Perturbations, Dispersion Relations, and the *n-p* Mass Difference*

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Two questions concerning the reliability of dispersion-theoretic treatments of the effect of perturbing forces on binding energy are investigated. First, the change in binding energy caused by variation of the mass of the exchanged particle responsible for the binding is considered. It is shown that the surprising sign of the Dashen-Frautschi result for this change is not physically absurd. Moreover, the methods used by Dashen and Frautschi generally give correct signs in potential-theory tests. Secondly, the special problems encountered in treating long-range or infinite-range $(1/r)$ perturbations are dealt with in detail. It is emphasized that the basic Dashen-Frautschi mass-shift formula tends to yield incorrect signs when applied to a long-range perturbation, even when no infrared divergences are involved. A modified technique, which is proved to be accurate in both sign and magnitude in s-wave potential theory, even for $1/r$ forces, is used to make a fully relativistic calculation of the electromagnetic driving term for the n-p mass difference. In agreement with physical expectations, the driving term turns out to be a negative contribution to M_n-M_p, suggesting that the experimental sign must be ascribed to other effects.

I. INTRODUCTION

DISPERSION relations have been used by Dashen and Frautschi (DF) to treat perturbations on hadron masses and coupling constants with great success.[1–6] The hadrons are assumed to be bootstrapped bound states. In any given calculation, the mass and coupling shifts δM and δg depend on driving terms, and also on themselves through self-consistency or "feedback" terms. For example, the mass shift of the ith particle is given to first order by a relation of the form

$$\delta M_i=\sum_j A_{ij}{}^{MM}\delta M_j+\sum_j A_{ij}{}^{Mg}\delta g_j+D_i{}^M, \quad (1.1)$$

where $D_i{}^M$ is the relevant driving term, and the A's are the feedback coefficients. Physically, the driving term represents the change in binding energy due to the new forces introduced when the perturbing interaction is turned on. The feedback terms give the change in binding energy caused by modifications in the forces originally present, resulting from the changes in all the masses and coupling constants. In addition, the sum on δM_j includes the contribution to δM_i of the change in the constituent particle masses.

In his calculation of the neutron-proton mass difference,[1,2] Dashen explicitly calculated the electromagnetic driving term, besides evaluating feedback coefficients, but most of Dashen and Frautschi's work depends only on the computation of the A_{ij} themselves.

Sawyer has noted[7] that, as calculated by DF, the coefficient $A_{be}{}^{MM}$, which describes the response of the binding energy of particle b to a shift in the mass of the particle e whose exchange is providing the binding, has a sign which seems physically very strange. If this sign is indeed wrong, then the bootstrap explanation of octet mass-splitting enhancement, as well as the calculation of the neutron-proton mass difference, become suspect. In Sec. II, we show that the sign DF obtain for $A_{be}{}^{MM}$, though indeed surprising at first, is *not* physically nonsensical. We go on to show that in potential theory, where the true response of binding energy to perturbations on the binding forces is known, N/D techniques of the sort used by DF generally give the correct sign for this response.

Turning to the neutron-proton mass difference, $M_n-M_p=\delta M_n-\delta M_p$, we focus on the driving term D_{n-p}, defined by $D_{n-p}\equiv D_n{}^M-D_p{}^M$. Apart from small feedback corrections, Dashen's result $M_n-M_p=+1.4$ MeV, which is in remarkable agreement with the observed value of $+1.3$ MeV, is equal to his calculated D_{n-p}. We ponder the fact that D_{n-p} turned out to be positive. In the DF model, the nucleon is taken to be a πN bound state, so that, with the appropriate Clebsch-Gordan coefficients,

$$\begin{aligned}|n\rangle&=-(\sqrt{\tfrac{2}{3}})|\pi^-p\rangle+(\sqrt{\tfrac{1}{3}})|\pi^0n\rangle,\\|p\rangle&=(\sqrt{\tfrac{2}{3}})|\pi^+n\rangle-(\sqrt{\tfrac{1}{3}})|\pi^0p\rangle.\end{aligned} \quad (1.2)$$

$D_n{}^M$ is simply $\frac{2}{3}$ times the change in binding of π^- to p due to photon exchange between them; similarly, $D_p{}^M$ and π^+n.[8] Now the electric force between π^- and p is obviously attractive. As Barton has pointed out,[9] so is the magnetic force (in the state with the spin parity of n, namely, $p_{1/2}$). By comparison, π^+ and n have no electrical interaction, and the magnetic force between them in the $p_{1/2}$ state also happens to be attractive. However, the proton's (total) magnetic moment is larger than the neutron's, so the magnetic attraction between π^- and p dominates over that between π^+ and n. The π^-p state has the electric attraction in addition, so one expects photon exchange to tend to make the neutron lighter

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[1] R. Dashen and S. Frautschi, Phys. Rev. **135**, B1190 (1964).
[2] R. Dashen, Phys. Rev. **135**, B1196 (1964).
[3] R. Dashen and S. Frautschi, Phys. Rev. **137**, B1318 (1965).
[4] R. Dashen and S. Frautschi, Phys. Rev. **137**, B1331 (1965).
[5] R. Dashen, S. Frautschi, and D. Sharp, Phys. Rev. Letters **13**, 777 (1964).
[6] R. Dashen, Y. Dothan, S. Frautschi, and D. Sharp, Phys. Rev. **143**, 1185 (1966).
[7] R. Sawyer, Phys. Rev. **142**, 991 (1966).
[8] One assumes, of course, that the π^0 does not emit photons.
[9] G. Barton, Phys. Rev. **146**, 1149 (1966).

than the proton. The question, then, is whether Dashen's positive D_{n-p} reflects quantum-relativistic effects neglected in the above argument, or is a spurious result.

The calculation of a photon-exchange driving term involves two difficulties. The first is the fact that the N/D method, with the Born approximation to the left cut, yields bound states whose binding energy tends to depend in the wrong sense on any long-range input forces present.[10] Here, "long range" simply means long by comparison with the separation between the bound particles. This problem is present even where no infrared divergences are involved, and would still exist if the photon had a mass of, say, 1 keV. In Sec. III, we demonstrate the incorrect dependence, and discuss how it can be overcome in perturbation calculations by using a modified technique originally proposed by DF[1] as a solution to the infrared difficulties. For the case of s-wave potential scattering, we prove that if the perturbing potential is nonsingular at short distances, then the modified method not only leads to correct signs, but results in a mass shift which agrees precisely with the correct first-order expression $\int_0^\infty dr\, \delta V(r)|\psi(r)|^2$ in the limit of a very short-range unperturbed potential.[11]

The second, not unrelated, difficulty is the spurious infrared divergence which appears in an approximate calculation of the driving term with the *unmodified* DF mass-shift expression. This is briefly discussed in Sec. IV. We recall that in calculating D_{n-p} Dashen actually used the divergent unmodified formula, subtracting an infinity at the end according to a prescription which DF present.[1,2] Barton has shown[9] that in potential theory this procedure can easily lead to a driving term of incorrect sign.

We argue that since in potential theory the *modified* formula treats the signs of long-range perturbations correctly, and is free of infrared divergences, this is the method which should be applied to the evaluation of the neutron-proton driving term. In Sec. V we calculate D_{n-p} with this approach using fully relativistic kinematics. Our result is

$$D_{n-p}=-5.7\text{ MeV}, \qquad (1.3)$$

whose sign agrees with the physical expectation, and contradicts the original calculation based on the unmodified formula. Speculations are made in Sec. VI.

II. RESPONSE OF BINDING ENERGY TO SHIFT IN EXCHANGED PARTICLE MASS

We consider, with Sawyer,[7] a static model of a meson-baryon bound state b of mass W_b. Denote the meson energy by ω, the external baryon mass by M, and suppose that in the partial wave where the bound state occurs, the input force cut is a single pole

$$R_p/(\omega-\omega_p) \qquad (2.1)$$

representing u-channel exchange of a particle e of mass M_e. Since (2.1) is assumed to produce a bound state, R_p must be attractive (i.e., positive).

Central to the DF mass-splitting calculations is the evaluation of the response coefficient $A_{be}{}^{MM}=dW_b/dM_e$ in such a model. Sawyer argues[7] that $A_{be}{}^{MM}$ must be positive, since an increase in M_e with all couplings held fixed would seem to be a decrease in the range of the input force, with the over-all strength held constant. Since the input force is attractive, this is a repulsive perturbation, so b is expected to become less tightly bound. In terms of the mathematics, an increase in M_e with fixed couplings corresponds to a change in the pole position ω_p, at fixed R_p. Sawyer notes that since the input pole is moving to the left, the attractive amplitude (2.1) is becoming numerically smaller at each energy ω above physical threshold. Thus, he reasons, the change in the input force is repulsive, so that again W_b is expected to increase. However, applying the DF mass-shift formula

$$\delta\omega_b=\frac{1}{R_b[D'(\omega_b)]^2}\frac{1}{\pi}\int_L\frac{D^2(\omega')\,\mathrm{Im}\delta f(\omega')}{\omega'-\omega_b}d\omega', \qquad (2.2)$$

and assuming a linear D function, Sawyer obtains the result $dW_b/dM_e<0$. [In (2.2), ω_b and R_b are the position and residue of the bound-state pole in the ω plane; $W_b=\omega_b+M$. $D(\omega)$ and $D'(\omega)$ are the unperturbed D function and its derivative. $f(\omega)$ is the partial wave of interest, and $\mathrm{Im}\delta f(\omega)$ the change in the left cut corresponding to the perturbation. $\int_L$ denotes an integral over left cuts.] A negative dW_b/dM_e also results from writing the unperturbed $f(\omega)$ as N/D and differentiating directly, provided that the unperturbed b is not too lightly bound.[7]

One is tempted to conclude that a calculation of dW_b/dM_e by the N/D method, or by the DF formula which is equivalent to it to first order, is simply wrong. However, we note that the contradiction between intuition and the N/D result also occurs in ordinary s-wave potential scattering. Consider a potential-theoretic s-wave amplitude f whose left cut is a single pole

$$R_p/(\nu-\nu_p) \qquad (2.1')$$

as before, and which has a bound state at $\nu=\nu_b$. (For nonrelativistic scattering we replace the energy variable ω by ν, the square of the center-of-mass momentum q.) Apply the same perturbation as before; namely, move the input pole to the left without changing the residue. Sawyer's argument can be made just as before, with the conclusion that this perturbation is repulsive. But again the DF formula (2.2) (ω replaced by ν), with the unperturbed D function taken to be a straight line, gives $\delta\nu_b<0$ (see Fig. 1). Of course, the equivalent differentiation in an explicit N/D representation for $f(\nu)$ also gives $\delta\nu_b<0$, provided as before that the unperturbed binding is not too small.

[10] B. Kayser (unpublished).

[11] This generalizes what DF find in Ref. 1 for a specific choice of perturbing potential $\delta V(r)$.

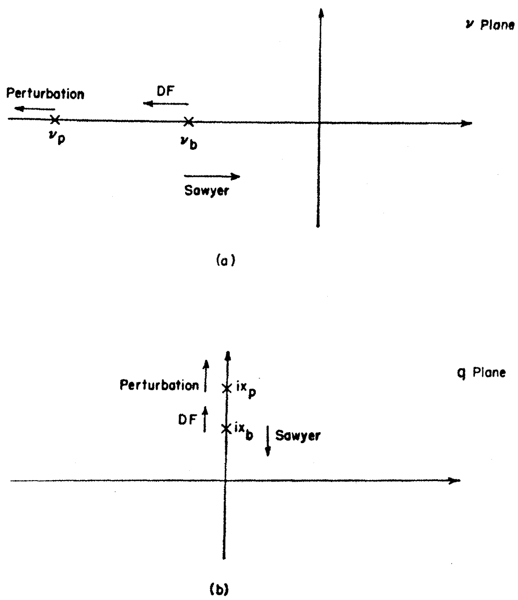

FIG. 1. Motion of the input and bound state poles of $f(\nu)$ in the ν plane (a) and the q plane (b). $\nu_{p,b}=-x_{p,b}^2$. The "Sawyer" arrow indicates the change in binding expected from Sawyer's argument, and the "DF" arrow the change given by the DF formula if a linear D is assumed.

Now in potential theory one knows that an N/D calculation with the exact left cut as input yields the same partial-wave amplitude as would be found by solving Schrödinger's equation. Hence, results obtained from the N/D representation, or the first-order DF formula (2.2), are necessarily correct. This means that the intuitive argument which seems to show that movement of the input pole to the left at fixed residue is a repulsive perturbation must be invalid. The left cut corresponding to some potential (or relativistic interaction) is not the potential itself, and what we learn here is that the often made identification between these two things cannot be pushed too far.

To find out what kind of a perturbation motion of the input pole to the left really is, one must find the potential $V(r)$ whose s-wave left cut is a single pole, and see what change in V corresponds to moving the pole. But before doing this, we determine the precise conditions under which $\delta\nu_b/\delta\nu_p$ has the "counter-intuitive" positive sign. From the knowledge that the only singularities of $f(\nu)$ on the left are the poles at ν_p and ν_b, and from the requirement that the S matrix $S(q)\equiv e^{2i\eta}$ have the usual analyticity and unitarity properties and approach unity as $q\rightarrow\infty$, we know immediately that

$$S(q)=\frac{q+ix_p}{q-ix_p}\frac{q+ix_b}{q-ix_b}. \tag{2.3}$$

Here $x_{p,b}\equiv|q_{p,b}|\equiv|\sqrt{\nu_{p,b}}|$. From (2.3), the residue R_p of the input pole of $f(\nu)=(S-1)/2iq$ at ν_p is

$$R_p=2x_p\frac{x_p+x_b}{x_p-x_b}. \tag{2.4}$$

Solving for x_b, we have

$$x_b=x_p\frac{R_p-2x_p}{R_p+2x_p}. \tag{2.5}$$

Motion of the pole (2.1′) to the left amounts to increasing x_p at fixed R_p (see Fig. 1). From (2.5) and (2.4), this will cause x_b to change by

$$\delta x_b=\left.\frac{\partial x_b}{\partial x_p}\right|_{R_p}\delta x_p=\frac{x_b^2+2x_px_b-x_p^2}{2x_p^2}\delta x_p. \tag{2.6}$$

As is well known, the bound state is constrained to lie to the right of the input pole in the ν plane, so that x_b is restricted to the interval $0<x_b<x_p$. From (2.6),

$$\left.\partial\nu_b/\partial\nu_p\right|_{R_p}\ \begin{array}{l}\text{is negative for } 0<x_b<(\sqrt{2}-1)x_p,\\ \text{is positive for } (\sqrt{2}-1)x_p<x_b<x_p.\end{array} \tag{2.7}$$

Lightly bound states act as though the perturbation were repulsive, and deeply bound ones act as though it were attractive.

To understand why this is so, one can find the potential which produces the S matrix (2.3), using the Gel'fand-Levitan-Marchenko equations.[12,13] Written in terms of the input pole parameters, this potential is

$$V(r)=\frac{-4R_px_pe^{-2x_pr}}{[(R_p/2x_p)e^{-2x_pr}+1]^2}. \tag{2.8}$$

$V(r)$ is nonsingular and attractive at all r. But note the involved way in which it depends on the pole position and residue. An increase in x_p at fixed R_p is evidently not a simple decrease in range at fixed strength. Rather, this increase corresponds to a change in the potential given by

$$\delta V(r)=\left.\frac{\partial V(r)}{\partial x_p}\right|_{R_p}\delta x_p=\frac{4R_pe^{-2x_pr}}{[(R_p/2x_p)e^{-2x_pr}+1]^3}\times\left\{\frac{-R_p}{2x_p}(2x_pr+3)e^{-2x_pr}-1+2x_pr\right\}\delta x_p. \tag{2.9}$$

$\delta V(r)$ is seen to be attractive near the origin, and repulsive at large distances. A graph is given in Fig. 2 for the case $R_p=6x_p$ (or $x_b/x_p=\frac{1}{2}$).

Whether the perturbation (2.9) will increase or de-

[12] V. Marchenko, Dokl. Akad. Nauk SSSR **104**, 433 (1955).

[13] There is actually a one-parameter infinity of potentials which all produce the same s-wave $S(q)$, (2.3) [V. Bargmann, Rev. Mod. Phys. **21**, 488 (1949)]. Any of them will lead to (2.7), which follows from $S(q)$ alone. We shall consider the particular one obtained from the Gel'fand-Levitan-Marchenko equations only to show in familiar terms how (2.7) can come about.

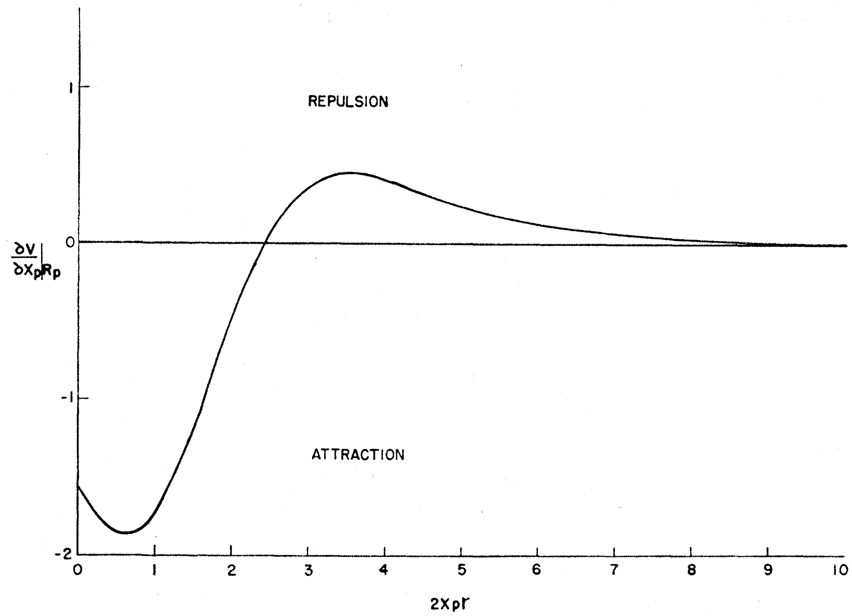

FIG. 2. Graph of $\delta V(r)$, in arbitrary units, for the case $x_b/x_p=\frac{1}{2}$.

crease the binding obviously depends on how tightly the particle is bound to begin with. The wave function of a deeply bound state will be concentrated near the origin, so that the attractive well of δV will be more strongly felt than the repulsive hump, and the binding will increase; similarly for the lightly bound state, whose wave function has a long tail. This explains the behavior summarized by (2.7).[14]

From this potential-theory discussion it is clear that the sign obtained for A_{be}^{MM} by the DF formula in the static model is *not physically absurd*, but one may still ask whether it is *correct*. One source of error is the fact that in any realistic situation the left cut is not a single pole, even though it is taken to be one by way of approximation. So let us see how the N/D result compares with the correct answer when a particle is bound by the potential

$$V(r)=-ge^{-\mu r}, \tag{2.10}$$

but the s-wave left cut is approximated by the single pole which happens to be the Born amplitude of this V. In the notation of (2.1′),[15]

$$\nu_p=-(\tfrac{1}{2}\mu)^2,\quad R_p=g/2\mu. \tag{2.11}$$

The perturbation of interest is still defined as a leftward shift of ν_p at constant R_p, and an N/D solution for $f(\nu)$ leads to (2.7) as before. But now the δV one has in mind when shifting ν_p (increasing x_p) is

$$\delta V(r)=\frac{\partial V}{\partial x_p}\bigg|_{R_p}\delta x_p=4R_pe^{-2x_pr}(2x_pr-1). \tag{2.12}$$

(Notice that here again motion of the input pole at fixed residue does not correspond to a change in the range of V at fixed strength.) The δV (2.12) is repulsive at large r and attractive at small, so the behavior (2.7) is qualitatively correct. For the true s-wave bound state produced by V,

$$\frac{\partial\nu_b}{\partial\nu_p}\bigg|_{R_p}\equiv\frac{\partial\nu_b}{\partial(-\mu^2/4)}\bigg|_{g/2\mu}$$

changes from negative to positive when $(\sqrt{g})/\mu$ exceeds 2.6.[16] By comparison, from (2.7), (2.5), and (2.11), $\partial\nu_b/\partial\nu_p|_{R_p}$ for the bound state computed by N/D changes sign when $\nu_b/\nu_p=(\sqrt{2}-1)^2=0.2$, corresponding to $(\sqrt{g})/\mu=2.2$.

In the relativistic problem, one is interested in the response to variation of an exchanged particle's mass, and it may be that the potential-theory perturbation which most closely parallels this is a variation of a range parameter at fixed potential strength, rather than the motion of an input pole at fixed residue. Suppose, then, that in the exponential potential (2.10) we let $\mu\to\mu+\delta\mu$, without changing g. This perturbation being unambiguously repulsive, the true bound state in $f(\nu)$ must move to the right. From (2.5) and (2.11), the bound state computed by N/D from the Born pole input occurs at

$$x_b=\tfrac{1}{2}\mu\frac{g-2\mu^2}{g+2\mu^2}. \tag{2.13}$$

Hence N/D gives

$$\partial x_b/\partial\mu\,|_g=[g^2-8g\mu^2-4\mu^4]/[2(g+2\mu^2)^2]. \tag{2.14}$$

[14] The conclusion $\delta\nu_b/\delta\nu_p>0$ from the DF formula (2.2) depends on the assumption that D is linear in the interval from ν_p to ν_b. The closer ν_b is to threshold, the larger this interval is, and from (2.7) the linear approximation is seen to become totally inadequate for $\nu_b>(\sqrt{2}-1)^2\nu_p$.

[15] The external reduced mass m is normalized to $2m=1$.

[16] This may be found by taking the appropriate derivative in a linear approximation to the s-wave bound state condition $J_{2x_b/\mu}[2(\sqrt{g}/\mu^2)]=0$ for an exponential potential. J is the Bessel function.

This derivative has the correct negative sign for $g/\sqrt{\mu}<2.9$, but for larger values is positive. $g/\sqrt{\mu}<2.9$ corresponds, from (2.13), to an N/D bound state with $\nu_b/\nu_p<0.38$. States more deeply bound than this move in the wrong direction.

As a final test, we consider the same perturbation $\mu\to\mu+\delta\mu$ at constant g for the Yukawa potential

$$V(r)=-g\frac{e^{-\mu r}}{r}. \tag{2.15}$$

In Born approximation, the s-wave amplitude is now

$$f_{\text{Born}}(\nu)=\frac{g}{4\nu}\ln\left(1+\frac{4\nu}{\mu^2}\right), \tag{2.16}$$

which has a cut running from $\nu=-\infty$ to $\nu=-\frac{1}{4}\mu^2$. We imagine that an N/D computation of $f(\nu)$ has been made with this cut as input, and has yielded a bound state at $\nu=\nu_b$. We shall suppose that g and μ are such that $-\frac{1}{4}\mu^2<\nu_b<0$.

If now $\mu\to\mu+\delta\mu$ at fixed g, the true bound state must move to the right. The behavior of the N/D bound state may be found from the DF formula (2.2) (with $\omega\to\nu$), which from its derivation[1] may be seen to give the shift in the bound-state pole resulting from a given change in the left cut, correct to first order in this change. In the present case, the perturbation of the left cut is

$$\operatorname{Im}\frac{g}{4\nu}\left[\ln\left(1+\frac{4\nu}{(\mu+\delta\mu)^2}\right)-\ln\left(1+\frac{4\nu}{\mu^2}\right)\right]=-\frac{\pi g}{4\nu},$$
$$-\tfrac{1}{4}(\mu+\delta\mu)^2<\nu<-\tfrac{1}{4}\mu^2. \tag{2.17}$$

Then

$$\delta\nu_b=\frac{-1}{R_b[D'(\nu_b)]^2}\int_{-(\mu+\delta\mu)^2/4}^{-\mu^2/4}\frac{D^2(\nu')g}{(\nu'-\nu_b)4\nu'}d\nu'. \tag{2.18}$$

The residue R_b of an s-wave bound state must be negative, so that for $-\frac{1}{4}\mu^2<\nu_b$, the N/D bound state responds correctly.

The potential-theory tests we have applied show that, in general, the N/D bound state based on an approximate left cut moves in the proper direction when the input pole is shifted and when the range of the potential is changed, at least if the unperturbed binding is not too great. This gives one hope that the reliability of relativistic calculations of $A_{be}{}^{MM}$ is not compromised by the use of crude left cuts. However, the relativistic application of the DF formula (2.2) involves a second source of error, namely, one's ignorance of the actual unperturbed D function.[17] Dashen and Frautschi always use a straight line, or else the Balázs-type D function

$$D(W)=(W-W_b)\frac{W_b-M'}{W-M'}, \tag{2.19}$$

with the parameter M' restricted to the range $M'=2W_b$ to $M'=\infty$.[2,4] We note that in the static model, where the left cut is a single pole, the choice of a straight-line D function or of (2.19) with M' in the indicated range immediately commits one to a negative $A_{be}{}^{MM}$. This follows from (2.2) ($\omega\to W$), according to which the sign of dW_b/dM_e depends on the choice of D through the factor

$$\frac{d}{dW}\left[\frac{D^2(W)}{W-W_b}\right]\Bigg|_{W_p}. \tag{2.20}$$

The denominator functions which DF use have been criticized by Shaw and Wong,[18] who give the result of calculating the derivative (2.20) from a number of more elaborate forms appropriate to the $p_{11}\,\pi N$ state. Although there are wide numerical differences, we observe that the sign never differs from that obtained from a straight line. This lends at least some support to the view that, so far as sign is concerned, the linear or Balázs-type D is adequate for the computation of $A_{be}{}^{MM}$.

III. TREATMENT OF LONG-RANGE FORCES

We turn now to a discussion of the electromagnetic driving term on which the bootstrap explanation of M_n-M_p is based. As mentioned in the Introduction, a major obstacle to the evaluation of this term is that N/D calculations based on the Born approximation to the left cut produce bound states whose binding energy tends to depend in the wrong sense on any long-range input forces. To exhibit this improper dependence, we give here a modified version of the simple potential-theory example originally presented in a previous paper.[10]

Suppose a pair of particles interact via the potential

$$V(r)=V_S(r)+V_L(r), \tag{3.1}$$

which has an attractive, unspecified short-range part $V_S(r)$, and a long-range part which for simplicity we take to be

$$V_L(r)=-ge^{-\mu r}, \tag{3.2}$$

with $g>0$ and μ small. Suppose further that V is strong enough to produce an s-wave bound state. We know that if g is increased the binding energy will increase.

Now imagine solving the problem by the N/D method, taking the input left cut from the s-wave Born amplitude. V_S being short range, its Born amplitude will only be singular well to the left of threshold. The Born term coming from V_L is just a single pole, located at a point $\nu=\nu_L$ quite near threshold if V_L is long range. By adjusting the strength of V_S, we can arrange that N/D have a bound-state pole between the V_S cut and the V_L pole (Fig. 3).

The residue R_L of the V_L pole must be positive, since

[17] All of our potential-theory tests used the actual D function corresponding to the unperturbed left cut.

[18] G. Shaw and D. Wong, Phys. Rev. **147**, 1028 (1966).

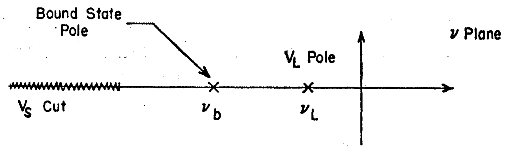

FIG. 3. The $\nu=q^2$ plane, with a bound-state pole at $\nu=\nu_b$ between the Born cut of V_S and the Born pole of V_L.

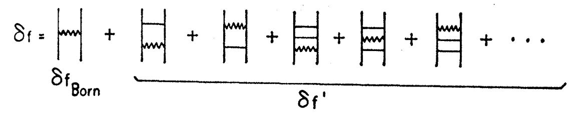

FIG. 4. Diagrams contributing to the first-order (in g) change in the s-wave amplitude. A wiggly line represents V_L acting once; a solid line, V_S acting once.

the Born amplitude corresponding to an attractive potential is positive in the physical region ($\nu>0$). If g (hence R_L) is increased slightly, the position ν_b of the bound-state pole will be shifted by an amount

$$\delta\nu_b=\frac{1}{R_b[D'(\nu_b)]^2}\frac{1}{\pi}\int_{-\infty}^{0}\frac{D^2(\nu')\,\mathrm{Im}\delta f(\nu')}{\nu'-\nu_b}d\nu', \quad (3.3)$$

with $\mathrm{Im}\delta f(\nu)$, the change in the left cut, given by

$$\mathrm{Im}\delta f(\nu)=\mathrm{Im}\left(\frac{\delta R_L}{\nu-\nu_L}\right)=-\pi\delta R_L\delta(\nu-\nu_L). \quad (3.4)$$

Since $\nu_L-\nu_b$ is positive (and the residue R_b of an s-wave bound state is negative), $\delta\nu_b$ is positive, which is the wrong sign. By imagining that g is gradually increased from zero to the value of interest, we can see from (3.3) and (3.4) that the entire effect of the attractive potential (3.2) on the binding energy is repulsive.[19]

In relativistic bootstrap models, the left cut is usually based on single-particle exchange diagrams, the higher-order contributions being neglected just as they were above. There can be little doubt that if the N/D method, with the Born approximation to the left cut, sometimes leads to completely incorrect dependence of binding energy on input parameters in potential theory, the same will happen in relativistic applications.[20] This suggests that when there is a bound state in the problem, especially a relatively deeply bound one whose pole lies near or atop the left cuts, an effort should be made to supply a better approximation to these cuts, or perhaps some technique other than the N/D method should be used.

The computation of the electromagnetic driving term corresponds, in the potential model, to the case where V_L is initially absent. One may imagine that the unperturbed problem has been solved with the exact left cut produced by V_S as input. Perturbation of the potential by the introduction of V_L, with a small value of g, will then result in a mass shift $\delta\nu_b$ which is given to first order in g by (3.3), provided that one supplies the change in the left cut to equal accuracy. Note that to first order in g (and all orders in V_S), the change in the left cut receives contributions from all the diagrams in Fig. 4.

Let us write the first-order change in the s-wave amplitude as $\delta f=\delta f_{\mathrm{Born}}+\delta f'$, where δf_{Born} is the Born amplitude for V_L, and $\delta f'$ is the sum of the remaining diagrams. In practice, the strong binding potential V_S is unknown, so the only part of δf which can be computed explicitly is δf_{Born}. Thus, a reliable approximate formula for the first order $\delta\nu_b$ that requires only a knowledge of δf_{Born} would be desirable. Unfortunately, if ν_L is to the right of ν_b, then (3.3) yields a mass shift of incorrect sign if one substitutes $\mathrm{Im}\delta f_{\mathrm{Born}}$, Eq. (3.4), for $\mathrm{Im}\delta f$.[21,22]

With V_S of short range, the neglected singularities $\mathrm{Im}\delta f'$ are far to the left. Hence a dispersion integral for $\delta\nu_b$ with faster convergence than that of (3.3) might help. Such an expression is the formula introduced by DF as a way of handling infrared divergences[23]:

$$\delta\nu_b=\frac{1}{R_b[D'(\nu_b)]^2}\frac{1}{\pi}\times\int_{-\infty}^{0}\frac{D^2(\nu')\,\mathrm{Im}[\delta f(\nu')-\delta f_{\mathrm{Born}}(\nu')S(\nu')]}{\nu'-\nu_b}d\nu'. \quad (3.5)$$

Here $S=e^{2i\eta}$ is the unperturbed S matrix. Like (3.3), (3.5) gives the mass shift to first order in the perturbation, if the full first-order change in the left cut, $\mathrm{Im}\delta f$, is supplied.

Consider an arbitrary long-range perturbing potential $\delta V(r)$ which, however, has no important short-range effects, so that one may assume that the cuts of the corresponding δf_{Born} extend only a finite distance into the left half-plane. Then (still assuming the unperturbed potential V_S to be short range) the cuts of δf_{Born} and S do not overlap, so that

$$\mathrm{Im}(\delta f-\delta f_{\mathrm{Born}}S) =(2x/D)(N\,\mathrm{Im}\delta f_{\mathrm{Born}}+\delta f_{\mathrm{Born}}\,\mathrm{Im}N)+\mathrm{Im}\delta f'. \quad (3.6)$$

Here we have used $(S-1)/2iq=N/D$, and written x

[19] From (3.3) it follows that if ν_b is to the left of ν_L for $g=0$, it will remain to the left for all values of g.

[20] A repulsive effect was noticed where an attractive one was expected in B. Kayser, Phys. Rev. **138**, B1244 (1965).

[21] That (3.3), with the approximation $\mathrm{Im}\delta f=\mathrm{Im}\delta f_{\mathrm{Born}}$, leads to a sign error when applied to a long-range perturbation has also been noted by Barton (Ref. 9).

[22] When $\mathrm{Im}\delta f$ is approximated by $\mathrm{Im}\delta f_{\mathrm{Born}}$, (3.3) can also yield an incorrect sign when the perturbing potential is not long range and all its Born cuts lie to the left of ν_b. Explicit examples can easily be constructed.

[23] We thank S. Frautschi for suggesting the application of this formula to long-range, nondivergent perturbations.

for $|q|$. Equation (3.5) now becomes

$$R_b[D'(\nu_b)]^2\delta\nu_b=\frac{1}{\pi}\int_{-\infty}^{0}\frac{2x'D(\nu')N(\nu')\,\mathrm{Im}\delta f_{\mathrm{Born}}(\nu')}{\nu'-\nu_b}d\nu'+\frac{1}{\pi}\int_{-\infty}^{0}\frac{2x'D(\nu')\delta f_{\mathrm{Born}}(\nu')\,\mathrm{Im}N(\nu')}{\nu'-\nu_b}d\nu'+\frac{1}{\pi}\int_{-\infty}^{0}\frac{D^2(\nu')\,\mathrm{Im}\delta f'(\nu')}{\nu'-\nu_b}d\nu'. \quad (3.7)$$

The last two integrals in this expression both represent "far left" contributions. But the last integral is not small, since it is precisely what was neglected when (3.3) was applied to the exponential perturbation with a resulting error in sign. In turns out, however, that if the original potential is of very short range, the two far left integrals cancel each other. We show this by proving that under this condition the first integral in (3.7) leads to the correct mass shift by itself.

With V_S very short range, one may approximate the unperturbed amplitude near threshold (a region including ν_b and the cuts of δf_{Born} but not those of f) by a zero-effective-range formula[1]:

$$f(\nu)\cong\frac{1}{iq_b-iq}=\frac{1}{x-x_b}, \quad (3.8)$$

where $x_b\equiv|q_b|\equiv|\sqrt{\nu_b}|$. From (3.8), $N=1$, $D=x-x_b$, and $R_b[D'(\nu_b)]^2=1/(-2x_b)$. Thus, keeping only the "nearby" integral of (3.7), we have

$$\delta\nu_b\cong-\frac{4x_b}{\pi}\int_{-\infty}^{0}\frac{x'(x'-x_b)\,\mathrm{Im}\delta f_{\mathrm{Born}}(\nu')}{\nu'-\nu_b}d\nu'. \quad (3.9)$$

This expression requires only a knowledge of the cuts of δf_{Born}. Further, if tested on a long-range exponential perturbation, it will obviously yield a $\delta\nu_b$ of correct sign whether the pole of δf_{Born} is to the right or left of ν_b, because the factor $x'-x_b$ (the D function) changes sign at the same point as $\nu'-\nu_b$ does.

We proceed to show that the mass shift given by (3.9) actually agrees with the exact first-order expression

$$\delta\nu_b=\int_0^{\infty}|\psi_b(r)|^2\delta V(r)dr, \quad (3.10)$$

where ψ_b is the bound-state wave function.[15] To this end, consider the function $F(\nu)$ defined by

$$F(\nu)=(\sqrt{-\nu})[(\sqrt{-\nu})-(\sqrt{-\nu_b})]\delta f_{\mathrm{Born}}(\nu). \quad (3.11)$$

For ν real, negative,

$$\mathrm{Im}F(\nu)=x(x-x_b)\,\mathrm{Im}\delta f_{\mathrm{Born}}(\nu), \quad (3.12a)$$

while for ν real, positive,

$$\mathrm{Im}F(\nu)=x_b(\sqrt{\nu})\delta f_{\mathrm{Born}}(\nu). \quad (3.12b)$$

Consistent with the idea that δV is not singular at short distances (for instance, if δV is a realistic representation of photon exchange, it will be cut off at small r by form factors), we *assume* that at large ν

$$\delta f_{\mathrm{Born}}(\nu)\to\delta A/\nu \quad (3.13)$$

(with δA possibly zero). Then $F(\nu)\to-\delta A$, $\nu\to\infty$, and we can write

$$F(\nu)=-\delta A+\frac{1}{\pi}\int_{-\infty}^{0}\frac{x'(x-x_b)}{\nu'-\nu}\,\mathrm{Im}\delta f_{\mathrm{Born}}(\nu')d\nu'+\frac{x_b}{\pi}\int_0^{\infty}\frac{(\sqrt{\nu'})\delta f_{\mathrm{Born}}(\nu')}{\nu'-\nu}d\nu'. \quad (3.14)$$

Since $F(\nu_b)$ vanishes, (3.14) enables us to rewrite (3.9) as

$$\delta\nu_b=-4x_b\delta A-\frac{4\nu_b}{\pi}\int_0^{\infty}\frac{(\sqrt{\nu'})\delta f_{\mathrm{Born}}(\nu')}{\nu'-\nu_b}d\nu'. \quad (3.15)$$

In terms of $\delta V(r)$, the s-wave Born amplitude δf_{Born} is

$$\delta f_{\mathrm{Born}}(\nu)=-\frac{1}{\nu}\int_0^{\infty}dr\,\delta V(r)\sin^2qr. \quad (3.16)$$

Substitution of this relation into the integral in (3.15) yields

$$-\frac{4\nu_b}{\pi}\int_0^{\infty}\frac{(\sqrt{\nu'})\delta f_{\mathrm{Born}}(\nu')}{\nu'-\nu_b}d\nu'=-2x_b\int_0^{\infty}dr\,\delta V(r)(1-e^{-2x_br}). \quad (3.17)$$

As for δA, (3.16) gives

$$\delta f_{\mathrm{Born}}=-\frac{1}{\nu}\int_0^{\infty}dr\delta V(r)\times\tfrac{1}{2}(1-\cos 2qr)\xrightarrow[\nu\to\infty]{}-\frac{1}{2\nu}\int_0^{\infty}dr\delta V(r), \quad (3.18)$$

assuming that δV is smooth in r so that the integral over the rapidly oscillating $\cos 2qr$ drops out. Thus

$$-4x_b\delta A=2x_b\int_0^{\infty}dr\,\delta V(r), \quad (3.19)$$

and the mass shift given by (3.9) is related to the perturbing potential by

$$\delta\nu_b=2x_b\int_0^{\infty}dr\,\delta V(r)e^{-2x_br}. \quad (3.20)$$

Since the unperturbed potential is very short range, the bound-state wave function may be taken to be

$$\psi_b(r)=(2x_b)^{1/2}e^{-x_br} \quad (3.21)$$

for all r, so that (3.20) and the exact first-order formula (3.10) are identical.[24,25]

This result suggests that the relativistic photon-exchange driving term could be calculated with fair accuracy using the first integral of (3.7), with some simple parametrization for the unperturbed N and D, and with x' replaced by the relativistically appropriate kinematical factor. Of course, the N^* exchange force which binds π and N together to make the unperturbed nucleon is not of *very* short range; its left cuts are not extremely far to the left of the nucleon bound-state pole. Furthermore, in potential theory the p-wave analog of (3.9), obtained by making the zero-range approximation to the unperturbed amplitude,

$$\frac{S(p-\text{wave})-1}{2iq^3}=\frac{N}{D}\cong\frac{1}{iq_b{}^3-iq^3}, \qquad (3.22)$$

does not yield a $\delta\nu_b$ which agrees exactly with $\int|\psi_b(\mathbf{r})|^2 \times\delta V(r)d^3r$, even when the unperturbed potential is of very short range.[9] However, the use of the first term of (3.7) will, at least, not lead to the sign error encountered when applying (3.3) to a long-range perturbation and keeping only the δf_{Born} cuts. This is because the integrand of this term involves only one power of D, which changes sign at $\nu'=\nu_b$, counteracting the sign change of $1/(\nu'-\nu_b)$.

IV. INFRARED DIVERGENCE

For the perturbation $e^{-\lambda r}/r$, the use of the δf_{Born} cut approximation to (3.3),

$$\delta\nu_b\cong\frac{1}{R_b[D'(\nu_b)]^2}\frac{1}{\pi}\int_{-\infty}^{0}\frac{D^2(\nu')\,\text{Im}\delta f_{\text{Born}}(\nu')}{\nu'-\nu_b}d\nu', \qquad (3.3')$$

not only gives wrong-sign contributions from the long-range parts of the potential, but leads to a spurious infrared divergence when $\lambda\to 0$. (The correct first-order shift, $\int|\psi_b|^2e^{-\lambda r}/rd^3r$, is finite in this limit.) DF suggest that (3.3′) can nevertheless be used to calculate electromagnetic mass shifts, provided that one drops an infinity at the end of the calculation according to the following prescription[1,2] (in potential-theory language):

(1) Express the partial-wave Born amplitude δf_{Coul} produced by the perturbation $e^{-\lambda r}/r$ in the form

$$\delta f_{\text{Coul}}=F(\nu)\ln\left[\frac{\lambda^2}{G(\nu)}\right]$$
$$+\text{terms which vanish as }\lambda\to 0. \qquad (4.1)$$

(2) Compute $\delta\nu_b$ by (3.3′), with δf_{Born} corresponding to a potential representing exchange of a photon of mass λ. (Since this potential should include effects of form factors, it will be more complicated than $e^{-\lambda r}/r$.)

(3) In the result of (2), drop the term diverging as $\ln[\lambda^2/|G(\nu_b)|]$, and take the limit $\lambda\to 0$.

Of course, the result obtained by subtracting the $\ln(\lambda^2/|G|)$ term could be of correct sign, even though the $\delta\nu_b$ given by (3.3′) before subtraction is not. However, Barton has tested this subtraction procedure in an s-wave potential-theory model in which photon exchange is represented by the potential[1]

$$\delta V(r)=-g\left[\frac{e^{-\lambda r}-e^{-mr}}{r}-\frac{m^2-\lambda^2}{2m}e^{-mr}\right], \qquad (4.2)$$

whose Born amplitude,

$$\sim\frac{m^2}{t-m^2}\frac{1}{t-\lambda^2}\frac{m^2}{t-m^2},$$

where t is the (momentum transfer)^2, is appropriate to exchange of a photon of mass λ with a form factor dominated by a particle of mass m at each vertex. He obtains[9]

$$\delta\nu_b=\frac{gx_b{}^2}{4R_b}\ln\left(\frac{4ex_b{}^2}{m^2}\right), \qquad (4.3)$$

where $e=2.718\cdots$. Noting that (4.2) is attractive at all r, we see that for the physically interesting case $m^2>4ex_b{}^2$, (4.3) has the wrong sign.

In applying (3.3′), Barton took $D(\nu)\cong\nu-\nu_b$, which parallels Dashen's relativistic procedure, but results in serious overweighting of δf_{Born} singularities far from ν_b. In order to assess the effect of improving the D function, we have repeated the calculation using the zero-effective-range expression $D(\nu)=x-x_b$. We find

$$\delta\nu_b=-x_bg\left\{\ln\frac{(x_b+x_m)^2}{x_bx_m}-\frac{1}{2}\frac{x_m-x_b}{x_m+x_b}\right\}, \qquad (4.4)$$

where $x_m\equiv\sqrt{(\tfrac{1}{4}m^2)}$. One may readily verify that this mass-shift has the correct negative sign for all x_m/x_b.

For an unperturbed potential of very short range, the actual first-order shift due to (4.2) is (as $\lambda\to 0$)

$$\delta\nu_b=2x_b\int_0^\infty e^{-2x_br}\delta V(r)dr$$
$$=-2x_bg\left\{\ln\frac{x_m+x_b}{x_b}-\frac{1}{2}\frac{x_m}{x_m+x_b}\right\}. \qquad (4.5)$$

[24] The assumptions made about the perturbation in demonstrating this equivalence are all satisfied by a finite linear superposition of Yukawas and exponentials, if the coupling constants of the Yukawa terms add up to zero so that the $1/r$ singularity at the origin is removed. The left cuts of δf_{Born} will then extend only a finite distance into the left half-plane, and δf_{Born} will behave as $1/\nu$ asymptotically. A potential of this type has been given as a model of photon exchange with form factors at the vertices by DF [see (4.2)].

[25] The cancellation between the two "far left" integrals of (3.7) which this implies can be seen explicitly in the model of J. Paton [Oxford report (unpublished)], who considers the perturbation δae^{-Kr} on the binding potential $ae^{-\mu r}$. N has a sequence of poles at $\nu_n=-\frac{1}{4}(n\mu)^2$, and $\delta f'$ a sequence of poles at $\nu_n'=-\frac{1}{4}(n\mu+K)^2$; $n=1, 2, \cdots$. From Paton's Eq. (A2.6), the contribution of the ν_n pole to the ImN integral cancels that of the ν_n' pole to the Im$\delta f'$ integral in the limit $K/\mu\to 0$.

If after letting $\lambda \to 0$ one lets $m \to 0$, δV vanishes, and (4.5) goes to zero as it should, but (4.4) diverges. This, of course, is just the spurious infrared divergence inherent in (3.3′), which appears here because the DF subtraction process affects only the contribution from the $e^{-\lambda r}/r$ term, not that from e^{-mr}/r. Thus (4.4) is poor for small values of m. At the opposite extreme, $m \to \infty$, (4.4) and (4.5) differ by a factor of $\frac{1}{2}$. For the nucleon bound state in the πN system, $x_b \approx$ (pion mass), so that if one takes $m \approx$ (ρ mass), $x_m/x_b = 7.2$ and $\delta\nu_b$ (4.4)/$\delta\nu_b$ (4.5) = 0.56.

The results (4.3) and (4.4) show that the $\ln(\lambda^2/G)$ subtraction procedure is not very accurate in any case. Whether Dashen's calculation of the neutron-proton driving term with this technique has at least correctly determined the sign obviously depends on the details of the relativistic problem. We note again that Dashen used a linear D function [or a D of the form (2.19) which leads to an almost identical answer], and that his result conflicts with the sign expected on simple physical grounds.

By contrast, a calculation of D_{n-p} based on the first integral I_1 of (3.7) would seem to be more trustworthy. Considering potentials which do not involve a $1/r$ divergence at large distances, we have shown that for the s wave this approach leads to the formula (3.9), which is accurate in magnitude as well as sign. But now observe that if δV does contain a term $e^{-\lambda r}/r$, the $\lambda \to 0$ limit of $\delta\nu_b$ as given by (3.9) must exist and equal the correct value,

$$\lim_{\lambda\to 0} 2x_b \int_0^\infty e^{-2x_b r} \delta V(r) dr,$$

since (3.9) gives precisely

$$2x_b \int_0^\infty e^{-2x_b r} \delta V(r) dr$$

for any value of λ greater than zero.[26] For the p wave, I_1 [with $\delta f_{\mathrm{Born}} \equiv \sin\eta^p{}_{\mathrm{Born}} \exp(i\eta^p{}_{\mathrm{Born}})/q^3$, and $x' \to -x'^3$] does not lead to a numerically precise mass shift, but one can have confidence in the sign for reasons already given. These reasons remain valid if δV contains an $e^{-\lambda r}/r$ term with small λ, and one may easily show that I_1 develops no infrared divergence as $\lambda \to 0$.[27]

V. DRIVING TERM FOR $M_n - M_p$

Assuming that the π^0 does not emit photons, the only πN states in (1.2) which contribute to the photon-exchange force are $\pi^- p$ and $\pi^+ n$. Since the anomalous magnetic moments of p and n are nearly equal and opposite, their contributions to $M_n - M_p$ cancel. Hence only photon exchange between π^- and p, with Dirac coupling to the proton, need be considered. If this produces a mass shift $\delta M_{\pi^- p}$, the driving term D_{n-p} for $M_n - M_p$ will be [cf. (1.2)]

$$D_{n-p} = \tfrac{2}{3} \delta M_{\pi^- p}. \tag{5.1}$$

The $p_{1/2}$ πN amplitude which is free of kinematical singularities is[28]

$$h(W) = \frac{\sin\eta e^{i\eta}}{q(E-M)/W} = \frac{N}{D}, \tag{5.2}$$

where W is the total center-of-mass energy, and q and E are the corresponding momentum and nucleon total energy. M is the nucleon mass. $h(W)$ has the nucleon pole at $W_b = M$, unitarity cuts for $W > M + \mu$ and $W < -(M+\mu)$, and the well-known dynamical singularities discussed in Ref. 28.

The differences between the relativistic situation and potential theory should not be minimized. The nucleon pole is at $W_b = 6.7$ (in pion mass units), while the important N^* exchange cut lies between $W = 3.8$ and $W = 5.0$, not too far to the left. If we write $\delta h = \delta h_{\mathrm{Born}} + \delta h'$, where δh_{Born} is the photon-exchange Born diagram, and δh is the sum of all diagrams involving one photon line which would contribute to the driving term in an exact first-order calculation, then the relativistic version of (3.6) is

$$\mathrm{Im}[\delta h - \delta h_{\mathrm{Born}} S] = (1/D)\,\mathrm{Im}[\delta h_{\mathrm{Born}}(-2i\rho)N] + \mathrm{Im}\delta h'. \tag{5.3}$$

In (5.3), $-2i\rho \equiv -2iq(E-M)/W$ has a left-hand cut, with an endpoint at $W = M - \mu$ (μ being the π mass), as well as a pole at $W = 0$. As $\lambda \to 0$, these kinematical singularities are infrared divergent because of the δh_{Born} multiplying $(-2i\rho)$. In addition, even if one assumes the cuts of N to have been produced by N^* exchange alone, these cuts do overlap with those of δh_{Born}, since both functions are singular on the imaginary W axis for $\lambda \neq 0$.

This last difficulty is not serious, since the singularity of δh_{Born} on the imaginary W axis comes from a term proportional to $(\lambda^2/4q^2)\ln(1+4q^2/\lambda^2)$. q^2 is not small anywhere on the imaginary axis, so this term vanishes there when $\lambda \to 0$. The contribution of the ρ singularities might not be negligible. The fact that they are infrared divergent is not alarming, however, since the far left integrals of (3.7), which cancel, are also infrared divergent.

Whereas $i\rho$, N, and $\delta h'$ all become singular at or not far to the left of $W = M - \mu$, the only consistent way to approximate the relativistic analog of (3.5) is to neglect all contributions to (5.3) coming from the left of this

[26] As $\lambda \to 0$, the two "far left" integrals in (3.7), which evidently continue to cancel each other, are separately divergent. The first of them diverges because δf_{Born} does. The second must be divergent because we know (3.3′) is, while (3.3), which is the sum of (3.3′) and the integral in question, equals the nondivergent $\delta\nu_b R_b [D'(\nu_b)]^2$.

[27] The nondivergent behavior of I_1 as $\lambda \to 0$ is a consequence of the phase-space factor x' (or x'^3), which vanishes at threshold to soften the divergent behavior of $\mathrm{Im}\delta f_{\mathrm{Born}}$ near that point.

[28] S. Frautschi and J. Walecka, Phys. Rev. **120**, 1486 (1960).

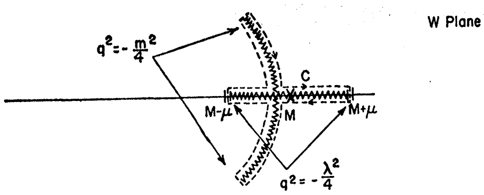

FIG. 5. The contour C around the p-wave cuts of δh_{Born}.

point. We hope that the *sign* of D_{n-p} will be correctly determined by the resulting relation:

$$\delta M_{\pi^- p} \cong \frac{1}{R_b[D'(W_b)]^2}\frac{1}{2\pi i} \times \oint_C \frac{D(W)[-2i\rho(W)]N(W)\delta h_{\text{Born}}(W)}{W-W_b}dW, \quad (5.4)$$

where the contour C (Fig. 5) is around the (p-wave[29]) cuts of the $\pi^- p$ photon-exchange amplitude δh_{Born}. As does the integral I_1 of (3.7), (5.4) involves only one power of $D(W)$, so that δh_{Born} cuts to the right of W_b contribute with the proper sign. Furthermore, because of the factor ρ in the integrand, (5.4) is not infrared divergent in the zero photon mass limit, as may be verified by explicit calculation.

In calculating $\delta h_{\text{Born}}(W,\lambda)$, we take both the π^- and p form factors to have the form $m^2/(m^2-t)$, with a common value of m. (This is motivated by the near equality of m_ω and m_ρ.) For the unperturbed N and D functions, the zero-effective-range parametrization corresponding to that based on (3.8) would be $N=1$, $D=i\rho(W_b)-i\rho(W)$. This would be a poor choice, because $i\rho$ is singular ($\sim[W-(M-\mu)]^{3/2}$) at $W=M-\mu$, where the true D is quite regular. What is worse, the vanishing of $i\rho$ at both $W=M+\mu$ and $W=M-\mu$ implies that the approximate $D'(W)$ would have a zero somewhere between these two points. If for a moment we regard the unperturbed nucleon bound-state position W_b as a variable, we see that for some value of W_b between $M\pm\mu$, $R_b[D'(W_b)]^2=[(N/D')D'^2]|_{W_b}=D'(W_b)$ would vanish. Unless the integral in (5.4) vanishes at the same point, the mass shift, as a function of W_b, will have a pole.

With $N=1$, $D=i\rho(W_b)-i\rho(W)$, (5.4) can be evaluated in closed form. We shall not quote the complicated result. The salient feature is that the integral itself is smooth and nonzero for W_b between $M\pm\mu$, so that the calculated mass shift does have a pole.[30]

A much better choice for N and D is simply $D(W)=W-W_b$, $N=R_b$. The contour integral of (5.4) is then easily performed, and yields

$$\delta M_{\pi^- p} = -\tfrac{1}{2}\alpha m\left[2+\frac{M}{[M^2-(\frac{1}{2}m)^2]^{1/2}}\right], \quad (5.5)$$

where $\alpha=1/137$ is the fine-structure constant.[31] This mass shift, and the corresponding driving term $D_{n-p}=\frac{2}{3}\delta M_{\pi^- p}$, have the sign expected from the simple physical argument. For $m=m_\rho=760$ MeV, (5.5) gives $D_{n-p}=-5.7$ MeV.

VI. SPECULATIONS

If one accepts the result $D_{n-p}<0$, then the fact that the neutron is heavier than the proton must be due to effects other than the electromagnetic forces between the π and N which make up the nucleon bound state. Perhaps the electromagnetic corrections to the strong forces such as N^* exchange play a major role. Estimates of these effects depend sensitively on strong-interaction approximations such as the choice of D function. With the elaborate D functions of Shaw and Wong,[18] the N^* mass splittings make a much larger contribution to M_n-M_p than that originally estimated by Dashen.[2,4] Another possibility is that channels other than πN which can couple to the nucleon are important. With very reasonable assumptions about the magnetic moments of the various charge states of the N^*, both the electric and magnetic forces in the πN^* channel make positive contributions to M_n-M_p.[32,33]

ACKNOWLEDGMENTS

It is a pleasure to acknowledge many useful conversations with colleagues at Berkeley, where much of this work was done. We also wish to thank Professor Steven Frautschi for several private communications.

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[29] Consistent with the neglect of all cuts to the left of $W=M-\mu$, we do not include the s-wave cuts of δh_{Born} in the left-half W plane.

[30] Barton (Ref. 9) has done the corresponding static-model calculation, choosing $\rho=q^3=(\omega^2-\mu^2)^{3/2}$, where $\omega\equiv W-M$, and taking $N=1$, $D=iq_b^3-iq^3$. Curiously, $D'(\omega_b)$ and the integral analogous to that of (5.4) vanish together at $\omega_b=0$, so that the mass shift is well behaved.

[31] In the limit $M\to\infty$, this result reduces to that obtained by Barton (Ref. 9) using the static model and a linear D.

[32] D. S. Beder, Lawrence Radiation Laboratory report (unpublished).

[33] In two recent approaches to the $n-p$ mass difference not based on the composite nucleon picture [H. Pagels, Phys. Rev. **144**, 1261 (1966); H. Fried and T. Truong, Phys. Rev. Letters **16**, 559 (1966)], the role of driving term is played by the γN intermediate-state contribution to the process (nucleon) $\to$ (nucleon $+$graviton), or to the nucleon self-energy. In both cases it is found that the driving term by itself leads to $M_n<M_p$, but that the inclusion of feedback effects can reverse the sign.