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Currents as Coordinates for Hadrons*

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We investigate the possibility that complete, dynamical theories can be formulated in terms of current densities and similar operators. Our approach to this problem is through the study of models like nonrelativistic quantum mechanics or the quark model. Generalizing what one learns from these models, it is evident that such a program is, in principle, feasible. Whether or not such a program will prove useful remains an open question. We do not attempt to maintain mathematical rigor. Rather, we try to isolate the physics upon which a formulation of hadron dynamics in terms of currents would be based.

I. INTRODUCTION

IN view of the remarkable success of current algebra in correlating various properties of the hadrons,[1] it is natural to ask how far the use of currents can be pushed. Is it possible, for example, to write a complete dynamical theory of hadrons in terms of the weak and electromagnetic hadron currents and, perhaps, a few additional operators? This is the question which is studied in this and the following two papers.[2–4]

A possible motivation for writing a theory in terms of currents may be found in the fact that among the hundred-odd known hadrons there are presently no candidates to play the role of an "elementary particle" or "building block." Since, traditionally, relativistic theories have always been written in terms of canonical fields whose quanta may be considered as the "building blocks" of matter, one is rather at a loss, when presented with the hadron spectrum, even to know where to start.[5] This situation can be summarized by the statement that we do not have a set of quantum-mechanical "coordinates," like the canonical fields of the traditional theories, with which to describe hadronic matter and in terms of which to write a Hamiltonian. One might hope that the currents, which treat all particles on an equal footing, could serve as coordinates to define a theory in which no hadron plays a special role.

We have found that it is, in fact, perfectly possible to write theories in terms of currents and similar operators. We do not, however, have anything, except for a few hints, on the question of whether an "elementary-particleless" theory like that discussed above can be constructed. Thus, the ideas expressed in the previous paragraph should be regarded largely as motivation for further research along the lines indicated here.

We have approached this problem mostly through the study of models. For example, we take a model like nonrelativistic quantum mechanics, rewrite it in terms of currents, and see what it looks like. Proceeding in this way, we have discovered what we suspect are the essential aspects of the problem.

Although we have tried to avoid writing equations that are pure mathematical fiction, we do not make any pretense at rigor. Rather, our goal is to isolate the basic physical questions. At the end of this paper there are a few remarks concerning the connection between our work and rigorous work on related problems.

This paper is organized as follows. In Sec. II we study nonrelativistic systems of many identical particles. There are several reasons for this. First, one has to get used to thinking in terms of currents rather than canonical fields, and nonrelativistic models are a good starting point. Secondly, these nonrelativistic theories are well defined mathematically so that we may manipulate operators with a relative impunity. Finally, the quark model is strikingly similar to these nonrelativistic models. Section III is devoted to the quark model. There, we first treat noninteracting quarks, in which case we know that the theory exists. We then turn to an interacting theory, and have to proceed in a purely formal manner. The interacting quark model gives one a rather good picture of how a theory based on currents would work. In Sec. IV we summarize our conclusions and point out a few more qualitative aspects of the problem.

In the following paper by one of us (DHS), a model based on a charged scalar meson theory is treated in more detail. Because of the relative mathematical

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[1] For a review of this subject, see R. F. Dashen, in *Proceedings of the Thirteenth Annual International Conference on High-Energy Physics, Berkeley, California, 1966* (University of California Press, Berkeley, Calif., 1967).
[2] D. H. Sharp, following paper, Phys. Rev. **165**, 1867 (1968).
[3] C. G. Callan, R. F. Dashen, and D. H. Sharp, second following paper, Phys. Rev. **165**, 1883 (1968).
[4] The related question of using currents to reconstruct theory from experiment has been studied by M. Halpern, Phys. Rev. (to be published).
[5] This situation would, of course, change radically if quarks were found.

simplicity of this model, it provides an excellent laboratory for the study of the problem at hand. In particular, this theory can be reduced to a set of c-number functional equations for the currents.

The third paper in this series deals with a solution of a two-dimensional relativistic model, closely related to the Thirring model, in terms of currents and the energy momentum tensor.

Before proceeding to the models, we would like to suggest that the reader keep in mind the following point. Many of our formulas and equations will look strange and often quite complicated. Basically, this is due to the fact that we will be expressing quantum-mechanical theories in terms of variables, or coordinates, that are not canonical. Physical theories which are written in terms of variables which are not canonical sometimes lack a certain mathematical elegance possessed by canonical theories. However, physics, rather than the elegance of canonical variables, is the final test. If one thinks about it, out of the totality of classical physics, only in a few cases are canonical variables the most natural means of describing a system; some theories, like hydrodynamics, have no simple canonical form. It is worthwhile to keep open the possibility that the natural description of hadrons may employ coordinates which are not canonical.

II. NONRELATIVISTIC QUANTUM MECHANICS

In this section, we show how nonrelativistic quantum mechanics can be formulated in terms of currents and charge densities. Many of the things which we will point out here have been known for years in the theory of liquid helium.[6] Others are, as far as we know, new. In any case, our purpose here is not to suggest a new approach to nonrelativistic problems, but rather to look for ideas that might prove fruitful in strong-interaction physics.

Throughout this and the subsequent sections we will often refer to a set of "quantum-mechanical coordinates" for a system. What we mean by coordinates is best seen through some simple examples.

(i) A single spinless particle is usually described by its position vector $\mathbf{r}$ and momentum $\mathbf{p}$, which satisfy the algebra

$$[r_i,p_j]=i\delta_{ij}. \tag{2.1}$$

The manifold of all states available to the particle spans a single irreducible representation of the algebra (2.1). This is sufficient to guarantee that any operator can, at least in principle, be expressed as a function of $\mathbf{r}$ and $\mathbf{p}$.[7] Thus, since any observable is a function of $\mathbf{r}$ and $\mathbf{p}$, one can specify the entire quantum mechanics of a particle in terms of these coordinates. In particular, the Hamiltonian H can always be written as a function of $\mathbf{r}$ and $\mathbf{p}$.

(ii) A spin-$\frac{1}{2}$ object at a fixed point in space is described by a spin $\mathbf{S}=\frac{1}{2}\boldsymbol{\sigma}$ which satisfies the algebra

$$[S_i,S_j]=i\epsilon_{ijk}S_k. \tag{2.2}$$

Again all possible states of the system form a single irreducible representation of the algebra, and any operator can be written as a function of the coordinates[8] $\mathbf{S}$.

We could continue in this way to discuss more complicated systems such as particles with spin and systems of particles, but this hardly seems worthwhile. The above examples should serve to bring out the essential property of a set of quantum mechanical coordinates. In cases like those above where the coordinates generate a relatively simple commutator algebra, the set of all states of the system must span a single irreducible representation of the algebra. The necessity of this condition follows from the fact that if the states span more than one irreducible representation of the algebra, then there will be operators which cannot be expressed in terms of the coordinates and a complete description of the system will not have been achieved.

According to Schur's lemma, the space of states will span a single irreducible representation if and only if every operator $\mathcal{O}$ which commutes with all the coordinates is a multiple of the identity. This is the criterion which we will use in what follows.

We now turn to the description of nonrelativistic systems in terms of charges and currents.

Consider a system of N identical spinless particles, either bosons or fermions. We will show that one can give a complete description of this system in terms of a charge density $\rho(\mathbf{x})$ and a current $\mathbf{J}(\mathbf{x})$. In the usual second-quantized formalism, these operators are given by

$$\begin{aligned}\rho(\mathbf{x})&=\psi^\dagger(\mathbf{x})\psi(\mathbf{x}),\\ \mathbf{J}(\mathbf{x})&=(1/2i)[\psi^\dagger(\mathbf{x})\nabla\psi(\mathbf{x})-\nabla\psi^\dagger(\mathbf{x})\psi(\mathbf{x})],\end{aligned} \tag{2.3}$$

where the fields $\psi^\dagger(\mathbf{x})$ and $\psi(\mathbf{x})$ satisfy canonical commutation or anticommutation relations. The physical interpretation of $\rho(\mathbf{x})$ and $\mathbf{J}(\mathbf{x})$ is left open: If the particles are charged, they could be the electric charge and current densities, or they might simply represent particle or mass density and flux. The algebra satisfied

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[6] For a review of this subject, see I. M. Khalatnikov, *An Introduction to the Theory of Superfluidity* (W. A. Benjamin, Inc., New York, 1965).

[7] On the level of mathematical rigor at which we are working, irreducibility is the same as completeness. A precise statement of the relationship between irreducibility and completeness was given by J. Von Neumann, Math. Ann. **104**, 570 (1931).

[8] This corresponds to the well-known fact that any 2×2 matrix is a linear combination of the unit matrix and components of **S**.

by $\rho(\mathbf{x})$ and $\mathbf{J}(\mathbf{x})$ is

$$[\rho(\mathbf{x}),\rho(\mathbf{y})]=0\,,$$
$$[\rho(\mathbf{x}),J_k(\mathbf{y})]=-i\frac{\partial}{\partial x^k}[\delta(\mathbf{x}-\mathbf{y})\rho(\mathbf{x})]\,,$$
$$[J_i(\mathbf{x}),J_j(\mathbf{y})]=-i\frac{\partial}{\partial x^j}[\delta(\mathbf{x}-\mathbf{y})J_i(\mathbf{x})]+i\frac{\partial}{\partial y^i}[\delta(\mathbf{x}-\mathbf{y})J_j(\mathbf{x})]. \tag{2.4}$$

Our first task is to establish that $\rho(\mathbf{x})$ and $\mathbf{J}(\mathbf{x})$ are a satisfactory set of coordinates. To this end, we will show that every operator $\mathcal{O}$ which commutes with $\rho(\mathbf{x})$ and $\mathbf{J}(\mathbf{x})$ is a multiple of the identity. The first step is to observe that any operator is a function $\mathcal{O}(\psi,\psi^\dagger)$ of the fields $\psi(\mathbf{x})$ and $\psi^\dagger(\mathbf{x})$.[9] Next we note that if $\mathcal{O}$ commutes with $\rho(\mathbf{x})$, then it is invariant under the unitary transformation

$$U(\lambda)=\exp\left(i\int\lambda(\mathbf{x})\rho(\mathbf{x})d^3x\right),$$

where λ is an arbitrary function. Under the transformation $U(\lambda)$, the fields satisfy

$$U(\lambda)\psi^\dagger(\mathbf{x})U^{-1}(\lambda)=e^{i\lambda(\mathbf{x})}\psi^\dagger(\mathbf{x})\,,$$
$$U(\lambda)\psi(\mathbf{x})U^{-1}(\lambda)=e^{-i\lambda(\mathbf{x})}\psi(\mathbf{x})\,, \tag{2.5}$$

and it is not hard to convince oneself that if

$$U(\lambda)\mathcal{O}(\psi^\dagger,\psi)U^{-1}(\lambda)=\mathcal{O}(\psi^\dagger,\psi)$$

for all λ, then $\mathcal{O}$ can depend on $\psi^\dagger(\mathbf{x})$ and $\psi(\mathbf{x})$ only through the combination $\psi^\dagger(\mathbf{x})\psi(\mathbf{x})=\rho(\mathbf{x})$. Thus, if $\mathcal{O}$ commutes with $\rho(\mathbf{x})$, it is a function only of $\rho(\mathbf{x})$ and the vanishing of $[\mathbf{J}(\mathbf{x}),\mathcal{O}]$ gives

$$[J_k(\mathbf{x}),\mathcal{O}]=-i\rho(\mathbf{x})\frac{\partial}{\partial x_k}\left(\frac{\delta\mathcal{O}}{\delta\rho(\mathbf{x})}\right)=0\,, \tag{2.6}$$

where $\delta\mathcal{O}/\delta\rho(\mathbf{x})$ denotes the functional derivative and we have used (2.4). To complete the proof, we note that the right side of (2.6) is equivalent to the statement that $\mathcal{O}$ depends only on the total charge

$$Q\equiv\int d^3x\,\rho(\mathbf{x})$$

and is therefore a c number.

The above result tells us two things. First, all the states of our system of N identical particles span a single irreducible representation of the algebra (2.4) and, secondly, every operator is a function of $\rho(\mathbf{x})$ and $\mathbf{J}(\mathbf{x})$. Leaving aside, for the moment, the question of how the proper irreducible representation is determined, let us see how a few operators look when expressed in terms of $\rho(\mathbf{x})$ and $\mathbf{J}(\mathbf{x})$. Aside from the currents and densities themselves, one is usually interested only in the total linear momentum $\mathbf{P}$, the total angular momentum $\mathbf{L}$, and the Hamiltonian H. The first two are trivial; one has

$$\mathbf{P}=\int\mathbf{J}(\mathbf{x})d^3x\,,$$
$$\mathbf{L}=\int\mathbf{x}\times\mathbf{J}(\mathbf{x})d^3x. \tag{2.7}$$

The expression for the Hamiltonian is more complicated. In terms of the fields $\psi(\mathbf{x})$ and $\psi^\dagger(\mathbf{x})$ it is given by

$$H=\tfrac{1}{2}\int\nabla\psi^\dagger(\mathbf{x})\cdot\nabla\psi(\mathbf{x})d^3x+\int\psi^\dagger(\mathbf{x})\psi(\mathbf{x})V(|\mathbf{x}-\mathbf{y}|)\psi^\dagger(\mathbf{y})\psi(\mathbf{y})d^3xd^3y\,, \tag{2.8}$$

where we have set the mass equal to unity and assumed that the particles interact through a central potential.[10] To write H in terms of $\rho(\mathbf{x})$ and $\mathbf{J}(\mathbf{x})$, we use the identities

$$\nabla\rho(\mathbf{x})+2i\mathbf{J}(\mathbf{x})=2\psi^\dagger(\mathbf{x})\nabla\psi(\mathbf{x})$$

and

$$\nabla\rho(\mathbf{x})-2i\mathbf{J}(\mathbf{x})=2\nabla\psi^\dagger(\mathbf{x})\psi(\mathbf{x})$$

to obtain

$$H=\tfrac{1}{8}\int[\nabla\rho(\mathbf{x})-2i\mathbf{J}(\mathbf{x})]\frac{1}{\rho(\mathbf{x})}[\nabla\rho(\mathbf{x})+2i\mathbf{J}(\mathbf{x})]d^3x+\int\rho(\mathbf{x})V(|\mathbf{x}-\mathbf{y}|)\rho(\mathbf{y})d^3xd^3y. \tag{2.9}$$

Having seen how to write operators in terms of $\rho(\mathbf{x})$ and $\mathbf{J}(\mathbf{x})$, we will now make a few qualitative remarks about representations of the algebra (2.4). To construct explicit representations, it is useful to represent the operators in terms of functions and functional derivatives. This technique is amply illustrated in the following paper and need not be discussed here. A more basic question is how one selects the particular representation that describes the system under consideration. For one thing, $Q=\int d^3\mathbf{x}\,\rho(\mathbf{x})$ commutes with both $\rho(\mathbf{x})$ and $\mathbf{J}(\mathbf{x})$ and is thus a constant in each irreducible representation. But Q is just the number of particles so that $Q=N$ picks out representations which describe systems with N particles. Next, representations describing bosons and fermions must differ in some

[9] This is a consequence of the fact that for the system under consideration the canonical fields $\psi^\dagger(\mathbf{x})$ and $\psi(\mathbf{x})$ are irreducible. Thus we are proving that irreducibility of the fields implies irreducibility of $\rho(\mathbf{x})$ and $\mathbf{J}(\mathbf{x})$.

[10] Actually, the order of operators in the second term of Eq. (2.8) should be $\psi^\dagger(\mathbf{x})\psi^\dagger(\mathbf{y})\psi(\mathbf{y})\psi(\mathbf{x})$. Thus the Hamiltonian given by Eq. (2.8) differs from the usual one by a constant term $\int V(0)\rho(\mathbf{x})d^3x=QV(0)$. One may, of course, subtract this (possibly infinite) constant from Eq. (2.8) and all subsequent expressions for H.

way. (Everything we have said up to this point works equally well in either case.) To see how this is done, we observe that if ΔV is some small volume containing the point $\mathbf{x}$, then the number of particles in ΔV is $\Delta V\rho(\mathbf{x})$. Now for fermions this number can only be zero or unity, and we have

$$[\Delta V\rho(\mathbf{x})]^2-\Delta V\rho(\mathbf{x})=0\,, \tag{2.10}$$

which, in the continuum limit, is

$$\rho^2(\mathbf{x})-\delta(\mathbf{0})\rho(\mathbf{x})=0\,. \tag{2.11}$$

For bosons there is no constraint.

One may verify that Eq. (2.11) is consistent with the commutation relations in the sense that if $|\Phi\rangle$ is a state satisfying $[\rho^2(\mathbf{x})-\delta(\mathbf{0})\rho(\mathbf{x})]|\Phi\rangle=0$, then $\rho(\mathbf{x})$ and $\mathbf{J}(\mathbf{x})$ acting on $|\Phi\rangle$ produce states which satisfy the same constraint. We suspect that the requirement $Q=N$, along with Eq. (2.11) in the case of fermions, picks out a unique representation[11,12] of the algebra (2.4).

We have seen how the quantum mechanics of N identical spinless particles can be formulated in terms of a charge and a current. If we have two species of particles, A and B, we could try to specify the theory in terms of two mutually commuting charges and currents $\rho_A,\mathbf{J}_A$ and $\rho_B,\mathbf{J}_B$. This will be adequate, provided that there is no observable operator which changes A into B. For example, we could satisfactorily describe electrons and protons in terms of two independent charges and currents. On the other hand, A and B might refer to two spin states of the same particle. In this case, a rotation takes A into B and the description in terms of two charges and two currents is inadequate. The description of particles with spin is studied below.

Consider a system of N identical spin-$\frac{1}{2}$ particles, either bosons or fermions. In the second-quantized formalism, this system is described by two-component fields $\psi_i^\dagger(\mathbf{x})$ and $\psi_i(\mathbf{x})$, $i=1,2$ which satisfy canonical commutation or anticommutation relations. Alternatively, one can formulate the theory in terms of operators $\rho(\mathbf{x})$, $\mathbf{J}(\mathbf{x})$, and $\mathbf{\Sigma}(\mathbf{x})$ which are related to the fields by

$$\begin{aligned}
&\rho(\mathbf{x})=\psi^\dagger(\mathbf{x})\psi(\mathbf{x})\,,\\
&\mathbf{\Sigma}(\mathbf{x})=\tfrac{1}{2}\psi^\dagger(\mathbf{x})\boldsymbol{\sigma}\psi(\mathbf{x})\,,\\
&\mathbf{J}(\mathbf{x})=(1/2i)[\psi^\dagger(\mathbf{x})\nabla\psi(\mathbf{x})-\nabla\psi^\dagger(\mathbf{x})\psi(\mathbf{x})]\,.
\end{aligned} \tag{2.12}$$

The physical meaning of $\rho(\mathbf{x})$ and $\mathbf{J}(\mathbf{x})$ is the same as before, while $\mathbf{\Sigma}(\mathbf{x})$ is a spin density.[13] The commutators of $\rho(\mathbf{x})$ and $\mathbf{J}(\mathbf{x})$ are the same as those in Eq. (2.4); the commutators involving $\mathbf{\Sigma}(\mathbf{x})$ are

$$\begin{aligned}
&[\Sigma_i(\mathbf{x}),\Sigma_j(\mathbf{y})]=i\epsilon_{ijk}\delta(\mathbf{x}-\mathbf{y})\Sigma_k(\mathbf{y})\,,\\
&[\rho(\mathbf{x}),\Sigma_i(\mathbf{y})]=0\,,\\
&[\Sigma_i(\mathbf{x}),J_k(\mathbf{y})]=-i\frac{\partial}{\partial x_k}[\delta(\mathbf{x}-\mathbf{y})\Sigma_i(\mathbf{x})]\,.
\end{aligned} \tag{2.13}$$

In order to establish that $\rho(\mathbf{x})$, $\mathbf{J}(\mathbf{x})$, and $\mathbf{\Sigma}(\mathbf{x})$ are an adequate set of coordinates, we need, as before, to show that any operator $\mathcal{O}$ which commutes with $\rho(\mathbf{x})$, $\mathbf{J}(\mathbf{x})$, and $\mathbf{\Sigma}(\mathbf{x})$ is a multiple of the identity. The proof is almost identical with that given above. By considering the unitary transformations

$$U(\lambda)=\exp\left(i\int\lambda(\mathbf{x})\rho(\mathbf{x})d^3x\right)$$

and

$$U(\mathbf{l})=\exp\left(i\int\mathbf{l}(\mathbf{x})\cdot\mathbf{\Sigma}(\mathbf{x})d^3x\right),$$

we conclude that any operator which commutes with both $\rho(\mathbf{x})$ and $\mathbf{\Sigma}(\mathbf{x})$ is a function only of $\rho(\mathbf{x})$. Equation (2.6) then states that if $\mathcal{O}$ also commutes with $\mathbf{J}(\mathbf{x})$, it is a function only of $Q=\int\rho(\mathbf{x})d^3x$ and is therefore a c number.

The analog of Eq. (2.11) which distinguishes bosons from fermions is particularly interesting here. To find it, we note that $\Delta V\rho(\mathbf{x})$ and $\Delta V\mathbf{\Sigma}(\mathbf{x})$ are, respectively, the total number of particles in ΔV and the total spin of these particles. (The point $\mathbf{x}$ is, of course, supposed to lie in ΔV.) Now if the particles are bosons, their spin wave function must be completely symmetrical (since they are all at the same space point) and $[\Delta V\mathbf{\Sigma}(\mathbf{x})]^2$ must therefore equal $\frac{1}{4}[\Delta V\rho(\mathbf{x})]^2+\frac{1}{2}\Delta V\rho(\mathbf{x})$.[14] Thus, we have, in the continuum limit

$$\mathbf{\Sigma}^2(\mathbf{x})=\tfrac{1}{4}\rho^2(\mathbf{x})+\tfrac{1}{2}\delta(\mathbf{0})\rho(\mathbf{x})\,,\quad\text{(bosons)}\,. \tag{2.14}$$

For fermions, similar reasoning leads to

$$\mathbf{\Sigma}^2(\mathbf{x})=\tfrac{3}{4}\rho^2(\mathbf{x})+\tfrac{3}{2}\delta(\mathbf{0})\rho(\mathbf{x})\,,\quad\text{(fermions)} \tag{2.15}$$

which implies that the number of particles in ΔV is not greater than two [since both $\mathbf{\Sigma}^2(\mathbf{x})$ and $\rho(\mathbf{x})$ are positive], and that the spin in ΔV is 0, $\frac{1}{2}$, or 0 if the number of particles is 0, 1, or 2, respectively. An alternative derivation of Eqs. (2.14) and (2.15) is obtained by multiplying the identity

$$\boldsymbol{\sigma}_{ab}\cdot\boldsymbol{\sigma}_{cd}=-\delta_{ab}\delta_{cd}+2\delta_{ad}\delta_{bc} \tag{2.16}$$

by $\psi_a^\dagger(\mathbf{x})\psi_b(\mathbf{x})\psi_c^\dagger(\mathbf{x})\psi_d(\mathbf{x})$ and performing the commutations or anticommutations required to bring the right side of Eq. (2.16) into the form $\rho^2(\mathbf{x})$ plus $\rho(\mathbf{x})$. Equations (2.14) and (2.15) are consistent with the commutators of $\rho(\mathbf{x})$, $\mathbf{J}(\mathbf{x})$, and $\mathbf{\Sigma}(\mathbf{x})$ in the same sense

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[11] By "unique" we mean, of course, unique up to a unitary transformation.

[12] One can construct formal representations of the algebra (2.4) in which Q is not an integer. We have no idea what such representations might be good for.

[13] Experimentally $\mathbf{\Sigma}(\mathbf{x})$ could be a magnetic moment density in, for example, a ferromagnet.

[14] This follows from the fact that the completely symmetrical state of n spin-$\frac{1}{2}$ particles has a total spin S_T equal to $\frac{1}{2}n$. Thus, $S_T^2=\frac{1}{4}n^2+\frac{1}{2}n$ in the symmetrical state.

that Eq. (2.11) is consistent with the commutators of $\rho(\mathbf{x})$ and $\mathbf{J}(\mathbf{x})$. It is expected that Eq. (2.14) or (2.15) along with the condition $Q=N$ completely specifies the representation of $\rho(\mathbf{x})$, $\mathbf{J}(\mathbf{x})$, and $\mathbf{\Sigma}(\mathbf{x})$ which describes N bosons or fermions.

As was the case for spinless particles, it is trivial to write the total linear momentum and angular momentum as functions of $\mathbf{\Sigma}(\mathbf{x})$, $\rho(\mathbf{x})$, and $\mathbf{J}(\mathbf{x})$. To obtain the Hamiltonian, however, requires a new trick. Here we will only illustrate the method, treating $\psi^\dagger(\mathbf{x})$ and $\psi(\mathbf{x})$ as classical, commuting fields. The reason for this is that keeping track of the order of operators is a tricky and tedious task that tends to obscure the basic idea. It will also be assumed that H is given by (2.8) with a summation over the two components of the field understood. We begin with the identity

$$\begin{aligned}4\nabla_i\Sigma^j(\mathbf{x})\nabla_i\Sigma^j(\mathbf{x})&=\nabla_i[\psi_a{}^\dagger(\mathbf{x})\sigma_{ab}{}^j\psi_b(\mathbf{x})]\nabla_i[\psi_c{}^\dagger(\mathbf{x})\sigma_{cd}{}^j\psi_d(\mathbf{x})\\&=-\nabla_i[\psi_a{}^\dagger(\mathbf{x})\psi_a(\mathbf{x})]\nabla_i[\psi_b{}^\dagger(\mathbf{x})\psi_b(\mathbf{x})]\\&\quad+2\nabla_i[\psi_a{}^\dagger(\mathbf{x})\psi_b(\mathbf{x})]\nabla_i[\psi_b{}^\dagger(\mathbf{x})\psi_a(\mathbf{x})],\end{aligned}\tag{2.17}$$

which follows directly from Eq. (2.16). Expanding the gradients on the right side of Eq. (2.17) and treating $\psi^\dagger(\mathbf{x})$ and $\psi(\mathbf{x})$ as commuting fields, one finds

$$4\nabla_i\Sigma^j(\mathbf{x})\nabla_i\Sigma^j(\mathbf{x})\sim4\nabla_i\psi_a{}^\dagger(\mathbf{x})\nabla_i\psi_a(\mathbf{x})\rho(\mathbf{x})-4J_i(\mathbf{x})J_i(\mathbf{x}).\tag{2.18}$$

Thus, H will have the form

$$H\sim\tfrac{1}{2}\int\frac{\nabla_i\Sigma^j(\mathbf{x})\nabla_i\Sigma^j(\mathbf{x})+J_i(\mathbf{x})J_i(\mathbf{x})}{\rho(\mathbf{x})}d^3x+\int\rho(\mathbf{x})\rho(\mathbf{y})V(|\mathbf{x}-\mathbf{y}|)d^3xd^3y,\tag{2.19}$$

where the $\sim$ signs in Eqs. (2.18) and (2.19) indicate that the right sides must be suitably ordered when the noncommutivity of operators is taken into account.[15]

We may summarize the result of this section with the statement that all of nonrelativistic quantum mechanics can be formulated in terms of charge densities, currents, spin densities, and so on. As stated above, we are not particularly interested in whether or not such a formulation would be useful. From our point of view, a more important question is how one would guess the equations of this section if a different form of the theory were not already at hand. First of all, the physical interpretation of $\rho(\mathbf{x})$, $\mathbf{J}(\mathbf{x})$, and $\mathbf{\Sigma}(\mathbf{x})$ along with the correspondence principle would lead to at least some of the commutation relations (2.4) and (2.13). The conditions (2.14) and (2.15), which distinguish between bosons and fermions, would appear when one studied the irreducible representations of the algebra of $\rho(\mathbf{x})$, $\mathbf{J}(\mathbf{x})$, and $\mathbf{\Sigma}(\mathbf{x})$. Finally, current conservation, $\rho(\mathbf{x})=-\nabla\cdot\mathbf{J}(\mathbf{x})$, is enough to tell us that H has the form

$$H=\tfrac{1}{2}\int J_i(\mathbf{x})\frac{1}{\rho(\mathbf{x})}J_i(\mathbf{x})d^3x+H',\tag{2.20}$$

where H' commutes with $\rho(\mathbf{x})$ and is therefore a function only of $\rho(\mathbf{x})$ and $\mathbf{\Sigma}(\mathbf{x})$. The remaining term H' would have to be determined by other considerations.

III. THE QUARK MODEL

In Sec. II, we showed how nonrelativistic quantum mechanics could be written in terms of charges and currents. Turning now to relativistic problems, we will see that the quark model can be treated in a similar fashion.

First we study noninteracting quarks whose Lagrangian density is

$$\mathcal{L}(x)=-\tfrac{1}{2}\bar{q}(-i\gamma\cdot\partial+m)q,\tag{3.1}$$

where $q(x)$ is the 12-component quark field. Our aim is, of course, to find a set of operators, like the charges and currents of Sec. II, which will serve as coordinates for the free quark system. Clearly, a good place to start is with the densities which can be formed by placing various matrices M between the fields $q^\dagger(x)$ and $q(x)$. We call them $\mathfrak{D}(\mathbf{x},M)$, i.e.,

$$\mathfrak{D}(\mathbf{x},M)=q^\dagger(\mathbf{x})Mq(\mathbf{x}),\tag{3.2}$$

where $q(x)$ is the quark field in the Schrödinger picture which is assumed to coincide with the Heisenberg picture at $t=0$. The $\mathfrak{D}$'s satisfy the algebra

$$[\mathfrak{D}(\mathbf{x},M),\mathfrak{D}(\mathbf{y},M')]=\delta(\mathbf{x}-\mathbf{y})\mathfrak{D}(\mathbf{x},[M,M']),\tag{3.3}$$

where we have ignored Schwinger terms since for free quarks they are (infinite) c numbers and will have no effect on what follows.

From Eq. (3.3), it is clear that $\mathfrak{D}(\mathbf{x},1)$ commutes with all the $\mathfrak{D}$'s. Thus, the $\mathfrak{D}$'s by themselves are not an adequate set of coordinates and we must add a further operator. From our experience with nonrelativistic models, it is evident that the needed operator is

$$Z^k(\mathbf{x})=(1/2i)[q^\dagger(\mathbf{x})\nabla^kq(\mathbf{x})-\nabla^kq^\dagger(\mathbf{x})q(\mathbf{x})].\tag{3.4}$$

It is straightforward to verify that the commutators of Z^k with a $\mathfrak{D}$ and of Z^k with itself are

$$\begin{aligned}[\mathfrak{D}(\mathbf{x},M),Z^k(\mathbf{y})]&=-i\frac{\partial}{\partial x^k}[\delta(\mathbf{x}-\mathbf{y})\mathfrak{D}(\mathbf{x},M)]\\ [Z^k(\mathbf{x}),Z^l(\mathbf{y})]&=-i\frac{\partial}{\partial x^l}[\delta(\mathbf{x}-\mathbf{y})Z^k(\mathbf{x})]\\&\quad+i\frac{\partial}{\partial y^k}[\delta(\mathbf{x}-\mathbf{y})Z^l(\mathbf{x})].\end{aligned}\tag{3.5}$$

Thus, Z^k and the $\mathfrak{D}$'s form a closed algebra. Further-

[15] The general pattern of ordering is indicated by Eq. (2.9). Note that Eq. (2.9) contains a term $\nabla\rho(1/\rho)\mathbf{J}-\mathbf{J}(1/\rho)\nabla\rho$ which would vanish if ρ and $\mathbf{J}$ commuted.

more, using the techniques of Sec. II, it is simple to show that every operator $\mathcal{O}$ which commutes with Z^k and the $\mathfrak{D}$'s is a function only of

$$\int \mathfrak{D}(\mathbf{x},1)d^3x = (\text{baryon number})$$

and is effectively a c number. The operators Z^k and $\mathfrak{D}(\mathbf{x},M)$ may, then, be taken as a set of coordinates which are sufficient to describe all the properties of a system of quarks.

It is well to ask about the physical interpretation of these coordinates. In the quark model, many of the $\mathfrak{D}(\mathbf{x},M)$'s are either components of the weak and electromagnetic currents or can be obtained from commutators of the latter. For example, $\frac{1}{2}\mathfrak{D}(\mathbf{x},\lambda_3)$ is the time component of the isospin current,[16] while $\mathfrak{D}(\mathbf{x},\boldsymbol{\sigma})$ shows up in the commutator of two space components of the weak current. Other $\mathfrak{D}$'s which correspond to S, P, and T densities in the quark model, cannot be obtained from the weak and electromagnetic currents. However, as we shall see below, such densities show up as pieces of the energy density. To find the physical significance of Z^k, we note that the space-time component of the energy momentum tensor θ^{0k} is

$$\begin{aligned}\theta^{0k} &= (1/2i)[q^\dagger(\mathbf{x})\nabla^k q(\mathbf{x}) - \nabla^k q^\dagger(\mathbf{x})q(\mathbf{x})] \\ &\quad + \tfrac{1}{2}\epsilon_{klm}\frac{\partial}{\partial x^l}q^\dagger(\mathbf{x})\sigma^m q(\mathbf{x}) \\ &= Z^k(\mathbf{x}) + \tfrac{1}{4}\epsilon_{klm}\frac{\partial}{\partial x^l}\mathfrak{D}(\mathbf{x},\sigma^m).\end{aligned} \tag{3.6}$$

Thus, Z^k is a piece of the momentum density.[17] Physically, Z^k is that part of θ^{0k} which gives rise to the orbital angular momentum L_i in the relation

$$J_i = \int d^3x\ \epsilon_{ijk}x_j\theta^{0k}(\mathbf{x}) = \int d^3x\ \epsilon_{ijk}x_jZ^k + \tfrac{1}{2}\int d^3x\ \mathfrak{D}(\mathbf{x},\sigma^i) = L_i + S_i. \tag{3.7}$$

Having found a set of operators which is suitable for describing a system of quarks, we should inquire as to which representations of the algebra (3.4) and (3.5) are allowed. The situation is much the same as that encountered in the nonrelativistic systems of the Sec. II. First, the baryon number $B = \int \mathfrak{D}(\mathbf{x},1)d^3x$ commutes with Z^k and all the $\mathfrak{D}$'s, so that each representation has a definite baryon number. Evidently, baryon number is the relativistic analog of particle number. To find the analog of Eq. (2.15), one uses the identity

$$\sum_{(i)} M_{ab}{}^{(i)}M_{cd}{}^{(i)} = 12\delta_{ad}\delta_{cb}, \tag{3.8}$$

where the $M^{(i)}$'s are any set of 144, 12×12 matrices satisfying $M^{(i)\dagger} = M^{(i)}$ and $\mathrm{Tr}(M^{(i)}M^{(j)}) = \delta_{ij}$. Sandwiching Eq. (3.8) between quark fields then leads to

$$\sum_i \mathfrak{D}(\mathbf{x},M^{(i)})^2 = -12[(\mathfrak{D}(\mathbf{x},1))^2 - 13\mathfrak{D}(\mathbf{x},1)\delta(\mathbf{0})]. \tag{3.9}$$

In interpreting Eq. (3.9) physically, one must take account of the fact that the $\mathfrak{D}$'s as defined in Eq. (3.2) are not normal ordered. If one normal orders the $\mathfrak{D}$'s in Eq. (3.9), one finds among other things that $\mathfrak{D}(\mathbf{x},1)\Delta V$ must lie between -6 and 6, which corresponds, of course, to the fact that no more than six quarks or antiquarks can be inside the small volume ΔV. Equation (3.9) is not, it turns out, sufficient to specify a unique representation of the algebra (3.3) and (3.5). There are more identities like (3.8) which involve products of three, four, and more M's.[18] These lead to further equations which are similar to (3.9) except that they involve higher powers of the $\mathfrak{D}$'s. We will not have any use for these additional equations.

One may also use the identity (3.8) to find an explicit formula for the Hamiltonian. Actually, it is more convenient to work with $\theta^{00}(x)$, the time-time component of the energy-momentum tensor whose space intergral is H. For free quarks, $\theta^{00}(x)$ is given by

$$\theta^{00}(\mathbf{x}) = (1/2i)[q^\dagger(\mathbf{x})\boldsymbol{\alpha}\cdot\boldsymbol{\nabla}q(\mathbf{x}) - \boldsymbol{\nabla}q^\dagger(\mathbf{x})\cdot\boldsymbol{\alpha}q(\mathbf{x})] + mq^\dagger(\mathbf{x})\beta q(\mathbf{x}). \tag{3.10}$$

The second term in $\theta^{00}(\mathbf{x})$ is just $m\mathfrak{D}(\mathbf{x},\beta)$. To obtain the first term, one proceeds in essentially the same way as we did for nonrelativistic particles with spin. For simplicity, we will again treat the quark fields as classical, commuting objects. The identity (3.8) leads to

$$\begin{aligned}&\sum_{(i)} \{q^\dagger(\mathbf{x})M^{(i)}q(\mathbf{x})\nabla_k[q^\dagger(\mathbf{x})M^{(i)}\alpha_k q(\mathbf{x})] \\ &\quad - \nabla_k[q^\dagger(\mathbf{x})M^{(i)}q(\mathbf{x})]q^\dagger(\mathbf{x})M^{(i)}\alpha_k q(\mathbf{x})\} \\ &\quad \sim 12[q^\dagger(\mathbf{x})\alpha_k\nabla_k q(\mathbf{x}) - \nabla_k q^\dagger(\mathbf{x})\alpha_k q(\mathbf{x})]q^\dagger(\mathbf{x})q(\mathbf{x}) \\ &\quad + 12[\nabla_k q^\dagger(\mathbf{x})q(\mathbf{x}) - q^\dagger(\mathbf{x})\nabla_k q(\mathbf{x})]q^\dagger(\mathbf{x})\alpha_k q(\mathbf{x}),\end{aligned} \tag{3.11}$$

and we see that $\theta^{00}(\mathbf{x})$ will have the form

$$\begin{aligned}\theta^{00}(\mathbf{x}) &\sim [24i\mathfrak{D}(\mathbf{x},1)]^{-1}\sum_{(i)}[\mathfrak{D}(\mathbf{x},M^{(i)})\nabla_k\mathfrak{D}(\mathbf{x},M^{(i)}\alpha_k) \\ &\quad - \nabla_k\mathfrak{D}(\mathbf{x},M^{(i)})\mathfrak{D}(\mathbf{x},M^{(i)}\alpha_k)] \\ &\quad + [\mathfrak{D}(\mathbf{x},1)]^{-1}Z^k(\mathbf{x})\mathfrak{D}(\mathbf{x},\alpha_k) + m\mathfrak{D}(\mathbf{x},\beta).\end{aligned} \tag{3.12}$$

The $\sim$ signs in (3.11) and (3.12) indicate, as before,

[16] Here we use the notation M = (Dirac matrix)[$SU(3)$ matrix]. The Dirac matrices $\boldsymbol{\alpha}$, $\boldsymbol{\sigma}$, and β are $\beta = \gamma^0$, $\boldsymbol{\alpha} = \gamma^0\boldsymbol{\gamma}$, and $\boldsymbol{\sigma} = -\gamma^0\gamma_5\boldsymbol{\gamma}$.

[17] We could have taken θ^{0k} rather than Z^k as a basic operator: however, the use of Z^k simplifies the algebra.

[18] For 12×12 matrices there are 11 nontrivial identities involving powers of M up to the twelfth.

that the right sides should be suitably ordered. Since we shall not make explicit use of Eq. (3.12), the precise ordering prescription will not be needed.

Thus far, we have been discussing noninteracting quarks. For these free quarks, everything which we have done could, with suitable modifications, be made mathematically respectable. When there are interactions, this will no longer be the case. The best we can do is to try to guess what sort of equations might continue to make sense.

The simplest and most naive way to introduce an interaction is to leave all our previous statements intact except for the addition of an interaction Hamiltonian density $\mathcal{H}_I(\mathbf{x})$ on the right side of Eqs. (3.10) and (3.12). The interaction density $\mathcal{H}_I$ should not contain any derivatives of the quark fields, since we want the currents and their commutation relations to remain unchanged.[19] Thus $\mathcal{H}_I(\mathbf{x})$ will be some explicit function of the $\mathfrak{D}(\mathbf{x},M)$'s; any dependence on Z^k or derivatives of the $\mathfrak{D}$'s would lead to dependence on derivatives of the quark fields. Also, since θ^{0k} is left unchanged,[19] $\mathcal{H}_I(\mathbf{x})$ must be a Lorentz scalar; hence $\mathcal{H}_I(\mathbf{x})$ will depend on the $\mathfrak{D}$'s only through manifestly Lorentz-invariant combinations such as $\mathfrak{D}(\mathbf{x},1)^2-\mathfrak{D}(\mathbf{x},\boldsymbol{\alpha})^2$. As an example of such a model, one could add a current-current interaction $\mathcal{H}_I(\mathbf{x})=-g[\mathfrak{D}(\mathbf{x},1)^2-\mathfrak{D}(\mathbf{x},\boldsymbol{\alpha})^2]$ to the right side of (3.11), leaving the commutation relations and the equations like (3.9) which specify the representation unchanged. This would, of course, produce a model which, insofar as it makes any sense, is completely equivalent to a normal quark field theory with current-current interaction.

The simple-minded insertion of an interaction as outlined above is almost certainly too naive. It is very likely that in order to accommodate interactions without running into mathematical catastrophes, one has to make the formalism somewhat less rigid. Let us see how this could be done. To begin with, the operators $\mathfrak{D}(\mathbf{x},M)$ and $Z^k(\mathbf{x})$, being closely related to observable quantities, must make sense: The question of precisely how these objects are to be interpreted as products of quark fields is not relevant. The fact that the $\mathfrak{D}$'s and Z^k are sensible objects does not, however, necessarily mean that the (equal-time) commutators in Eqs. (3.3) and (3.5) continue to make sense. Nevertheless, it is clear that our program will fail at the very start unless the commutators are meaningful, and we will assume that they are.[20] We will also assume that all the states of the system continue to span one irreducible representation of the algebra. It may not be necessary to demand, however, that this representation is one for which equations like (3.9) hold. If we give up (3.9), the only bad thing that happens is that our explicit expression (3.12) for the Hamiltonian is no longer correct. Later on, however, we will see how the Hamiltonian can be specified in a more general manner which is independent of any particular representation of the algebra.

A motivation for giving up (3.9) may be seen in the following chain of thought. When equations like (3.9) hold, we are working with a representation which is equivalent to that realized by free quarks. If we now take an interaction $-g[\mathfrak{D}^2(\mathbf{x},1)-\mathfrak{D}^2(\mathbf{x},\alpha)]$, and expand in powers of g, we will in effect be doing the usual perturbation theory. However, in perturbation theory it is well known that products of operators at a point, such as appear in Eq. (3.9), are so singular that, for practical purposes, they are meaningless, as would also be expected on the basis of Haag's theorem. Thus, it is very likely that in the presence of interaction, Eq. (3.9) is self-contradictory.

Actually, the idea of working with representations other than that realized by free quarks might make good physical sense for a quite different reason. In many ways, the hadrons look as though they were made out of quarks, but as yet no quarks have been found. One might be tempted to think, then, that the world is some kind of solution to the quark model in which the quarks have, loosely speaking, picked up an infinite mass and no longer exists as physical particles. In a normal field theory of quarks, this would be a limit in which the field operator $q(\mathbf{x})$ goes away, but bilinears such as $q^\dagger(\mathbf{x})q(\mathbf{x})$ remain finite, well-defined quantities.[21] Clearly, this would be a situation where non-quark representations of the algebra of the $\mathfrak{D}$'s and Z^k appear.

The possibility that nonquark representations of the algebra might be interesting brings up a further point. We have set up a formalism that is not manifestly covariant. Thus, Lorentz invariance can be demonstrated only by showing that there exist Lorentz-transformation generators $\mathbf{\Lambda}$ which have the proper commutation relations with the various operators. In the quark representation satisfying (3.9) we know that such generators must exist, since in that representation our equations are simply a rewriting of a manifestly covariant field theory. The remainder of this section is devoted to the question of Lorentz invariance in more general representations.

The generators $\mathbf{\Lambda}$ are given by

$$\mathbf{\Lambda}=\int \mathbf{x}\theta^{00}(\mathbf{x})d^3x. \tag{3.13}$$

Thus, the Lorentz-transformation properties of any operator are determined by its commutator with the energy density $\theta^{00}(\mathbf{x})$. For example, the known transformation property of $\theta^{00}(\mathbf{x})$ itself leads to the familiar

[19] In Lagarangian field theory, an interaction which contains no derivatives does not change the functional form of either the currents or θ^{0k}.

[20] We are implicitly assuming that the Schwinger terms remain infinite c numbers and may be ignored. In Ref. 3, the question of Schwinger terms is discussed in more detail.

[21] Rigorously, this must be a limit in which either $q(\mathbf{x})$ vanishes identically, or in which the asymptotic states cease to be complete. In either case $q^\dagger(\mathbf{x})q(\mathbf{x})$ must have a sensible limit.

result[22]

$$[\theta^{00}(\mathbf{x}),\theta^{00}(\mathbf{y})]=i\left(\frac{\partial}{\partial y^k}-\frac{\partial}{\partial x^k}\right)[\delta(\mathbf{x}-\mathbf{y})\theta^{0k}(\mathbf{x})], \quad (3.14)$$

where it has been assumed that no terms more singluar than the first derivative of a δ function appear in the commutator. Let us see what restrictions relativity imposes on the commutator of $\theta^{00}(\mathbf{x})$ with the $\mathfrak{D}$'s. First we list the Lorentz-transformation properties of the $\mathfrak{D}$'s; they are

$$[\Lambda_i,\mathfrak{D}(\mathbf{x},M)]=x_i[H,\mathfrak{D}(\mathbf{x},M)]+\tfrac{1}{2}i\mathfrak{D}(\mathbf{x},\{\alpha_i,M\}), \quad (3.15)$$

where the first term on the right comes from transforming the coordinate $\mathbf{x}$ and the second is an "index rotation." For example, when $M=\beta$ so that $\mathfrak{D}$ is a scalar, the anticommutator of M with α_i vanishes and the second term is absent. Next, we make the assumption that the commutator of $\theta^{00}(\mathbf{y})$ and $\mathfrak{D}(\mathbf{x},M)$ is no more singular than a δ function or its first derivative. The general form of the commutator is then

$$[\theta^{00}(\mathbf{y}),\mathfrak{D}(\mathbf{x},M)]=\delta(\mathbf{x}-\mathbf{y})\mathcal{O}(\mathbf{x}) +i\frac{\partial}{\partial x^k}[\delta(\mathbf{x}-\mathbf{y})P^k(\mathbf{x})], \quad (3.16)$$

where $\mathcal{O}(\mathbf{x})$ and $P^k(\mathbf{x})$ are as yet unrestricted operators. Multiplying (3.15) by y_i, integrating over d^3y, and comparing with Eq. (3.15), one finds that $\mathcal{O}(\mathbf{x})$ remains unrestricted, but that $P^k(\mathbf{x})$ must equal $\frac{1}{2}\mathfrak{D}(\mathbf{x},\{\alpha_i,M\})$. Thus we can write the commutator of $\theta^{00}(\mathbf{y})$ and $\mathcal{O}(\mathbf{x})$ in the convenient form

$$[\theta^{00}(\mathbf{y}),\mathfrak{D}(\mathbf{x},M)]=\delta(\mathbf{x}-\mathbf{y})\left[\theta^{00}(\mathbf{y}),\int d^3x\ \mathfrak{D}(\mathbf{x},M)\right] +\tfrac{1}{2}i\frac{\partial}{\partial x^k}[\delta(\mathbf{x}-\mathbf{y})\mathfrak{D}(\mathbf{x},\{\alpha_k,M\})]. \quad (3.17)$$

To be complete, we need the commutator of $\theta^{00}(\mathbf{x})$ with Z^k. It is easier to give the commutator with

$$\theta^{0k}(\mathbf{x})=Z^k+\tfrac{1}{4}\epsilon_{klm}\frac{\partial}{\partial x^l}\mathfrak{D}(\mathbf{x},\sigma^m),$$

which is

$$[\theta^{00}(\mathbf{y}),\theta^{0k}(\mathbf{x})]=i\frac{\partial}{\partial x^l}[\delta(\mathbf{x}-\mathbf{y})\theta^{lk}(\mathbf{x})] -i\frac{\partial}{\partial y^k}[\delta(\mathbf{x}-\mathbf{y})\theta^{00}(\mathbf{x})], \quad (3.18)$$

where it has again been assumed that the commutator is no more singular than the first derivative of a δ function.

We can now state the requirements of relativity as follows: *The only allowed representations of the algebra are those in which there exists an operator* $\theta^{00}(\mathbf{y})$ *which satisfies Eqs. (3.14) and (3.17).* We need not consider Eq. (3.18) because its only role is to define $\theta^{kl}(\mathbf{x})$. In the quark representation which satisfies Eq. (3.9), the expression for $\theta^{00}(\mathbf{x})$ given in (3.12) satisfies these requirements.[23] Unfortunately, we have not been able to find any other representation in which we have been able to construct a $\theta^{00}(\mathbf{x})$ which satisfies the constraints.

While we have not demonstrated the existence of an energy density $\theta^{00}(\mathbf{x})$ in any interesting representation, we have found that (3.14) and (3.17) are enough largely to determine $\theta^{00}(\mathbf{x})$ when it does exist. Thus, we need not use singular-looking expressions like Eq. (3.12), but can work with the presumably more sensible commutation relations (3.14) and (3.17). The reason for this is as follows. Suppose that in a given irreducible representation of the algebra generated by the $\mathfrak{D}$'s and Z^k, there are two solutions to (3.14) and (3.17). The difference Δ between the two θ^{00}'s will evidently satisfy

$$[\Delta(\mathbf{y}),\mathfrak{D}(\mathbf{x},M)]=\delta(\mathbf{x}-\mathbf{y})\left[\Delta(\mathbf{y}),\int\mathfrak{D}(\mathbf{x},M)d^3x\right] \quad (3.19)$$

and

$$[\Delta(\mathbf{x}),\Delta(\mathbf{y})]+[\Delta(\mathbf{x}),\theta^{00}(\mathbf{y})]+[\theta^{00}(\mathbf{x}),\Delta(\mathbf{y})]=0, \quad (3.20)$$

where the $\theta^{00}(\mathbf{x})$ in the latter equation is one of the two solutions. Now in an irreducible representation, Δ must be a function of Z^k and the $\mathfrak{D}$'s, and Eq. (3.19) is then sufficient to guarantee that $\Delta(\mathbf{x})$ depends only on $\mathfrak{D}(\mathbf{x},M)$'s at the same point $\mathbf{x}$: Any dependence on Z^k or derivatives of the $\mathfrak{D}$'s would lead to gradients of δ functions in (3.19). Next, it is easy to see that $[\Delta(\mathbf{x}),\Delta(\mathbf{y})]$ must vanish. Since Δ does not contain Z^k or derivatives of the $\mathfrak{D}$'s the commutator $[\Delta(\mathbf{x}),\Delta(\mathbf{y})]$ must be proportional to $\delta(\mathbf{x}-\mathbf{y})$, but the coefficient of $\delta(\mathbf{x}-\mathbf{y})$ must vanish since $[\Delta(\mathbf{x}),\Delta(\mathbf{y})]$ is antisymmetric in $\mathbf{x}$ and $\mathbf{y}$. Using this information, Eq. (3.20), and our usual assumption that $[\Delta(\mathbf{x}),\theta^{00}(\mathbf{y})]$ is no more singular than a δ function or its first derivative, we find that

$$[\theta^{00}(\mathbf{x}),\Delta(\mathbf{y})]=\delta(\mathbf{x}-\mathbf{y})[H,\Delta(\mathbf{y})], \quad (3.21)$$

which implies

$$[\Lambda_i,\Delta(\mathbf{y})]=y_i[H,\Delta(\mathbf{y})]. \quad (3.22)$$

The interpretation of Eq. (3.22) is, of course, that Δ is a Lorentz scalar. But a Lorentz-scalar function of the $\mathfrak{D}$'s, containing no derivatives or Z^k, must be a manifestly Lorentz-invariant function of invariant products like $\mathfrak{D}^2(\mathbf{x},1)-\mathfrak{D}^2(\mathbf{x},\alpha)$. Thus, the commutation relations (3.14) and (3.17) determine $\theta^{00}(\mathbf{x})$ up to a (manifest) Lorentz-scalar function of the $\mathfrak{D}(\mathbf{x},M)$'s. This is the same amount of freedom as is present when one simply adds an interaction in the usual quark model.

We may summarize the last half of this section with the statements: (i) In interacting theories, one will probably have to avoid equations like (3.9). (ii) When such equations do not hold, one cannot simply take a

[22] J. Schwinger, in *Theoretical Physics* (International Atomic Energy Agency, Vienna, 1963), pp. 89–134; Phys. Rev. **130**, 406 (1963); **130**, 800 (1963).

[23] Note that the $\theta^{00}(\mathbf{x})$ of Eq. (3.12) will *not* satisfy either (3.14) or (3.17) unless Eq. (3.9) holds.

field-theoretic Hamiltonian and rewrite it, in a straightforward way, as a function of the currents. (iii) Nevertheless, we have found that the requirements of relativity are such that $\theta^{00}(\mathbf{x})$ is always determined up to a relatively simple "interaction term."

The reader will recall that we have consistently assumed that the commutator of $\theta^{00}(\mathbf{x})$ with various operators is no more singular than a δ function or its first derivative. This is the minimum amount of singularity which is consistent with relativity. Without this "minimum-singularity" assumption, our conclusions would have been considerably weaker.

IV. DISCUSSION

In Secs. II and III we have seen that quantum-mechanical theories, both relativistic and nonrelativistic, can be written in terms of coordinates like charge and current densities. In each of the cases studied, the one thing which seems least satisfactory is the quite awkward expression for the kinetic-energy part of the Hamiltonian. We saw, however, in the relativistic case that the essential properties of the kinetic part of H are completely determined by the requirements of Lorentz invariance. Thus, there is reason to expect that the singular expressions in our formulas for H can be replaced by simpler statements such as (3.14) and (3.17).

As far as relativistic theories are concerned, we limited ourselves to the quark model. However, our conclusions are by no means peculiar to this one model. In the following paper, a theory of charged scalar mesons is studied. Again it is found that a complete formulation of the theory can be based on operators like charges and currents.

To summarize, we know that it is possible to write a theory in terms of currents. The next question is: Is it a useful thing to do? Obviously we are not going to try to answer this question. If we knew the answer, we would probably have enough information to present a complete theory of strong interactions. Nevertheless, it probably is worth making one point. It does not seem likely that a theory written in terms of currents will be particularly useful unless it is somehow qualitatively different from a normal canonical field theory. Thus we do not expect that rewriting the quark model in terms of currents will cure the ills of Lagrangian field theory. One way in which a qualitative difference could occur is in the choice of a representation of the current algebra. This has already been discussed in the quark model. Another, more speculative, possibility is that a theory, for which there is no analog in canonical field theory, can be written in terms of the known weak and electromagnetic currents along with some additional operators like θ^{0k}. Presumably one would postulate some algebra for these operators and assume that all hadron states span a single irreducible representation of it. The requirements of relativity would have to be investigated. If they can be satisfied, it is likely that $\theta^{00}(\mathbf{x})$ would be determined up to some simple "interaction." If such a theory exists, it would describe a world in which there are no elementary particles, only elementary currents. In fact, this is about the only way in which we can imagine writing a concrete, dynamical theory without recourse to some kind of elementary particle.

The question of elementary particles brings up another point. Where do the particles, namely, the physical hadrons, show up in a theory based on currents? In attempting to answer this question, we prefer to discuss nonrelativistic theories, where we know that our statements make mathematical sense. Consider the algebra (2.4) and (2.13), which, according to our claims, describes N nonrelativistic spin-$\frac{1}{2}$ particles, either bosons or fermions. How, in fact, do we find these particles if we start directly with the algebra of Eqs. (2.4) and (2.15)? The answer is that we have to look at representations which satisfy (2.14) or (2.15). In these representations, Eqs. (2.14) and (2.15) tell us that there exist states in which $\rho(\mathbf{x})$, $\mathbf{\Sigma}(\mathbf{x})$, and $\mathbf{J}(\mathbf{x})$ are all localized at set of points[24] $x_1, x_2, \cdots, x_N$. Such states could be interpreted as states with particles localized at $x_1, x_2, \cdots, x_N$. We might call such particles "elementary" to distinguish them from bound states which cannot be found in this way, since the charge density of a bound state is spread out rather than localized at one point. The bound states could only be found by diagonalizing the Hamiltonian. The implications of these remarks for the relativistic case should be obvious.

The reader may also wonder how the S matrix could be constructed in a theory based on currents. In practice, this would be very difficult, but in principle there is no problem. There are two ways to see this. First, it is known[25,26] that the matrix elements of Heisenberg currents like $\mathfrak{D}(\mathbf{x},t,M)$, taken at all times t, contain a quantity of information which is sufficient to determine the particle cross sections. If, as in the models discussed here, one has the Hamiltonian and the currents at $t=0$, one can clearly calculate the currents for all times, and the cross sections can then be calculated, in principle. An easier way to arrive at the same conclusion is to note that in any theory with an explicit Hamiltonian, a set of formal scattering equations can be set up: The solution of these equations yields the S matrix.

So far, we have always been talking about how this or that can be calculated in principle. If some sort of theory built on currents does exist, one would be faced with the problem of making practical, approximate calculations. We do not really have any good ideas as to how this would be done. There are two approaches which would seem natural, but it is not clear at this point how they could actually be implemented.

In the theory of liquid helium, the use of currents

[24] For example, the expectation value of $\rho(\mathbf{x})$ will have the form $\langle\rho(\mathbf{x})\rangle \propto \sum_{i=1}^{N}\delta(\mathbf{x}-\mathbf{x}_i)$.

[25] H. Araki and R. Haag, Comm. Math. Phys. 4, 77 (1967).

[26] J. Langerholc and B. Schroer, Comm. Math. Phys. 4, 123 (1967).

and densities as basic coordinates for the system has led to physically significant results. The reason for this may be seen by referring to the Hamiltonian (2.9). If, as happens in liquid helium, $\rho(\mathbf{x})$ is nearly a constant, one can set the factor $\rho^{-1}(\mathbf{x})$ in the first term in H equal to $\rho_0^{-1}(\mathbf{x})$, where $\rho_0(\mathbf{x})$ is the ground-state expectation value of ρ. The resulting Hamiltonian is bilinear in $\rho(\mathbf{x})$ and $\mathbf{J}(\mathbf{x})$ and is therefore much easier to handle. Let us see what happens if we try an analogous approximation in the quark-model Hamiltonian (3.12). The vacuum state is what corresponds in field theory to the ground state of a quantum fluid, so what we would want to say is that the baryon number density $\mathfrak{D}(\mathbf{x},1)$ is, apart from small fluctuations, equal to its vacuum expectation value. The trouble with this is, of course, that $\mathfrak{D}(\mathbf{x},1)$ is the time component of a 4-vector, so that setting $\mathfrak{D}(\mathbf{x},1)$ equal to a constant c number makes no sense *unless* there is some preferred frame of reference. Also, it is by no means clear *a priori* that $\mathfrak{D}(\mathbf{x},1)$ should vary only slightly from its vacuum expectation value.

In view of this, it is interesting to note that there is a physical problem where the baryon number density is, apart from small fluctuations, a large constant, so that a preferred frame of reference does exist. This is a very dense neutron star[27] (density $\gg 10$ times nuclear density). Thus, neutron stars might turn out to be relativistic analogs of a tank of liquid helium. Actually, this could make physical sense. For a sufficiently dense neutron star, the usual description of matter in terms of individual nucleons and hyperons is likely to be meaningless: The properties of nuclear matter in such a highly compressed state might be more appropriately described[28] in terms of quantities such as the baryon number per unit volume, hypercharge per unit volume, etc., which enter in a natural way in theories based on currents.

A second possible approximation scheme would seem to be contained in the suggestion[29] that infinite sets of baryon and meson resonances form simple representations of the algebra generated by $\mathfrak{D}(\mathbf{x},\lambda_i)$ and $\mathfrak{D}(\mathbf{x},\gamma_5\lambda_i)$. While this scheme may lead to a useful phenomenological model of the hadrons, it is not clear that such an approach would be a useful way to work towards the solution of a complete theory. In order to see what the problem is, it is necessary to understand the relation between a complete theory and the model suggested in Ref. 29. For the sake of argument, let us suppose that the hadrons actually are described by the $\mathfrak{D}(\mathbf{x},M)$'s and $Z^k(\mathbf{x})$ of the quark model, along with some $\theta^{00}(\mathbf{x})$ built out of them. In such a world, the set of all states with a given baryon number form one irreducible representation of the algebra (3.3) and (3.5). In the phenomenological model, on the other hand, one works only with the subalgebra generated by $\int \mathfrak{D}(x_1,x_2,x_3,\lambda_i)dx_3$ and $\int \mathfrak{D}(x_1,x_2,x_3,\gamma_5\lambda_i)dx_3$. Clearly, the set of all states with a given third component of momentum P_3 form a representation of this subalgebra. In general, this representation will be complicated and highly reducible. The hypothesis of Ref. 28 is that for $P_3\to\infty$, infinite sets of meson and baryon resonances which form simple but still reducible representations of the subalgebra are singled out. Just which representations occur and the mass spectrum within a representation are, according to Ref. 29, to be determined by the requirements of relativity, self-consistency, and a certain amount of input from experiment. Now it is clear that our hypothetical complete theory built on all the $\mathfrak{D}$'s and Z^k would, in principle, determine which representatiosn and what mass spectrum are to be used in the phenomenological model. However, if one were given a specific $\theta^{00}(\mathbf{x})$ written as a function of the $\mathfrak{D}$'s and Z^k, it is not at all obvious how one would go about calculating the parameters of the phenomenological model. It may well be that the complete theory and the model would be brought together only after the former had been solved completely.

In closing we would like to remark on the connection between the present work and the branch of axiomatic field theory which deals with "rings of local observables." The local observables are just the objects which we are working with, namely, currents, components of the energy momentum tensor, and so on. A nonobservable is, for example, a fermion field. The difference between the axiomatic approach and ours is simply that the axiomaticians prove rigorous statements about general theories without any explicit properties or interactions, while we have been concerned with mathematically dubious statements about concrete, specific theories. As an example of the difference between the axiomatic approach and ours, we have specified theories in terms of a set of densities taken at a single time (we have been implicitly setting $t=0$) with definite commutation relations and a Hamiltonian written in terms of the densities. On the other hand, it may be proved from the axioms that knowing the matrix elements of only one current for all times is sufficient to define the theory. That these statements are not incompatible is easily seen if we consider a simple classical system with one degree of freedom. If one is given p and q at one time, say $t=0$, and a Hamiltonian $H(p,q)$, it is possible to compute p and q for all times. On the other hand, a knowledge of the coordinate q for all times is also sufficient, since from the trajectory $q(t)$ one could compute both p and H.

It is unfortunate that, in our present state of knowledge of field theory, mathematical rigor and concrete theories with interaction are mutually exclusive. It would be nice to think that the axiomatic approach and that which we have used here will some day converge and produce a real theory.

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[27] For a recent review of the subject of "superdense stars" see J. A. Wheeler, Ann. Rev. Astron. Astrophys. 4, 393 (1966).

[28] One of the authors (DHS) would like to thank M. Gell-Mann for a stimulating conversation on this topic.

[29] R. Dashen and M. Gell-Mann, Phys. Rev. Letters **17**, 340 (1966); M. Gell-Mann in *1966 International School of Physics "Ettore Majorana," Erice, 1966*, edited by A. Zichichi (Academic Press Inc., New York, 1966).