be present if there were no interaction with the electromagnetic field.

The energy-momentum tensor can be expressed in terms of the "coordinates" introduced above, as before, by introducing

$$\mathcal{J}_\mu(x)=\partial_\mu S-ij_\mu=2\varphi^*(x)(\partial\varphi(x)/\partial x_\mu+ie_0A_\mu(x)\varphi(x))$$

and

$$\mathcal{J}_\mu{}^\dagger(x)=\partial_\mu S+ij_\mu=2(\partial\varphi^*(x)/\partial x_\mu-ie_0A_\mu(x)\varphi^*(x))\varphi(x). \tag{3.12}$$

Then we find

$$\theta_{\mu\nu}(x)=\tfrac{1}{4}\mathcal{J}_\mu{}^\dagger\frac{1}{S}\mathcal{J}_\nu+\tfrac{1}{4}\mathcal{J}_\nu{}^\dagger\frac{1}{S}\mathcal{J}_\mu-\tfrac{1}{2}[F_{\mu\alpha}F_\nu{}^\alpha+F_{\nu\alpha}F_\mu{}^\alpha]-g_{\mu\nu}\left[\tfrac{1}{4}\mathcal{J}_\alpha{}^\dagger\frac{1}{S}\mathcal{J}_\alpha-\mu^2S-\tfrac{1}{4}F_{\alpha\beta}F^{\alpha\beta}\right]. \tag{3.13}$$

It is amusing to note that $\theta_{\mu\nu}$ looks, formally, like the sum of an energy-momentum tensor for a free charged scalar field plus that for a free electromagnetic field. That $\theta_{\mu\nu}$ above does not involve the interaction is, however, illusory. The effect of the interaction comes in when one solves the equal-time current algebra, in particular through the commutator of $j_k(x)$ and $F_{0i}(x)$ which is zero if there is no interaction and is given by Eq. (3.11c) if there is an electromagnetic interaction.

One way to see this is to try to introduce a functional representation which solves the current algebra, as in Sec. IIF. This is easily done, but it is clear that in order

[29] The fields A_μ and $\dot{A}_\mu$ are not really a canonically conjugate set and, as is well known, the following set of commutators, Eq. (3.5), have some paradoxical implications if taken literally. For this reason, it should be mentioned that all the results of this section remain unchanged if, instead of proceding as here with the Lorentz gauge, we (a) introduce the radiation gauge and then carry out the calculation, or (b) first write the field theory in a gauge-invariant but path-dependent way [following Mandelstam (Ref. 30)], obtaining the commutation relations using the Peierls method (Ref. 31), and then introduce the currents and their commutation relations.

[30] S. Mandelstam, Ann. Phys. (N. Y.) **19**, 1 (1962).

[31] R. Peierls, Proc. Roy. Soc. (London) **213**, 143 (1952).

[32] G. Wentzel (Ref. 8).

[33] This is a case where the Hamiltonian density $\mathcal{H}(x)$, if used, does not coincide with $\theta_{00}(x)$. However, it remains true that $\int\mathcal{H}(x)d^3x=\int\theta_{00}(x)d^3x$.

[34] In this equation, e_0 is the bare electric charge. It occurs in two places; once as a multiple of the 4-vector potential $A_\mu(x)$, and again as an over-all factor in the expression for the current. In the latter case, we will replace it by unity, as before (see Sec. IIA). We will, however, continue to display e_0 whenever it appears in its dynamical role as a coefficient of $A_\mu(x)$.

to satisfy Eq. (3.11c), $j_k(x)$ and $F_{0i}(x)$ must have some pieces, at least, which are composed of a common set of canonically commuting variables. Otherwise they would commute. When $\theta_{\mu\nu}(x)$ is written out in terms of a functional representation having this property, the interaction term reappears in an explicit form.

To conclude, we may briefly discuss the equations of motion. The equations obtained by commuting ρ, S, and $\dot{S}$ with $\theta_{00}(x)$ remain unchanged in form, while that obtained by commuting $j_k(x)$ with $\theta_{00}(x)$ picks up an additional term from the commutator of $j_k(x)$ with $F_{0i}F^{0i}$. It becomes

$$\frac{\partial j_k(x)}{\partial t}=\tfrac{1}{2}i\left[\left(\mathcal{J}_0{}^\dagger\frac{1}{S}\mathcal{J}_k\right)-\left(\mathcal{J}_0{}^\dagger\frac{1}{S}\mathcal{J}_k\right)^\dagger\right]+\frac{\partial\rho(x)}{\partial x_k}-2e_0S(x)F_{0k}(x). \quad (3.14)$$

It remains only to write the equations governing the electromagnetic field. The Maxwell equations divide into two groups: a group of six equations involving the time derivatives $\dot{F}_{0i}$ and $\dot{F}_{kl}$; and two equations involving just space derivatives, $\nabla\cdot\mathbf{E}$ and $\nabla\cdot\mathbf{B}$.

It is not difficult to verify that the six equations of motion follow directly by commutation of F_{0i} and F_{kl} with $\theta_{00}(x)$, as given by Eq. (3.13). On the other hand, the remaining two Maxwell equations are obtained by commuting F_{0i} and F_{ij} with θ_{0k}. The resulting set of eight equations may be summarized as usual in the form

$$\begin{aligned}\epsilon_{\alpha\beta\mu\nu}\partial F_{\mu\nu}/\partial x_\beta&=0\,,\\ \partial F^{\mu\nu}/\partial x_\nu&=e_0 j^\mu\,.\end{aligned} \quad (3.15)$$

It should be emphasized that the above theory is on a different footing from that described in Sec. II insofar as the matter and the radiation field are treated differently; we permit ourselves to describe the radiation by a quantized field, but the matter is described by currents. We do not see that there is necessarily any inconsistency in this procedure. It does, however, complicate the problem with respect to renormalization. What quantities are to be renormalized, and how, is now a more subtle question, the key to which lies in the relationship of the bare charge e_0, introduced above, to the physical charge. We have not yet resolved this point.

IV. SUMMARY AND CONCLUSION

We would like to comment here on some of the foregoing results.

(1) We have now seen how a relativistic model of bosons can be formulated as a set of relations between hadron currents and the energy-momentum tensor, supplementing what we learned about the quark model in I. Evidently, one could generalize what we have done in a straightforward way to include other kinds of theories, if desired.

(2) It has again turned out that $\theta_{\mu\nu}(x)$ can be defined by its equal-time commutators. Here we have found (Sec. IIE and Appendix B) that $\theta_{\mu\nu}(x)$ is completely determined if the particular form of all its equal-time commutation relations is specified. In I, we saw in the quark model that $\theta_{\mu\nu}(x)$ was determined up to a manifestly covariant Lorentz scalar (an "interaction" term) by just those properties of its equal-time commutators which express relativistic invariance requirements.

(3) The introduction of functional representations brings theories formulated in terms of currents closer to a concrete, numerical form. Similar functional representations can presumably also be employed in other cases, such as the quark model treated in I, where the resulting equations are, of course, of much greater potential interest.

(4) We suspect that most of the conclusions we have formed on the basis of the charged scalar model will turn out to hold quite generally for theories formulated in terms of currents. However, the model undoubtedly represents an oversimplification of the real state of affairs in that it has been so easy to maintain manifest covariance here, for example, in the expression for $\theta_{\mu\nu}$ as a function of the currents. This is a consequence of the simple spin structure of the theory, which results in correspondingly simple relativistic transformation properties.

(5) We have seen that electromagnetism can be incorporated into strong interaction theories of this type, by adding as additional "coordinates" the electromagnetic field $F_{\mu\nu}(x)$. We see no logical inconsistency in this procedure in the electromagnetic case.

But suppose the strong interactions are mediated by a basic Yukawa interaction of some kind? Our treatment of electromagnetism indicates how such an interaction could be included in the theory, but in this case the logic involved appears more questionable. An effective Yukawa interaction could, however, emerge in a theory which involves just the currents. Here one would think of using the partially conserved axial-vector current hypothesis (PCAC) to define a field with the quantum numbers of the pion as the divergence of an axial-vector current.

At this juncture, let us see if we can abstract from the models studied in these papers a tentative set of working hypotheses. It would appear that we can suppose:

(i) that the weak and electromagnetic hadron currents, the hadron energy-momentum tensor $\theta_{\mu\nu}(x)$, and operators defined in terms of these by equal-time commutation are quantities which are measurable in principle;

(ii) that the above operators include a complete set of observables for the hadrons; and

(iii) that a representation of the equal-time algebra of the currents and $\theta_{\mu\nu}$, which satisfies the conditions of Lorentz invariance and positivity of the energy spectrum, determines all the matrix elements of these operators, including their time dependence.

Some strong points of this approach have been mentioned in I. Here, we wish to add the following:

(a) As commented in Sec. I, it is desirable in any theory dealing with complicated systems to have a formulation which incorporates as much as possible that is true at the outset. This approach seems admirable from this point of view in that strong-interaction symmetries and sum rules are expressed rather directly through the equal-time current algebra, which is the starting point of this formulation. Also, theories built on currents are consistent with the idea of having no elementary strongly interacting particles.

(b) In the present formulation, simple fundamental interactions, for example, those of the current x current form, can be introduced in a simple way. This is not always the case with theories based on observables. For example, the S-matrix theory shares property (a) with theories based on currents, but not (b).

(c) In discussing the quark model in terms of currents, one comes fairly close to writing a theory of strong interactions that is based on observables, which one could imagine as being complete, and that is described by definite dynamical laws.[35] It is probably worthwhile to be able to look at strong interactions within such a concrete dynamical framework, even if one does not entertain great hopes of making it mathematically consistent, let alone solving the dynamics, overnight.

To counterbalance such optimism, one can well ask: Is it likely that this formulation actually circumvents any of the really central difficulties which confront one in strong-interaction dynamics? Hardly. It has not been our experience that any of the many reformulations of the problem of strong-interaction dynamics have "swept all before them." However, the different approaches have sometimes led to new insights which have proven of value. Perhaps one can hope as much for the approach discussed in these papers.

ACKNOWLEDGMENTS

It is a pleasure for the author to thank Professor R. F. Dashen for many stimulating and clarifying conversations on the subject of this paper. He would also like to thank Professor S. L. Adler, Professor R. F. Dashen, and Professor A. M. Jaffe for numerous helpful comments on the manuscript. Finally, he would like to acknowledge several lively and interesting conversations on these topics with Professor R. P. Feynman and Professor M. Gell-Mann.

APPENDIX A: IRREDUCIBILITY OF THE EQUAL-TIME CURRENT ALGEBRA

Suppose there is an operator[21] $\hat{A}(x)$, which commutes at a given time with $\hat{\jmath}_\mu(x)$, $\hat{S}(x)$, and $\partial\hat{S}(x)/\partial t$;

$$[\hat{A}(\mathbf{x}),\hat{S}(\mathbf{y})]=0\,, \tag{A1a}$$

$$[\hat{A}(\mathbf{x}),\partial\hat{S}(\mathbf{y})/\partial t]=0\,, \tag{A1b}$$

$$[\hat{A}(\mathbf{x}),\hat{\rho}(\mathbf{y})]=0\,, \tag{A1c}$$

$$[\hat{A}(\mathbf{x}),\hat{\jmath}_k(\mathbf{y})]=0\,, \tag{A1d}$$

but which is otherwise unspecified. Here we shall sketch a heuristic argument to show that $\hat{A}(\mathbf{x})$ is a function only of the total charge, $\hat{Q}=\int\hat{\rho}(\mathbf{x})d^3x$. Since the charge operator is a multiple of the identity acting on any given charge sector, this result means that $\hat{\jmath}_\mu(x)$, $\hat{S}(x)$, and $\partial\hat{S}(x)/\partial t$ form an irreducible set of operators, at a given time, when acting on a space of states all having the same total charge.

For our present purposes, we will assume that the fields $\hat{\varphi}(x)$, $\hat{\varphi}^*(x)$, $\hat{\pi}(x)$, and $\hat{\pi}^*(x)$ are irreducible at a given time; that the operators $\hat{\jmath}_\mu(x)$, $\hat{S}(x)$, and $\partial\hat{S}(x)/\partial t$ are defined in terms of the fields by Eqs. (2.6), (2.8d), and (2.9); and that they satisfy the equal-time current algebra, Eqs. (2.8) and (2.10).

The plan of the argument is as follows.[36] Each state which is a simultaneous eigenstate of $\hat{\varphi}(\mathbf{x})$ and $\hat{\varphi}^*(\mathbf{x})$ is also an eigenstate of $\hat{S}(\mathbf{x})$. (Not conversely.) We first find the set of eigenstates of $\hat{\varphi}(\mathbf{x})$ and $\hat{\varphi}^*(\mathbf{x})$ corresponding to a fixed eigenvalue[37] $S(\mathbf{x})$ of $\hat{S}(\mathbf{x})$. These span a certain subspace $\mathcal{H}_S$ of the space of eigenstates of $\hat{\varphi}(x)$ and $\hat{\varphi}^*(x)$. Since $\hat{A}(\mathbf{x})$ commutes with $\hat{S}(\mathbf{x})$, by hypothesis, it maps $\mathcal{H}_S$ into itself. The operator $\hat{A}(\mathbf{x})$ may then be represented, in general, by giving its matrix elements first with respect to each pair of independent vectors in $\mathcal{H}_S$ corresponding to a fixed eigenvalue $S(\mathbf{x})$, and then taking the direct integral over each distinct eigenvalue $S(\mathbf{x})$ of $\hat{S}(\mathbf{x})$. [Naturally the matrix elements of $\hat{A}(\mathbf{x})$ with respect to the independent eigenvectors in $\mathcal{H}_S$ will depend on $S(\mathbf{x})$.] With $\hat{A}(\mathbf{x})$ represented in this form, the effect of the restrictions imposed by Eqs. (A1b)–(A1d) can be seen rather directly, and it is found that $\hat{A}$ is a function only of the charge operator $\hat{Q}$.

We first use the fact that $\hat{S}(\mathbf{x})$, $\hat{\varphi}(\mathbf{x})$, and $\hat{\varphi}^*(\mathbf{x})$ commute among themselves at equal times. This implies, first, that $\hat{\varphi}(\mathbf{x})$ and $\hat{\varphi}^*(\mathbf{x})$ may be simultaneously diagonalized, and secondly, that their eigenvalues are complex conjugates of one another. Moreover, each

[35] Aside from Lagrangian field theories, this is the only way we know of to write, in explicit form, a complete dynamical theory.

[36] This argument follows the same logical pattern as that employed in the preceding paper. We are simply trying to fill in here some of the details which were passed over in Ref. 1.

[37] It may be helpful at this point to refer to the material on functional representations in Sec. IIF, or to the books on this subject listed in Ref. 19.

eigenstate of $\hat{\varphi}(\mathbf{x})$ and $\hat{\varphi}^*(\mathbf{x})$ is an eigenstate of $\hat{S}(\mathbf{x})$ with eigenvalue $\varphi^*(\mathbf{x})\varphi(\mathbf{x})$.

Next, we characterize all those eigenstates of $\hat{\varphi}(\mathbf{x})$ and $\hat{\varphi}^*(\mathbf{x})$ corresponding to a fixed eigenvalue $S(\mathbf{x})$ of $\hat{S}(\mathbf{x})$. We have, at each point $\mathbf{x}$, and for fixed eigenstates $|\varphi,\varphi^*\rangle$ of $\hat{\varphi}(\mathbf{x})$, $\hat{\varphi}^*(\mathbf{x})$,

$$\hat{\varphi}(\mathbf{x})|\varphi,\varphi^*\rangle=\varphi(\mathbf{x})|\varphi,\varphi^*\rangle\,, \tag{A2a}$$

$$\hat{\varphi}^*(\mathbf{x})|\varphi,\varphi^*\rangle=\varphi^*(\mathbf{x})|\varphi,\varphi^*\rangle\,, \tag{A2b}$$

and

$$\hat{S}(\mathbf{x})|\varphi,\varphi^*\rangle=\varphi^*(\mathbf{x})\varphi(\mathbf{x})|\varphi,\varphi^*\rangle\,. \tag{A2c}$$

Hence, what we need are all eigenstates of $\hat{\varphi}(\mathbf{x})$ whose eigenvalues have a fixed value for their square modulus; that is, all eigenstates of $\hat{\varphi}(\mathbf{x})$ whose eigenvalues differ by a (generally space-dependent) phase factor of modulus unity.

These can be generated as follows. From the commutation relation

$$[\hat{\rho}(\mathbf{x}),\hat{\varphi}(\mathbf{y})]=-\hat{\varphi}(\mathbf{y})\delta(\mathbf{x}-\mathbf{y}) \tag{A3}$$

one finds

$$[\hat{\rho}(\lambda),\hat{\varphi}(\mathbf{y})]=-\lambda(\mathbf{y})\hat{\varphi}(\mathbf{y})\,, \tag{A4}$$

with

$$\hat{\rho}(\lambda)=\int\lambda(\mathbf{x})\hat{\rho}(\mathbf{x})d^3x\,. \tag{A5}$$

Here λ is an arbitrary real function of $\mathbf{x}$. From Eqs. (A4) and (A5) it follows that

$$\exp[i\hat{\rho}(\lambda)]\hat{\varphi}(\mathbf{y})\exp[-i\hat{\rho}(\lambda)]=\exp[-i\lambda(\mathbf{y})]\hat{\varphi}(\mathbf{y})\,. \tag{A6}$$

Applied to an eigenstate of $\hat{\varphi}(\mathbf{x})$, the relationship (A.6) gives

$$\hat{\varphi}(\mathbf{x})\left[\exp\left(-i\int\lambda(\mathbf{y})\hat{\rho}(\mathbf{y})d^3y\right)|\varphi,\varphi^*\rangle\right]$$
$$=e^{-i\lambda(\mathbf{x})}\varphi(\mathbf{x})\left[\exp\left(-i\int\lambda(\mathbf{y})\hat{\rho}(\mathbf{y})d^3y\right)|\varphi,\varphi^*\rangle\right]\,. \tag{A7}$$

The state $\exp[-i\int\lambda(\mathbf{y})\hat{\rho}(\mathbf{y})d^3y]|\varphi,\varphi^*\rangle$ is therefore also an eigenstate of $\hat{\varphi}(\mathbf{x})$ with the eigenvalue changed at each point $\mathbf{x}$ by $\exp[-i\lambda(\mathbf{x})]$.

The set of states $\exp[-i\hat{\rho}(\lambda)]|\varphi,\varphi^*\rangle$, associated with a fixed set of eigenvalues φ, φ^*, clearly span the space $\mathcal{H}_S$ of eigenstates of $\hat{S}(\mathbf{x})$ having a fixed eigenvalue $S(x)$. That is, an arbitrary eigenstate of $\hat{S}(\mathbf{x})$, corresponding to the above eigenvalue $S(\mathbf{x})$, can be written as a linear superposition of eigenstates $|\varphi,\varphi^*\rangle$ of $\hat{\varphi}(\mathbf{x})$ and $\hat{\varphi}^*(\mathbf{x})$ with arbitrary coefficients $g(\lambda,\varphi,\varphi^*)$ as follows:

$$|S\rangle=\int\exp[-i\hat{\rho}(\lambda)]|\varphi,\varphi^*\rangle$$
$$\times g(\lambda(\mathbf{y}),\varphi(\mathbf{y}),\varphi^*(\mathbf{y}))\mathfrak{D}\lambda(\mathbf{y})\,, \tag{A8}$$

where $\mathfrak{D}\lambda(\mathbf{y})$ represents a functional integral over $\lambda(\mathbf{y})$.

The operator $\hat{A}(\mathbf{x})$ commutes with $\hat{S}(\mathbf{x})$, and it therefore carries $\mathcal{H}_S$ into itself. It can be specified by giving for each $S(\mathbf{x})$ its matrix elements between every pair of eigenstates belonging to a set which spans $\mathcal{H}_S$. Therefore, we can write

$$\hat{A}(\mathbf{x})=\int\exp[-i\hat{\rho}(\lambda)]|\varphi(\mathbf{x}),\varphi^*(\mathbf{x})\rangle\langle\varphi(\mathbf{x}),\varphi^*(\mathbf{x})|$$
$$\times\exp[i\hat{\rho}(\lambda')]F(\lambda,\lambda',\varphi,\varphi^*)\mathfrak{D}\lambda\mathfrak{D}\lambda'\mathfrak{D}\varphi\mathfrak{D}\varphi^*\,, \tag{A9}$$

where $\mathfrak{D}\varphi$ and $\mathfrak{D}\varphi^*$ represent integrals over the eigenfunctions $\varphi(\mathbf{x})$ and $\varphi^*(\mathbf{x})$.

The above representation is deficient in that one can not integrate over all $\varphi(\mathbf{x})$ and $\varphi^*(\mathbf{x})$ without counting some states more than once. We therefore label states by $|\varphi(S),\varphi^*(S)\rangle$; corresponding to a fixed $S(\mathbf{x})$ we choose $\varphi(\mathbf{x})$ and $\varphi^*(\mathbf{x})$ to be some definite function of S. (The choice is, of course, not unique, but it is only necessary to be able to do it in some way.) All the other states that go with this value of $S(\mathbf{x})$ are generated by the operation

$$\exp[-i\hat{\rho}(\lambda)]|\varphi(S),\varphi^*(S)\rangle\,. \tag{A10}$$

With this modification we can then integrate over all $S(\mathbf{x})$ in Eq. (A9). Thus we can write

$$\hat{A}(\mathbf{x})=\int\exp[-i\hat{\rho}(\lambda)]|\varphi(S),\varphi^*(S)\rangle\langle\varphi(S),\varphi^*(S)|$$
$$\times\exp[i\hat{\rho}(\lambda')]F(\lambda,\lambda',\varphi(S),\varphi^*(S))\mathfrak{D}\lambda\mathfrak{D}\lambda'\mathfrak{D}S \tag{A11}$$

$$=\int\mathfrak{D}\lambda\mathfrak{D}\lambda'\exp[-i\hat{\rho}(\lambda)]A(\lambda,\lambda',\hat{\varphi}(\mathbf{x}),\hat{\varphi}^*(\mathbf{x}))$$
$$\times\exp[+i\hat{\rho}(\lambda')]\,, \tag{A12}$$

where $A(\lambda,\lambda',\hat{\varphi}(\mathbf{x}),\hat{\varphi}^*(\mathbf{x}))$ depends on the functions $\lambda(\mathbf{y})$ and $\lambda'(\mathbf{y})$ and the operators $\hat{\varphi}(\mathbf{x}),\hat{\varphi}^*(\mathbf{x})$.

To further restrict $\hat{A}(\mathbf{x})$, we must use the fact that it commutes with $\hat{\rho}$, $\partial\hat{S}/\partial t$, and $\hat{\jmath}_k$. Since

$$[\hat{\varphi}(\mathbf{x}),\hat{\rho}(\mathbf{y})]=\hat{\varphi}(\mathbf{x})\delta(\mathbf{x}-\mathbf{y})\,, \tag{A13a}$$

$$[\hat{\varphi}(\mathbf{x}),\partial\hat{S}(\mathbf{y})/\partial t]=i\hat{\varphi}(\mathbf{x})\delta(\mathbf{x}-\mathbf{y})\,, \tag{A13b}$$

$$[\hat{\varphi}^*(\mathbf{x}),\hat{\rho}(\mathbf{y})]=-\hat{\varphi}^*(\mathbf{x})\delta(\mathbf{x}-\mathbf{y})\,, \tag{A13c}$$

$$[\hat{\varphi}^*(\mathbf{x}),\partial\hat{S}(\mathbf{y})/\partial t]=i\hat{\varphi}^*(\mathbf{x})\delta(\mathbf{x}-\mathbf{y})\,, \tag{A13d}$$

it follows that

$$[A(\hat{\varphi}(\mathbf{x})),\hat{\rho}(\mathbf{y})]=\varphi(\mathbf{y})\frac{\delta A(\varphi(\mathbf{x}))}{\delta\varphi(\mathbf{y})}\,.$$

Consequently, commuting $\hat{A}(\mathbf{x})$ with $\partial\hat{S}(\mathbf{y})/\partial t+i\hat{\rho}(\mathbf{y})$ and $\partial\hat{S}(\mathbf{y})/\partial t-i\hat{\rho}(\mathbf{y})$, we find that Eqs. (A1b) and (A1c) imply

$$\int\mathfrak{D}\lambda\mathfrak{D}\lambda'\exp[-i\hat{\rho}(\lambda)]\varphi(\mathbf{y})\frac{\delta A(\lambda,\lambda',\varphi(\mathbf{x}),\varphi^*(\mathbf{x}))}{\delta\varphi(\mathbf{y})}$$
$$\times\exp[i\hat{\rho}(\lambda')]=0 \tag{A14a}$$

and

$$\int \mathfrak{D}\lambda\mathfrak{D}\lambda' \exp[-i\hat{\rho}(\lambda)]\varphi^*(\mathbf{y})\frac{\delta A(\lambda,\lambda',\varphi(\mathbf{x}),\varphi^*(\mathbf{x}))}{\delta\varphi^*(\mathbf{y})} \times\exp[i\hat{\rho}(\lambda')]=0, \quad \text{(A14b)}$$

since $\hat{\rho}(\lambda)$ commutes with $\hat{\rho}(\mathbf{y})$ and $\partial\hat{S}(\mathbf{y})/\partial t$.

To analyze this equation we take its matrix element between eigenstates of $\hat{\rho}(\mathbf{x})$, finding:

$$\int \mathfrak{D}\lambda\mathfrak{D}\lambda' \exp\left(-i\int\lambda(\mathbf{y})\rho(\mathbf{y})d^3y\right) \times\varphi(\mathbf{x})\frac{\delta A(\lambda,\lambda',\varphi(\mathbf{x}),\varphi^*(\mathbf{x}))}{\delta\varphi(\mathbf{x})} \times\exp\left(+i\int\lambda'(\mathbf{z})\rho(\mathbf{z})d^3z\right)=0, \quad \text{(A15a)}$$

with a similar result for Eq. (A14b). Since these integrals must now vanish for arbitrary $\rho(\mathbf{y})$, it is probably safe to conclude that the integrands themselves must vanish. Thus we must have

$$\varphi(\mathbf{x})\delta A(\varphi,\varphi^*)/\delta\varphi(\mathbf{x})=0 \quad \text{(A15b)}$$

and

$$\varphi^*(\mathbf{x})\delta A(\varphi,\varphi^*)/\delta\varphi^*(\mathbf{x})=0. \quad \text{(A15c)}$$

The precise character of the solutions of such equations is a rather subtle matter. Roughly, they imply that $A(\varphi,\varphi^*)$ is functionally independent of φ and φ^* except for functionals that are discontinuous at a zero eigenvalue of $\hat{\varphi}(\mathbf{x})$ or $\hat{\varphi}^*(\mathbf{x})$. For the time being, we shall disregard the latter possibility.[38] Then we have

$$\hat{A}(\mathbf{x})=\int\mathfrak{D}\lambda\mathfrak{D}\lambda' A(\lambda,\lambda')\exp[i\hat{\rho}(\lambda'-\lambda)] =\int\mathfrak{D}\mu\mathfrak{D}\nu A(\mu,\nu)\exp[i\hat{\rho}(\mu)], \quad \text{(A16)}$$

with $\mu=\lambda-\lambda'$ and $\nu=\lambda+\lambda'$.

It remains to impose Eq. (A1d). From Eqs. (2.8b) and (A5) it is clear that

$$[\hat{\rho}(\mu),\hat{\jmath}_k(\mathbf{y})]=-2i[\partial\mu(\mathbf{y})/\partial y_k]\hat{S}(\mathbf{y}). \quad \text{(A17)}$$

Since $\hat{\rho}(\mathbf{x})$ and $\hat{S}(\mathbf{x})$ commute, it is also clear that

$$[\exp(i\hat{\rho}(\mu)),\hat{\jmath}_k(\mathbf{y})]=2[\partial\mu(\mathbf{y})/\partial y_k]\hat{S}(\mathbf{y}). \quad \text{(A18)}$$

Therefore the hypotheses that $\hat{A}$ commutes with $\hat{\jmath}_k(\mathbf{y})$, Eq. (A1d), implies that

$$2\int d\mu d\nu A(\mu,\nu)\frac{\partial\mu(\mathbf{y})}{\partial y_k}\hat{S}(\mathbf{y})=0. \quad \text{(A19)}$$

Since this equation holds for each point $\mathbf{y}$, it can in general be satisfied only if $\partial\mu(\mathbf{y})/\partial y_k=0$. Equation (A16) then states that $\hat{A}=F(\hat{Q})$. Acting on a set of states all having some fixed total charge, $\hat{Q}$ is a multiple of the identity; hence the operators $\hat{\jmath}_\mu(x)$, $\hat{S}(x)$, and $\partial\hat{S}(x)/\partial t$ are irreducible at a fixed time in a given charge sector of the charged scalar theory.

APPENDIX B: EQUAL-TIME COMMUTATION RELATIONS OF THE ENERGY-MOMENTUM TENSOR

In this Appendix we first list the equal-time commutation relations between the components of $\theta_{\mu\nu}(x)$, and between $\theta_{\mu\nu}(x)$ and the operators $j_\mu(x)$, $S(x)$, and $\dot{S}(x)$. In the second part of the Appendix, we suppose that there are two expressions for $\theta_{\mu\nu}(x)$ which are both compatible with this set of commutation relations, and show that the two expressions can differ only by a multiple of the identity. The relevance of this result to the question of how the equal-time commutation relations of $\theta_{\mu\nu}(x)$ carry dynamical information is discussed in Sec. IIE.

The commutation relations between $\theta_{\mu\nu}(x)$ and the currents are

$$\text{(a)}\quad i[\theta_{00}(\mathbf{x}),\rho(\mathbf{y})]=\partial[j_k(\mathbf{y})\delta(\mathbf{x}-\mathbf{y})]/\partial y_k, \quad \text{(B1a)}$$

$$i[\theta_{00}(\mathbf{x}),\mathbf{j}(\mathbf{y})]=[\partial\mathbf{j}(\mathbf{y})/\partial t]\delta(\mathbf{x}-\mathbf{y}), \quad \text{(B1b)}$$

$$i[\theta_{00}(\mathbf{x}),S(\mathbf{y})]=\dot{S}(\mathbf{y})\delta(\mathbf{x}-\mathbf{y}), \quad \text{(B1c)}$$

$$i[\theta_{00}(\mathbf{x}),\dot{S}(\mathbf{y})]=[2\mu^2S(\mathbf{x})-\theta_\mu{}^\mu(\mathbf{x})]\delta(\mathbf{x}-\mathbf{y}) +\partial[\partial_kS(\mathbf{y})\delta(\mathbf{x}-\mathbf{y})]/\partial y_k, \quad \text{(B1d)}$$

---

[38] A similar situation arises if one studies the following problem in nonrelativistic quantum mechanics in one dimension. Suppose an operator $\hat{A}$ satisfies $[\hat{A},\hat{x}^2]=0$, $[\hat{A},\hat{x}\hat{p}+\hat{p}\hat{x}]=0$, with $[\hat{x},\hat{x}]=[\hat{p},\hat{p}]=0$; $[\hat{x},\hat{p}]=i$. What is $\hat{A}$, supposing $\hat{x}$ and $\hat{p}$ complete? Although one might think that $\hat{A}$ must be a linear combination of just the identity and parity ($\hat{P}$) operators, this is not, as a matter of fact, correct, as pointed out to me by Professor V. Bargmann. In fact, $\hat{A}$ can also depend on $\hat{\sigma}$, defined by

$$\hat{\sigma}\Psi=\sigma(x)\Psi;\quad \sigma(x)=+1,\ x>0;\ =0,\ x=0;\ =-1,\ x<0$$

and on $\hat{\sigma}\hat{P}$. In one way of solving this problem, the possibility of $\hat{\sigma}$ and $\hat{\sigma}\hat{P}$ as solutions arises precisely from the nonuniqueness of solutions of equations of the same form as those in the text. What this means is that $\hat{x}^2$ and $\hat{x}\hat{p}+\hat{p}\hat{x}$ are irreducible on a fixed parity subspace of Hilbert space, $\hat{\sigma}$ being an operator that takes one from one parity subspace to another.

In the neutral scalar theory, the operator corresponding to $\hat{\sigma}$ is $\hat{\sigma}(\varphi(\mathbf{x}))$—the "field amplitude parity." One might expect that the blineiar operators $\hat{S}=\hat{\varphi}^2$ and $\partial\hat{S}/\partial t=\hat{\varphi}\hat{\pi}+\hat{\pi}\hat{\varphi}$ appropriate to the neutral theory would be irreducible only on a fixed amplitude parity subspace of Hilbert space. This is really no restriction in this case, however, because in a world with just self-interacting spinless bosons there is a superselection rule saying that only pairs of bosons can be produced. Hence one cannot distinguish between scalar and pseudoscalar mesons, which is what $\hat{\sigma}(\varphi(\mathbf{x}))$ allows one to do. So in fact one can forget that $\hat{\sigma}(\varphi)$ is a solution of the algebraic problem, if one imposes the superselection rule, and regard $\hat{S}(\mathbf{x})$ and $\partial\hat{S}/\partial t$ as irreducible on the whole Hilbert space.

In the charged scalar theory, the corresponding "ambiguity" has to do with the phase of a field at a point [namely, the point where the (complex) field amplitude has a zero eigenvalue] instead of just the sign of the field. However, in a self-interacting charged scalar theory one can not determine the absolute phase of a field, so that the question of irreducibility should not be effected by this.

The author would like to take this opportunity to thank Professor V. Bargmann for a very instructive discussion of this problem, as well as Professor G. Tiktopoulos and G. Svetlichney for many helpful comments.

(b)
$$i[\theta_{kl}(\mathbf{x}),\rho(\mathbf{y})]=\partial[j_k(\mathbf{y})\delta(\mathbf{x}-\mathbf{y})]/\partial y_l+\partial[j_l(\mathbf{y})\delta(\mathbf{x}-\mathbf{y})]/\partial y_k-\delta_{kl}i[\theta_{00}(\mathbf{x}),\rho(\mathbf{y})],\quad \text{(B2a)}$$

$$i[\theta_{kl}(\mathbf{x}),\mathbf{j}(\mathbf{y})]=\delta_{kl}i[\theta_{00}(\mathbf{x}),\mathbf{j}(\mathbf{y})],\quad \text{(B2b)}$$

$$i[\theta_{kl}(\mathbf{x}),S(\mathbf{y})]=\delta_{kl}i[\theta_{00}(\mathbf{x}),S(\mathbf{y})],\quad \text{(B2c)}$$

$$i[\theta_{kl}(\mathbf{x}),\dot{S}(\mathbf{y})]=\delta_{kl}i[\theta_{00}(\mathbf{x}),\dot{S}(\mathbf{y})]+2[\delta_{kl}\theta_{00}(\mathbf{x})-\theta_{kl}(\mathbf{x})+\delta_{kl}\lambda S^2(\mathbf{x})]\delta(\mathbf{x}-\mathbf{y})-2[\partial[\partial_l S(\mathbf{y})\delta(\mathbf{x}-\mathbf{y})]/\partial y_k+\partial[\partial_k S(\mathbf{y})\delta(\mathbf{x}-\mathbf{y})]/\partial y_l],\quad \text{(B2d)}$$

(c)
$$i[\theta_{0k}(\mathbf{x}),\rho(\mathbf{y})]=-\rho(\mathbf{x})\partial\delta(\mathbf{x}-\mathbf{y})/\partial x_k,\quad \text{(B3a)}$$

$$i[\theta_{0k}(\mathbf{x}),j_l(\mathbf{y})]=[\partial j_l(\mathbf{x})/\partial x_k]\delta(\mathbf{x}-\mathbf{y})-\partial[j_k(\mathbf{x})\delta(\mathbf{x}-\mathbf{y})]/\partial x_l,\quad \text{(B3b)}$$

$$i[\theta_{0k}(\mathbf{x}),S(\mathbf{y})]=[\partial S(\mathbf{x})/\partial x_k]\delta(\mathbf{x}-\mathbf{y}),\quad \text{(B3c)}$$

$$i[\theta_{0k}(\mathbf{x}),\dot{S}(\mathbf{y})]=-\dot{S}(\mathbf{x})\partial\delta(\mathbf{x}-\mathbf{y})/\partial x_k.\quad \text{(B3d)}$$

Next, we list the commutation relations between the various components of $\theta_{\mu\nu}(x)$. We find[39]

$$i[\theta_{00}(\mathbf{x}),\theta_{00}(\mathbf{y})]=-[\theta_{0k}(\mathbf{x})+\theta_{0k}(\mathbf{y})]\frac{\partial}{\partial x_k}\delta(\mathbf{x}-\mathbf{y}),\quad \text{(B4a)}$$

$$i[\theta_{00}(\mathbf{x}),\theta_{0k}(\mathbf{y})]=-\left\{\theta_{kl}(\mathbf{x})\frac{\partial}{\partial x_l}\delta(\mathbf{x}-\mathbf{y})+\frac{\partial}{\partial x_l}[\theta_{00}(\mathbf{x})\delta_{kl}\delta(\mathbf{x}-\mathbf{y})]\right\},\quad \text{(B4b)}$$

$$i[\theta_{00}(\mathbf{x}),\theta_{kl}(\mathbf{y})]=\dot{\theta}_{kl}(\mathbf{x})\delta(\mathbf{x}-\mathbf{y})-\frac{\partial}{\partial x_l}[\theta_{0k}(\mathbf{x})\delta(\mathbf{x}-\mathbf{y})]-\frac{\partial}{\partial x_k}[\theta_{0l}(\mathbf{x})\delta(\mathbf{x}-\mathbf{y})],\quad \text{(B4c)}$$

$$i[\theta_{0k}(\mathbf{x}),\theta_{0l}(\mathbf{y})]=\theta_{0k}(\mathbf{y})\frac{\partial}{\partial y_l}\delta(\mathbf{x}-\mathbf{y})-\theta_{0l}(\mathbf{x})\frac{\partial}{\partial x_k}\delta(\mathbf{x}-\mathbf{y}).\quad \text{(B4d)}$$

Now let us see how the above commutation relations determine $\theta_{\mu\nu}(x)$. First, note that since $j_\mu(x)$, $S(x)$, and $\dot{S}(x)$ are supposed to form a complete set, any operator can be written in terms of them. We assume that this means that a solution to Eqs. (B1)–(B4) for $\theta_{\mu\nu}(x)$ in terms of $j_\mu(x)$, $S(x)$, and $\dot{S}(x)$ exists.[16] The question is then whether this solution is unique. To answer this we suppose that, corresponding to a single, specific choice for $j_\mu(x)$, $S(x)$, and $\dot{S}(x)$, there are two energy-momentum tensors $\theta_{\mu\nu}{}^1(x)$ and $\theta_{\mu\nu}{}^2(x)$ satisfying the equal-time algebra, Eqs. (B1)–(B4). We then form their difference $\Delta_{\mu\nu}(x)=\theta_{\mu\nu}{}^1(x)-\theta_{\mu\nu}{}^2(x)$ and consider its commutation relations with $j_\mu(x)$, $S(x)$, and $\dot{S}(x)$.

First consider $\Delta_{0k}(x)$. From Eqs. (B3) it is clear that $\Delta_{0k}(x)$ commutes with each of the operators $j_\mu(x)$, $S(x)$, and $\dot{S}(x)$ at equal times. We see from Sec. IIC and Appendix A, therefore, that $\Delta_{0k}(x)$ must be a multiple of the identity on a fixed charge sector of the theory.[40]

The situation for $\Delta_{kl}(x)$, $k\neq l$, is also simple. From Eqs. (B2a)–(B2c) we see that it commutes with $S(x)$, $\rho(x)$, and $j_k(x)$, and it is therefore a function only of $S(x)$. Its commutator with $\dot{S}(x)$ is

$$[\Delta_{kl}(\mathbf{x}),\dot{S}(\mathbf{y})]=2i\Delta_{kl}(\mathbf{x})\delta(\mathbf{x}-\mathbf{y}),\quad k\neq l.\quad \text{(B5)}$$

Since $\Delta_{kl}(\mathbf{x})$ must be made up out of $S(\mathbf{x})$, it must involve terms like δ_{kl} times a Lorentz scalar formed from $S(\mathbf{x})$, or some symmetric space derivatives of $S(\mathbf{x})$, again times a Lorentz scalar made out of $S(\mathbf{x})$. The former possibility is ruled out because $k\neq l$, while terms of the latter type can not satisfy Eq. (B5). Hence Δ_{kl} $(k\neq l)$ must also be a multiple of the identity.

The situation for $\Delta_{00}(x)$ and $\Delta_{kl}(x)$ $(k=l)$ is somewhat more involved, because the commutator of $\theta_{00}(x)$ with $j_k(x)$ is not known,[41] but is rather the new operator $\partial\mathbf{j}(x)/\partial t$. We therefore reason as follows.

First, it is clear that $\Delta_{00}(x)$ and $\Delta_{kk}(x)$ (no summation on k here) both commute with $S(x)$ and $\rho(x)$, so we have

$$\Delta_{00}(x)=\Delta_{00}(S,\rho),\qquad \Delta_{kk}(x)=\Delta_{kk}(S,\rho).\quad \text{(B6)}$$

Next we look at the commutator:

$$[\Delta_{kk}(\mathbf{x}),\mathbf{j}(\mathbf{y})]=\delta_{kk}[\Delta_{00}(\mathbf{x}),\mathbf{j}(\mathbf{y})]\ (k\text{ not summed})\quad \text{(B7)}$$

obtained from Eq. (B2b). We see from Eq. (B7) that the difference between $\Delta_{kk}(x)$ and $\delta_{kk}\Delta_{00}(x)$ commutes with $S(x)$, $\rho(x)$, and $\mathbf{j}(x)$, so it can only be a function of $S(x)$. Thus

$$\delta_{kk}\Delta_{00}(S,\rho)=\Delta_{kk}(S,\rho)+F_{kk}(S).\quad \text{(B8)}$$

The commutators of $\Delta_{00}(x)$ and $\Delta_{kk}(x)$ with $\dot{S}(x)$ are

$$[\Delta_{00}(\mathbf{x}),\dot{S}(\mathbf{y})]=i\Delta_\mu{}^\mu(\mathbf{x})\delta(\mathbf{x}-\mathbf{y}),\quad \text{(B9)}$$

$$[\Delta_{kk}(\mathbf{x}),\dot{S}(\mathbf{y})]=i[\delta_{kk}\Delta_\mu{}^\mu(\mathbf{x})-2\delta_{kk}\Delta_{00}(\mathbf{x})+2\Delta_{kk}(\mathbf{x})]\delta(\mathbf{x}-\mathbf{y}),\quad \text{(B10)}$$

[39] The commutators $[\theta_{0h}(\mathbf{x}),\theta_{kl}(\mathbf{y})]$ and $[\theta_{kl}(\mathbf{x}),\theta_{ij}(\mathbf{y})]$ have been omitted from this list because they are complicated and not very instructive. However, like the commutators listed, they involve no inverse operators and no operator products other than in the interaction term λS^2.

[40] The basic fact that we use here and below is that an operator is determined if its commutator with each of a complete set of operators is known.

[41] Of course, we do know the commutator $[\theta_{00}(\mathbf{x}),\mathbf{j}(\mathbf{y})]$ in terms of $j_\mu(x)$, $S(x)$ and $\dot{S}(x)$; it is essentially given in Eq. (2.19). However, we have found no way to express it in terms of $\theta_{\mu\nu}(x)$ and the currents in a linear fashion, so we must give up this information if we pursue the viewpoint of Sec. IIE.

(sum over μ in the above equations). Using Eq. (B8), these equations take the form

$$[\Delta_{kk}(\mathbf{x})+F_{kk}(\mathbf{x}),\dot{S}(\mathbf{y})] = i[\Delta_{kk}(\mathbf{x})+F_{kk}(\mathbf{x})-\delta_{kk}\Delta(\mathbf{x})]\delta(\mathbf{x}-\mathbf{y}), \quad (B11)$$

$$[\Delta_{kk}(\mathbf{x}),\dot{S}(\mathbf{y})] = i[\Delta_{kk}(\mathbf{x})-F_{kk}(\mathbf{x})-\delta_{kk}\Delta(\mathbf{x})]\delta(\mathbf{x}-\mathbf{y}), \quad (B12)$$

where

$$\Delta(\mathbf{x})=\sum_{l=1}^{3}\Delta_{ll}(\mathbf{x}).$$

Subtracting Eq. (B12) from Eq. (B11), we find $[F_{kk}(\mathbf{x}),\dot{S}(\mathbf{y})]=2iF_{kk}(\mathbf{x})\delta(\mathbf{x}-\mathbf{y})$. Since F_{kk} depends only on $S(x)$, the solution of this equation is $F_{kk}(S)=\alpha_{kk}S(x)$, where the three coefficients α_{kk} must be c numbers. Using this result, and summing Eq. (B12) over k, we find

$$[\Delta(\mathbf{x}),\dot{S}(\mathbf{y})]=-i[\alpha S(\mathbf{x})+2\Delta(\mathbf{x})]\delta(\mathbf{x}-\mathbf{y}), \quad (B13)$$

with $\alpha=\sum_k \alpha_{kk}$. Solving this equation for Δ gives [recall that $\Delta_{kk}=\Delta_{kk}(S,\rho)$]

$$\Delta(x)=\beta(1/S)f(\rho)-\tfrac{1}{4}\alpha S, \quad (B14)$$

where β is a c number.

Next, return to Eq. (B8) and sum over k to find

$$3\Delta_{00}(x)=\sum_k \Delta_{kk}+\sum_k F_{kk} =\beta(1/S)f(\rho)+\tfrac{3}{4}\alpha S. \quad (B15)$$

We have now essentially exhausted the commutation relations of $\theta_{00}(x)$ and θ_{kk} with the currents. To further restrict $\Delta_{00}(x)$ and $\Delta_{kk}(x)$, we must make use of Lorentz invariance.

A simple way to do this is to argue as follows. Since $\Delta_{00}(x)$ is the (0,0) component of a Lorentz tensor, and $\rho(x)$ is the time component of a Lorentz 4-vector, one must in fact have in Eq. (B15) that $f(\rho)\propto\rho^2+\gamma g_{00}$ and $\alpha\propto g_{00}$. Thus Eqs. (B15) and (B14) must have the form

$$\Delta(x)=\beta'\frac{1}{S}\rho^2+\gamma' g_{00}\frac{1}{S}-\tfrac{1}{4}\alpha' g_{00}S, \quad (B14')$$

$$3\Delta_{00}(x)=\beta'\frac{1}{S}\rho^2+\gamma' g_{00}\frac{1}{S}+\tfrac{3}{4}\alpha' g_{00}S. \quad (B15')$$

This, however, is impossible. The quantity $\Delta_\mu{}^\mu(x)=\Delta_{00}(x)-\Delta(x)$ must be a Lorentz scalar, while from Eqs. (3.14′) and (3.15′) it is clear that each term in $\Delta_{00}(x)$, $\Delta(x)$ and in their difference transforms like the (0,0) component of a Lorentz tensor. Hence we must have $\alpha'=\beta'=\gamma'=0$, which implies that $\Delta_{00}(x)$ and $\Delta(x)$ must either vanish or be multiples of the identity.

This leaves us with Δ_{kk}, which from Eq. (B8) and the above results must have the form $\Delta_{kk}(x)=-\alpha_{kk}S(x)$. However, the only way one can add a term proportional to $S(x)$ to $\Delta_{kl}(x)$ in such a way that $\Delta_{kl}(x)$ maintains its correct Lorentz transformation properties is to have α_{kk} proportional to δ_{kk}. But we cannot choose $\alpha_{kk}\propto\delta_{kk}$, for then $\sum_k \alpha_{kk}=\alpha\propto\alpha'\neq 0$, in contradiction to what we found above.

Alternatively, we may argue directly from Eqs. (B4a) and (B4b). First, Eq. (B4a), integrated over y, implies that

$$\dot{\theta}_{00}(x)+\theta_{0k,k}(x)=0 \quad (k \text{ summed}). \quad (B16)$$

Therefore the difference of two solutions $\Delta_{00}(x)=\theta_{00}{}^1(x)-\theta_{00}{}^2(x)$ satisfies

$$\dot{\Delta}_{00}(x)=0. \quad (B17)$$

For $\Delta_{00}(x)$ of the form (B15), this equation implies

$$\tfrac{1}{3}\beta\frac{\partial}{\partial t}\left(\frac{1}{S}f(\rho)\right)+\tfrac{3}{4}\alpha\frac{\partial S}{\partial t}=0$$

or

$$\tfrac{1}{3}\beta\left[\frac{1}{S}g(\nabla\cdot j)+f(\rho)\frac{\partial}{\partial t}\left(\frac{1}{S}\right)\right]+\tfrac{3}{4}\alpha\dot{S}=0. \quad (B18)$$

Since we are considering the possibility of different solutions $\theta_{\mu\nu}{}^1(x)$, $\theta_{\mu\nu}{}^2(x)$ corresponding to an arbitrary but fixed choice of $j_\mu(x)$, $S(x)$, and $\dot{S}(x)$, we are not free to adjust these operators so as to satisfy the above equation. Thus we must have $\alpha=\beta=0$, as before. Finally, we use Eq. (B4b), recalling that we already know that $\Delta_{0k}(x)$, $\Delta_{kl}(k\neq l)$ and $\Delta_{00}(x)$ either vanish or are multiples of the identity. We can then write

$$[\Delta_{00}(\mathbf{x}),\Delta_{0k}(\mathbf{y})]=i\left\{\Delta_{kl}(\mathbf{x})\frac{\partial}{\partial x^l}\delta(\mathbf{x}-\mathbf{y}) +\frac{\partial}{\partial x^l}[\Delta_{00}(\mathbf{x})\delta_{kl}\delta(\mathbf{x}-\mathbf{y})]\right\}. \quad (B19)$$

Integrating over $\mathbf{x}$, we find

$$0=-i\partial\Delta_{kl}(\mathbf{y})/\partial y_l, \quad (B20)$$

which implies that each coefficient $\alpha_{kk}=0$, using the facts that $\Delta_{kk}(x)=\alpha_{kk}S(x)$ and $\Delta_{kl}(k\neq l)$ is a multiple of the identity.

We have shown, therefore, that $\Delta_{\mu\nu}(x)$ must be zero or a multiple of the identity. Thus, the commutation relations (B1)–(B4) suffice to determine $\theta_{\mu\nu}(x)$ uniquely (up to inessential phase factors), once a representation of the equal-time current algebra is specified.

In the above, we have supposed that the interaction term had the specific form $\lambda\Theta(x)=\lambda S^2(x)$. This was clearly inessential; all that is needed is that the commutators of the interaction term with the complete set of operators $j_\mu(x)$, $S(x)$, and $\dot{S}(x)$ be known and specified, as in Eq. (2.20).