Strong-Coupling Theory of Fermion Interactions*

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A strong-coupling approach to the fermion interactions is proposed. This is useful in the determination of the mass of the bound state when the kinetic energy of the constituents is smaller than their rest mass. The quark model is studied by this method. In the strong-coupling limit, the following statement is proved: If we assume (i) Lorentz-invariant and local (nonderivative) interactions, (ii) $SU(3)$ invariance, and (iii) quark and antiquark number conservations in the limit when the quarks are at rest, then (a) the system does not have, in general, any higher invariance than $SU(3)$, (b) nevertheless, every localized eigenstate of the Hamiltonian belongs to the irreducible representation of the nonchiral $U^{(+)}(6)\times U^{(-)}(6)$. Furthermore, if (iv) one restricts oneself to the four-fermion interactions, then (c) the system is exactly solvable, and the mass formula is given by

$$M=m_0N_Q+a(N_Q-6)^2+b[\mathbf{G}^2(3)-4\mathbf{G}^{(+)}(3)\cdot\mathbf{G}^{(-)}(3)]+c[\mathbf{G}^2(6)+2\mathbf{S}^2+6B^2],$$

where m_0, a, b, and c are arbitrary real constants, and N_Q, B, $\mathbf{G}^{(\pm)}(6)$ $[\mathbf{G}(6)=\mathbf{G}^{(+)}(6)+\mathbf{G}^{(-)}(6)]$, $\mathbf{G}^{(\pm)}(3)$ $[\mathbf{G}(3)=\mathbf{G}^{(+)}(3)+\mathbf{G}^{(-)}(3)]$, and $\mathbf{S}$ are the quark and antiquark number, the baryon number, the generators of $SU^{(+)}(6)\times SU^{(-)}(6)$, those of $SU^{(+)}(3)\times SU^{(-)}(3)$, and the spin operator, respectively. It is also shown that only a restricted class of irreducible representations is realized in the quark system. The three-body forces among quarks and the three-triplet model proposed by Han and Nambu are briefly discussed.

I. INTRODUCTION

SINCE the $SU(6)$-group invariance of the hadron interactions conflicts with Lorentz invariance,[1] one of the most natural views on this difficulty may be that $SU(6)$ is a dynamical symmetry group of some particular problem, for example, energy-level classification in the static limit of quark motion. In fact, the success of the nonrelativistic, phenomenological quark-model approach,[2] which has recently attracted considerable attention, seems to be based on the notion that the quark interaction is so strong that the rest mass is mostly compensated for by the potential energy, while the kinetic energy is very small compared with the former. This suggests that one should adopt a quite different approach from that of the conventional perturbation theory, in which one takes the free particle as an unperturbed state.

In this paper, a new approach to the strongly interacting fermion system is presented.

In a given Hamiltonian (or Lagrangian) divided into two parts, the kinetic-energy and the interaction terms, the latter is diagonalized using the current-algebra technique. The energy spectrum is obtained in terms of group generators. Subsequently, the kinetic-energy part is taken into account as a perturbation. This method is especially useful in obtaining the s-wave bound states of particles.

In the case of fermion-boson or boson-boson interactions, the idea of the strong-coupling approach[3] is not new and has been reported upon extensively in literature,[4] while the fermion-fermion interaction has been treated only in some very special contexts.[5] Recently, however, the study of the algebraic structure of fermion interactions has attracted considerable interest, stimulated by current algebra. Okubo, Marshak, Goldberg, and Ryan,[6] and Bardakci, Cornwall, Freund, and Lee[7] have attempted to find a larger group invariance than $SU(3)$ by analyzing the interaction-energy part of the Hamiltonian. This is regarded as the first step toward the strong-coupling theory.

In this paper, however, we do not try to find any larger invariance group of the Hamiltonian. On the contrary, starting with the minimum number of assumptions which are required from Lorentz invariance and the phenomenological quark-model analysis,[2] we determine the algebraic characteristics of the Hamiltonian. The Hamiltonian so obtained is solved exactly in the strong-coupling limit.

We know that the Fermi-interaction model suffers from the serious difficulty of divergences or unrenormalizability. Nevertheless, we hope that the algebraic characteristics of the Hamiltonian will survive in a complete theory. Therefore, we do not do anything

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about this point; instead, we pay attention only to the algebra of field quantities.[8]

In Sec. II, we show a simple example of the fermion-interaction model, in which the kinetic energy is taken into account up to second order in the perturbation. A condition for applicability of the strong-coupling approach will be shown.

The rest of the sections are devoted to the applications of the theory to the hadron system. In Sec. III the quark model is studied, and the statement cited in the abstract is proved. The mass formula so obtained shows that the three-body force is necessary, not only to raise the masses of the particles with nonzero triality,[9] but also to explain the boson (0^- and 1^- particles) mass levels.

In Sec. IV the three-body interactions among quarks are calculated. In Sec. V the physical meaning of some identities (the Fierz transformation relations) is studied and the restrictions on the realizable irreducible representations (i.r.) are shown.

The three-triplet model[10] is discussed briefly in Sec. VI with a few additional pertinent remarks at the end.

II. STRONG-COUPLING APPROACH

This section is devoted to a simple model in order to clarify the idea of the strong-coupling approach, and to get the condition of its applicability. Consider an isoscalar fermion field ψ and a Hamiltonian

$$H=\int d^3x\left[\bar{\psi}\gamma_i\partial_i\psi+m_0\bar{\psi}\psi-\frac{g}{m_0^2}(\bar{\psi}\psi)^2\right]\equiv K+H_I, \quad (2.1)$$

where

$$K=\int d^3x\,\bar{\psi}\gamma_i\partial_i\psi. \quad (2.2)$$

To quantize the field, we follow the procedure used by Heisenberg and Pauli.[11] That is, we divide the space into small lattice boxes with the volume ΔV, and associate each box with $\psi(\mathbf{l})$ and $\psi^*(\mathbf{l})$, where

$$\psi(\mathbf{l})=\lim_{\Delta V\to 0}\frac{1}{\sqrt{\Delta V}}\psi(\mathbf{x},t),$$

and $\mathbf{l}$ specifies the corresponding box containing the position coordinate $\mathbf{x}$. Finally the limit $\Delta V\to 0$ is taken at the end of the calculations. We do not specify the time variable in ψ from now on, because it is fixed at some definite point, say, at $t=0$. The commutators are introduced by

$$\{\psi(\mathbf{l}),\psi^*(\mathbf{m})\}=\delta_{\mathbf{l},\mathbf{m}},$$
$$\{\psi(\mathbf{l}),\psi(\mathbf{m})\}=\{\psi^*(\mathbf{l}),\psi^*(\mathbf{m})\}=0. \quad (2.3)$$

The field is decomposed into particle and antiparticle parts as

$$\psi=\begin{pmatrix}\psi^{(+)}\\ \psi^{(-)*}\end{pmatrix},\quad \psi^*=(\psi^{(+)*},\psi^{(-)}),$$

or

$$\psi^{(+)}=\tfrac{1}{2}(1+\beta)\psi,\quad \psi^{(-)}=\tfrac{1}{2}(1-\beta)\psi^*. \quad (2.4)$$

In fact, $\psi^{(+)}(\psi^{(-)})$ and $\psi^{(+)*}(\psi^{(-)*})$ coincide with the particle (antiparticle) annihilation and creation operators, respectively, in the limit when the particles are at rest. The vacuum state is introduced by

$$\psi^{(+)}|0\rangle=\psi^{(-)}|0\rangle=0. \quad (2.5)$$

The kinetic energy K and the interaction energy H_I are written in a difference form as

$$K=\tfrac{1}{2}\sum_{\mathbf{k}}\frac{1}{\Delta k_i}[\bar{\psi}(\mathbf{k})\gamma_i\psi(\mathbf{k}+\Delta\mathbf{k})-\bar{\psi}(\mathbf{k}+\Delta\mathbf{k})\gamma_i\psi(\mathbf{k})]$$
$$=\tfrac{1}{2}\sum_{\mathbf{k}}\frac{-i}{\Delta k_i}[\psi^{(-)}(\mathbf{k})\sigma_i\psi^{(+)}(\mathbf{k}+\Delta\mathbf{k})$$
$$-\psi^{(-)}(\mathbf{k}+\Delta\mathbf{k})\sigma_i\psi^{(+)}(\mathbf{k})+\text{H.c.}], \quad (2.6)$$

and

$$H_I=\sum_{\mathbf{l}}\left[m_0\bar{\psi}(\mathbf{l})\psi(\mathbf{l})-\frac{g}{\Delta V m_0^2}(\bar{\psi}(\mathbf{l})\psi(\mathbf{l}))^2\right]$$
$$=\sum_{\mathbf{l}}\left[m_0S(\mathbf{l})-\frac{g}{\Delta V m_0^2}S^2(\mathbf{l})\right], \quad (2.7)$$

where $S(\mathbf{l})$, the scalar density, is related to $N(\mathbf{l})$, the particle-number operator, by

$$S(\mathbf{l})=\bar{\psi}(\mathbf{l})\psi(\mathbf{l})$$
$$=\sum_{r=1,2}[\psi_r^{(+)*}(\mathbf{l})\psi_r^{(+)}(\mathbf{l})+\psi_r^{(-)*}(\mathbf{l})\psi_r^{(-)}(\mathbf{l})-1]$$
$$=N(\mathbf{l})-2. \quad (2.8)$$

The suffix r refers to the spin direction of the particle.

We shall first diagonalize H_I, and then take K into account.

Unperturbed part H_I

Since H_I commutes with $N(\mathbf{l})$, the particle number is a good quantum number. The system concerned has a number density operator at each lattice box, and each box has discrete energy levels as a function of the eigenvalue $n(\mathbf{l})$ of $N(\mathbf{l})$, which takes the values $0,1,\cdots,4$ according to the Pauli principle (see Fig. 1).

The vacuum of H_I, the lowest energy state, is given when all $n(\mathbf{l})$'s are zero. The one-fermion state occurs

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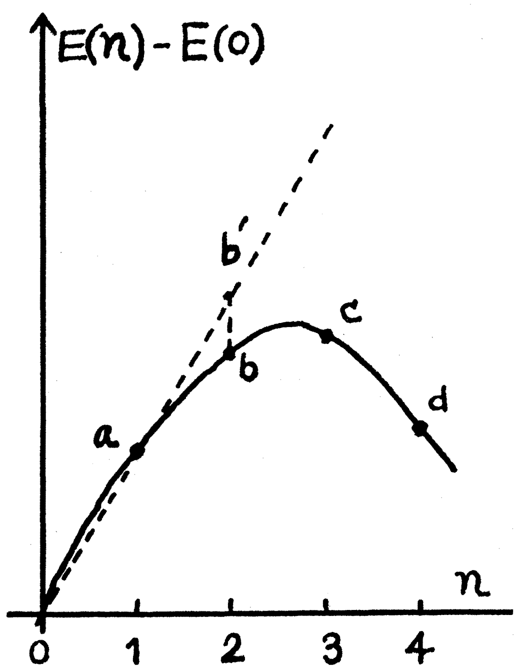

FIG. 1. Masses of the physical particles. The distance bb' stands for the binding energy of the two-particle bound states.

when one of the $n(\mathbf{l})$'s is unity and the others are zero; the one-boson state (two-particle bound state) occurs when one of the $n(\mathbf{l})$'s is two and the others are zero; and so on. Therefore, the rest masses of the fermion and the boson are given by, respectively,

$$m=E(1)-E(0)=m_0(1+3g/\rho^3) \tag{2.9}$$

and

$$\mu=E(2)-E(0)=2m_0(1+2g/\rho^3), \tag{2.10}$$

where $E(n)$ stands for the eigenvalue of H_I for $N(\mathbf{l})=n(\mathbf{l})$, and $\rho^3=\Delta V m_0^3$. The binding energy of the boson is, for example, $2m-\mu=2g/\rho^3$. All possible levels and degeneracies are shown in Table I.

Kinetic-Energy Perturbation

Defining the annihilation operators of the propagating states by

$$a_r(\mathbf{p})=\frac{1}{\sqrt{V}}\sum_{\mathbf{l}} u_r^*(\mathbf{p})\psi(\mathbf{l})e^{-i\mathbf{p}\cdot\mathbf{l}},$$

$$b_r(\mathbf{p})=\frac{1}{\sqrt{V}}\sum_{\mathbf{l}} \psi^*(\mathbf{l})v_r(\mathbf{p})e^{i\mathbf{p}\cdot\mathbf{l}}, \tag{2.11}$$

where a_r and b_r refer to particle and antiparticle, respectively, we proceed with the usual perturbation program by taking K as a perturbation.

As we can see from (2.6), the kinetic-energy term K brings a pair creation or annihilation into the system. Therefore, the first-order perturbation vanishes:

$$\langle 1|K|1\rangle=0, \tag{2.12}$$

where

$$|1\rangle=a_r^*(\mathbf{p})|0\rangle \quad\text{or}\quad b_r^*(\mathbf{p})|0\rangle.$$

The second-order perturbation is given by

$$\Delta E=-\sum_n \frac{|\langle n|K|1\rangle|^2}{E_n-E_1}$$

$$=\delta_{\mathbf{p}\cdot\mathbf{q}}u_s^*(\mathbf{p})\left[\frac{\mathbf{p}^2}{2m_0}-\frac{\partial^2}{\partial\mathbf{m}\cdot\partial\mathbf{k}}\left(\frac{1}{E(3)-E(1)}\right)+i(\gamma_i p_i\cdot\gamma_j)\frac{\partial}{\partial k_j}\left(\frac{1}{E(3)-E(1)}\right)\right]u_r(\mathbf{q}). \tag{2.13}$$

Assuming a continuous energy function

$$E(3)-E(1)=(4gm_0/\rho^3)\times[(\mathbf{l}-\mathbf{k})^2+(\mathbf{k}-\mathbf{m})^2+(\mathbf{m}-\mathbf{l})^2]+2m_0,$$

which satisfies the conditions obtained above [Table I], and working in the rest system of the particle, we get the mass of the one-fermion state:

$$M=m_0(1+3g/\rho^3-3g/\rho^5). \tag{2.14}$$

In obtaining (2.14) we have added the unperturbed mass term (2.9) to (2.13).

Now, it is clear in what situation the strong-coupling approach should be used. The mass equation (2.14) is the Taylor expansion with respect to the parameter $1/\rho=1/(\Delta l\times m_0)$, where $\Delta l^3=\Delta V$; we therefore get a condition

$$\rho=\Delta l\times m_0\sim m_0/\Delta p\gg 1. \tag{2.15}$$

The characteristic momentum of the particle must be small compared with the rest mass.

When we apply the technique to the quark model, we are interested in the bound-state mass rather than the fermion (or the quark) mass itself. But the situation is the same for the former as far as the applicability condition is concerned. Details will be given in the Appendix.

III. QUARK MODEL

In this section local quark interactions are studied. As in Sec. II, the quark field $q(x)$ is replaced by $q(\mathbf{l})$, the total Hamiltonian H is divided into K and H_I, and the algebraic properties of the latter are discussed under some assumptions. The index $\mathbf{l}$ which specifies

TABLE I. Masses of the physical particles and the number of degeneracies.

n	Baryon No.	Spin	$E(n)-E(0)$	Degeneracy No.
1	±1	$\frac{1}{2}$	$m_0(1+3g/\rho^3)$	4
2	$\begin{cases}0\\ \pm2\end{cases}$	$\begin{matrix}1,0\\ 0\end{matrix}$	$2m_0(1+2g/\rho^3)$	6
3	$\begin{cases}\pm1\\ \pm3\end{cases}$	$\frac{1}{2}$	$3m_0(1+g/\rho^3)$	4
4	0	0	$4m_0$	1

the lattice boxes will be dropped in the following, because we are concerned with a definite local point, for example, $\mathbf{l}=\mathbf{l}_0$.

One of the characteristics, which is assumed throughout the nonrelativistic quark-model analysis,[2] is quark and antiquark number conservation. We call the meson a bound state of a quark and an antiquark, the baryon that of three quarks, etc. In a relativistic theory, however, the quark and antiquark numbers are not separately defined concepts. We therefore define the quark number in the limit when the quarks are at rest. Algebraically this is expressed by the scalar density

$$N_Q'=(\sqrt{\tfrac{3}{2}})S_0=(\sqrt{\tfrac{3}{2}})\tfrac{1}{2}\bar{q}\lambda_0 q\,, \tag{3.1}$$

which is proportional to $(q^{(+)*}q^{(+)}+q^{(-)*}q^{(-)})$, while the baryon number is proportional to $(q^{(+)*}q^{(+)}-q^{(-)*}q^{(-)})$. Here we have introduced $q^{(\pm)}$ by

$$q=\begin{pmatrix} q^{(+)} \\ q^{(-)*} \end{pmatrix},\quad q^*=(q^{(+)*},q^{(-)})\,,$$

or

$$q^{(+)}=\tfrac{1}{2}(1+\beta)q\,,\quad q^{(-)}=\tfrac{1}{2}(1-\beta)q^*. \tag{3.2}$$

The algebraic properties of the bilinear forms of the quark fields, which are listed in Table II, are summarized as follows. The $\rho\otimes\sigma$ representation[7] for the Dirac γ matrix for convenience is used. The positive-parity group of the bilinear forms constitutes a nonchiral group $U^{(+)}(6)\times U^{(-)}(6)$, where $(\pm)$ refer to the quark and the antiquark. In fact, defining

TABLE II. The bilinear forms of the quark fields and the generators of $U^{(+)}(6)\times U^{(-)}(6)$.

Positive-parity forms	Negative-parity forms
$S_\alpha=\frac{1}{2}\bar{q}\lambda_\alpha q=\frac{1}{2}q^*\rho_3\sigma_0\lambda_\alpha q$	$P_\alpha=\frac{1}{2}i\bar{q}\gamma_5\lambda_\alpha q=\frac{1}{2}q^*\rho_2\sigma_0\lambda_\alpha q$
$V_{0,\alpha}=\frac{1}{2}\bar{q}\beta\lambda_\alpha q=\frac{1}{2}q^*\rho_0\sigma_0\lambda_\alpha q$	$V_{i,\alpha}=\frac{1}{2}i\bar{q}\gamma_i\lambda_\alpha q=\frac{1}{2}q^*\rho_1\sigma_i\lambda_\alpha q$
$A_{i,\alpha}=\frac{1}{2}i\bar{q}\gamma_5\gamma_i\lambda_\alpha q=\frac{1}{2}q^*\rho_0\sigma_i\lambda_\alpha q$	$A_{0,\alpha}=\frac{1}{2}\bar{q}\gamma_5\beta\lambda_\alpha q=\frac{1}{2}q^*\rho_1\sigma_0\lambda_\alpha q$
$T_{ij,\alpha}=\frac{1}{2}\bar{q}\sigma_{ij}\lambda_\alpha q=\frac{1}{2}\epsilon_{ijk}q^*\rho_3\sigma_i\lambda_\alpha q$	$T_{0i,\alpha}=\frac{1}{2}i\bar{q}\sigma_{0i}\lambda_\alpha q=\frac{1}{2}q^*\rho_2\sigma_i\lambda_\alpha q$
Generators of $U^{(+)}(6)\times U^{(-)}(6)$:	
$G_{\mu,\alpha}{}^{(+)}=\frac{1}{2}q^{(+)*}\sigma_\mu\lambda_\alpha q^{(+)}$,	$G_{\mu,\alpha}{}^{(-)}=\frac{1}{2}q^{(-)}\sigma_\mu\lambda_\alpha q^{(-)*}$.

$$\begin{aligned} G_{\mu,\alpha}{}^{(\pm)}&=G_{0,\alpha}{}^{(\pm)}=\tfrac{1}{2}(V_{0,\alpha}\pm S_\alpha)\,,\quad (\text{for }\mu=0)\\ &=G_{i,\alpha}{}^{(\pm)}=\tfrac{1}{2}(A_{i,\alpha}\pm\tfrac{1}{2}\epsilon_{ijk}T_{jk,\alpha})\,,\\ &\qquad (\text{for }\mu=1,2,3) \end{aligned} \tag{3.3}$$

we can easily show that

$$\begin{aligned} [G_{\mu,\alpha}{}^{(\pm)},G_{\nu,\beta}{}^{(\pm)}]&=iF_{\mu\alpha,\nu\beta,\omega\gamma}G_{\omega,\gamma}{}^{(\pm)}\,,\\ [G_{\mu,\alpha}{}^{(\pm)},G_{\nu,\beta}{}^{(\mp)}]&=0\,, \end{aligned} \tag{3.4}$$

where $F_{\mu\alpha,\nu\beta,\omega\gamma}$ is the structure constant of the $U(6)$ group. For future convenience we define the following quantities: the $U(6)$ generators

$$G_{\mu,\alpha}(6)=G_{\mu,\alpha}{}^{(+)}+G_{\mu,\alpha}{}^{(-)};\quad (\mu=0,\cdots,3;\ \alpha=0,\cdots,8) \tag{3.5}$$

the $U(3)$ generators

$$G_\alpha(3)=G_{0,\alpha}{}^{(+)}+G_{0,\alpha}{}^{(-)};\quad (\alpha=0,\cdots,8) \tag{3.6}$$

the spin operators

$$S_i=S_i{}^{(+)}+S_i{}^{(-)}\quad\text{and}\quad S_i{}^{(\pm)}=(\sqrt{\tfrac{3}{2}})G_{i,0}{}^{(\pm)};\quad (i=1,2,3) \tag{3.7}$$

the quark number N_Q

$$N_Q'=(\sqrt{6})(G_{0,0}{}^{(+)}-G_{0,0}{}^{(-)})=N_Q-6; \tag{3.8}$$

and the baryon number B

$$B'=(\sqrt{6})\tfrac{1}{3}(G_{0,0}{}^{(+)}+G_{0,0}{}^{(-)})=B+6. \tag{3.9}$$

We shall prove the following statement about H_I. Statement I. If we assume:

(i) Lorentz-invariant and local (nonderivative) interactions in H_I,
(ii) $SU(3)$ invariance,
(iii) quark number conservation,

$$[H_I,N_Q]=0\,, \tag{3.10}$$

then

(a) the system does not have, in general, any higher invariance group than $SU(3)$, and
(b) nevertheless, every localized eigenstate of H_I belongs to some irreducible representation (i.r.) of the nonchrial $U^{(+)}(6)\times U^{(-)}(6)$.

In addition to the above assumptions, if we assume that
(iv) the interactions are restricted to the four-fermion type, then
(c) the system is exactly solvable and the mass formula for the localized states is

$$\begin{aligned} M=m_0N_Q&+a(N_Q-6)^2\\ &+b[\mathbf{G}^2(3)-4\mathbf{G}^{(+)}(3)\cdot\mathbf{G}^{(-)}(3)]\\ &\quad+c[\mathbf{G}^2(6)+2\mathbf{S}^2+6B^2]\,, \end{aligned} \tag{3.11}$$

where $\mathbf{G}\cdot\mathbf{G}$ means the inner product of the $SU(6)$ or $SU(3)$ group generators [not $U(6)$ or $U(3)$].

We shall first prove (c). For this we have to find the four-fermion type interactions which commute with $N_Q\sim\rho_3\sigma_0\lambda_0$. They must have the structure of $\rho_3{}^2$, $\rho_0{}^2$, or $\rho_1{}^2+\rho_2{}^2$ in the ρ-spin space[12] because they must commute with ρ_3. From Table II, we can easily show that the possible candidates are

$$S_\alpha{}^2\sim(\tfrac{1}{2}\rho_3\sigma_0\lambda_\alpha)^2\,, \tag{3.12}$$

$$P_\alpha{}^2+(iA_\alpha)^2-S_\alpha{}^2\sim(\tfrac{1}{2}\rho_0\sigma_\mu\lambda_\alpha)^2-(\tfrac{1}{2}\rho_\mu\sigma_0\lambda_\alpha)^2\,, \tag{3.13}$$

$$\begin{aligned} V_\alpha{}^2+T_\alpha{}^2\sim(\tfrac{1}{2}\rho_0\sigma_\mu\lambda_\alpha)^2&+(\tfrac{1}{2}\rho_\mu\sigma_0\lambda_\alpha)^2\\ &-(\tfrac{1}{2}\rho_\nu\sigma_\mu\lambda_\alpha)^2+2(\tfrac{1}{2}\rho_3\sigma_i\lambda_\alpha)^2\,, \end{aligned} \tag{3.14}$$

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[12] We use a convention of notations: $(\rho_\mu\sigma_\nu\lambda_\alpha)^2$ means

$$\textstyle\sum_{\mu,\nu,\alpha}(q^*\rho_\mu\sigma_\nu\lambda_\alpha q)^2,$$

so that

$$\rho_1{}^2+\rho_2{}^2\sim(q^*\rho_1\sigma_0\lambda_0 q)^2+(q^*\rho_2\sigma_0\lambda_0 q)^2,\ \text{etc.}$$

together with S_0^2, $P_0^2+(iA_0)^2-S_0^2$, and $V_0^2+T_0^2$,[13] where μ and ν run from 0 to 3, and α runs from 0 to 8. The above six interactions are not all independent. Because of the Fierz transformation relations

$$S_\alpha^2+P_\alpha^2+(iA_\alpha)^2+V_\alpha^2+T_\alpha^2 = -12S_0^2+c\text{ number},$$

$$S_0^2+P_0^2+(iA_0)^2+V_0^2+T_0^2 = -\tfrac{4}{3}S_\alpha^2+c\text{ number}, \quad (3.15)$$

any two of them, say, $V_0^2+T_0^2$ and $V_\alpha^2+T_\alpha^2$, can be written as combinations of the others. Furthermore, we have another identity:

$$P_\alpha^2+(iA_\alpha)^2-S_\alpha^2 = 3[P_0^2+(iA_0)^2-S_0^2]+c\text{ number}. \quad (3.16)$$

These three relations can be directly proved[14] by the use of the identities

$$(\rho_\mu)_{ab}(\rho_\mu)_{cd}=2(\rho_0)_{ad}(\rho_0)_{cb},$$
$$(\sigma_\mu)_{ab}(\sigma_\mu)_{cd}=2(\sigma_0)_{ad}(\sigma_0)_{cb},$$
$$(\lambda_\alpha)_{ij}(\lambda_\alpha)_{kl}=3(\lambda_0)_{il}(\lambda_0)_{kl}. \quad (3.17)$$

As a consequence, we shall take S_0^2, S_α^2, and $P_\alpha^2+(iA_\alpha)^2-S_\alpha^2$ as the independent interactions. S_0^2 is nothing but the quark-number operator by (3.8). The other two are rewritten from (3.12) and (3.13) as follows:

$$\begin{aligned} S_\alpha^2 &\sim [\tfrac{1}{2}(\tfrac{1}{2}(1+\rho_3)\sigma_0\lambda_\alpha-\tfrac{1}{2}(1-\rho_3)\sigma_0\lambda_\alpha)]^2 \\ &= [G_\alpha^{(+)}(3)-G_\alpha^{(-)}(3)]^2 \\ &= \mathbf{G}^2(3)-4\mathbf{G}^{(+)}(3)\cdot\mathbf{G}^{(-)}(3)+\tfrac{1}{6}(N_Q-6)^2, \end{aligned} \quad (3.18)$$

$$\begin{aligned} P_\alpha^2+(iA_\alpha)^2-S_\alpha^2 &\sim (\tfrac{1}{2}\rho_0\sigma_\mu\lambda_\alpha)^2+3(\tfrac{1}{2}\rho_0\sigma_\mu\lambda_0)^2 \\ &= G_{\mu,\alpha}^2+G_{\mu,0}^2=\mathbf{G}^2(6)+2\mathbf{S}^2+6B^2. \end{aligned} \quad (3.19)$$

Here, in going from (3.13) to the first line of (3.19), we have used (3.17) on the second term of the right side of (3.13). From (3.18) and (3.19), the general local interaction H_I of four-fermion type is written by (3.11). The first term in (3.11) comes from the quark mass term.

Statement (a) is obvious from (3.11).

To prove statement (b) let us first look at (3.11), where each eigenstate belongs to some irreducible representation of $U^{(+)}(6)\times U^{(-)}(6)$, although the mass levels are nondegenerate within a multiplet.

In general, the interaction is composed of two kinds of bilinear combinations of $q^{(\pm)}$ and $q^{(\pm)*}$. The first is $q^{(+)*}Oq^{(+)}$ or $q^{(-)}Oq^{(-)*}$, where O is an arbitrary matrix of $\sigma\otimes\lambda$, each of which is a generator of $U^{(+)}(6)\times U^{(-)}(6)$.

[13] Here we have dropped the suffixes μ and ν from the Lorentz tensors V_μ, A_μ, and $T_{\mu\nu}$. S_0, P_0, V_0, etc., mean the zeroth components of the $U(3)$ tensors.

[14] On the right sides of (3.15), the minus signs come from the anticommutation property of q and q^*. The bilinear forms of the quark field, which arise in (3.15) and (3.16) because of the noncommutability of q and q^*, are cancelled if we use the charge-symmetric version of the vector current. Hence, throughout this work we imply that the fourth component of the vector current B' should be replaced by B of (3.9).

The second is $q^{(+)*}Oq^{(-)*}$ or $q^{(-)}Oq^{(+)}$. But because of the quark-number conservation, the former should be accompanied by the latter in an interaction term; take the interaction of order n, for example,

$$(q^{(+)*}O_1q^{(-)*})\cdots(q^{(-)}O_2q^{(+)})\cdots(\ ).$$

Making use of Fierz transformation about $q^{(-)*}$ and $q^{(+)}$, we have

$$(q^{(+)*}O_1'q^{(+)})\cdots(q^{(-)}O_2'q^{(-)*})\cdots(\ ) +(\text{terms of less than order } n-2).$$

To the quantity so obtained the same procedure may be applied, if necessary, until all terms are rewritten in terms of the generators of $U^{(+)}(6)\times U^{(-)}(6)$. If we choose a state vector which belongs to an irreducible representation of $U^{(+)}(6)\times U^{(-)}(6)$, the state vector continues to remain in the i.r. representation when multiplied by H_I. Therefore statement (b) has been proved.

The interaction Hamiltonian for the arbitrary (nonlocalized) state is given by

$$H_I=\sum_{\mathbf{l}} M(\mathbf{l})=\int d^3x\frac{M(\mathbf{x})}{\Delta V}, \quad (3.20)$$

where $M(\mathbf{l})$ is given by (3.11).

It is interesting here to require $SU(6)$ invariance on M. Then, from (3.11), $b=c=0$, which shows the difficulty noted by Ohnuki and Toyoda,[15] that the system is invariant under a larger group which contains the baryon-number-changing operator. We want to emphasize, however, that even if the interaction is not invariant under $SU(6)$, the state vector can still be an eigenstate which belongs to an i.r. representation of $U^{(+)}(6)\times U^{(-)}(6)$.

Finally, we apply the mass formula (3.11) to the meson mass of $\mathbf{(6,6^*)=1+35}$ to obtain

$$V(1)-P(1)=4[V(8)-P(8)],$$

where $P(8)$ $(P(1))$ and $V(8)$ $(V(1))$ represent the average mass of the pseudoscalar octet (singlet) meson, and the vector octet (singlet) meson, respectively. This formula is in serious contradiction to the experimental situation if $P(1)=960$ MeV. As is well known, the four-fermion interaction (two-body force) is not good enough to explain the high masses of the triality-nonzero particles,[9] so that one usually assumes a simple three-body force, say, $S_0^3\sim(N_Q-6)^3$. We want to stress that the three-body force is necessary to explain the mass levels of the mesons as well as those of the triality-nonzero particles (see Sec. IV).

The mass formula (3.11) should not be applied directly to the baryon mass, because in the usual quark model it has a totally antisymmetric wave function about three quark coordinates and it vanishes at the

[15] Y. Ohnuki and A. Toyoda, Nuovo Cimento **36**, 1405 (1965).

local point limit. The baryon mass formula will be discussed in Sec. VI.

IV. THREE-BODY FORCES

In Sec. III we found that the four-fermion interaction (two-body force) does not explain the boson average mass as well as the mass of the particles with nonzero triality. To overcome these difficulties we shall investigate the general form of the three-body interactions.

All of the possible three-body interactions, which are local, Lorentz-invariant, and commuting with N_Q, are

$$\begin{aligned}&S_0^3,\quad S_0S_\alpha^2,\quad S_0[P_\alpha^2+(iA_\alpha)^2-S_\alpha^2],\\&S_\alpha(P_\alpha P_0+i^2A_\alpha A_0-S_\alpha S_0),\\&d_{\alpha\beta\gamma}S_\alpha S_\beta S_\gamma,\quad d_{\alpha\beta\gamma}(P_\alpha P_\beta+i^2A_\alpha A_\beta-S_\alpha S_\beta)S_\gamma,\end{aligned}\tag{4.1}$$

where $d_{\alpha\beta\gamma}=\mathrm{Tr}\frac{1}{4}\lambda_\alpha\{\lambda_\beta,\lambda_\gamma\}$. The other combinations, $(V_\alpha V_\beta+T_\alpha T_\beta)S_\gamma d_{\alpha\beta\gamma}$, etc., can be written as some combinations of the above expressions. The interactions (4.1) can be expressed again in terms of the $U^{(+)}(6)\times U^{(-)}(6)$ generators.

The eigenvalues of the three-body interactions are easily obtained, especially for some low-dimensional i.r. representations. To do this let us first prove some simple relations among $G^{(\pm)}(3)$. For the state vectors of **(6,1)**, **(1,6*)**, and **(6,6*)**,

$$\begin{aligned}&G^{(+)}(3)_a{}^bG^{(+)}(3)_c{}^d|\,\rangle=\delta_c{}^bG^{(+)}(3)_a{}^d|\,\rangle,\\&G^{(-)}(3)_a{}^bG^{(-)}(3)_c{}^d|\,\rangle=-\delta_a{}^dG^{(-)}(3)_c{}^b|\,\rangle,\end{aligned}\tag{4.2}$$

where $G^{(\pm)}(3)_a{}^b\sim\rho_\pm\sigma_0\lambda_a{}^b$ [16] and $|\,\rangle$ stands for any one of the states mentioned above. Using the explicit form of the generators, we can show that

$$G^{(+)}{}_a{}^bG^{(+)}{}_c{}^d=\delta_c{}^bG^{(+)}{}_a{}^d+:G^{(+)}{}_a{}^bG^{(+)}{}_c{}^d:,$$

where : : means the normal product. Because every term in the normal product contains two annihilation operators $q^{(+)}q^{(+)}$, it vanishes when operated on $q^{(\pm)*}|0\rangle\sim$ **(6,1)** or **(1,6*)** and $q^{(+)*}q^{(-)*}|0\rangle\sim$ **(6,6*)**. Hence we obtain the first relation of (4.2). The same is true for the second.

Taking the interaction $d_{\alpha\beta\gamma}S_\alpha S_\beta S_\gamma$, as an example, we find

$$\begin{aligned}&d_{\alpha\beta\gamma}S_\alpha S_\beta S_\gamma|\,\rangle\\&=d_{\alpha\beta\gamma}(G_\alpha{}^{(+)}-G_\alpha{}^{(-)})(G_\beta{}^{(+)}-G_\beta{}^{(-)})(G_\gamma{}^{(+)}-G_\gamma{}^{(-)})|\,\rangle\\&\propto\{G_\alpha{}^{(+)2}+G_\alpha{}^{(-)2}-6G_\alpha{}^{(+)}G_\alpha{}^{(-)}\}|\,\rangle\\&=\{2N_Q-3\mathbf{G}^2(3)-\tfrac{9}{2}B^2+c\text{ number}\}|\,\rangle.\end{aligned}\tag{4.3}$$

In obtaining (4.3), use of (4.2) has been made. For the other interactions in (4.1), calculations are performed in a similar way. Combining these calculations together with (3.11), we get

$$M=f(N_Q,B)+\tilde{a}\mathbf{S}^2+\tilde{b}\mathbf{G}^2(3)+\tilde{c}\cdot\mathbf{G}^2(6),\tag{4.4}$$

where $\tilde{a}$, $\tilde{b}$, and $\tilde{c}$ are certain polynomials in N_Q, and $f(N_Q,B)$ is a polynomial in N_Q and B. The mass levels of the bosons **(6,6*)** and the quarks **(6,1)** are obtained by choosing definite $\tilde{a}$, $\tilde{b}$, etc. Unfortunately, there are so many parameters that no sum rule can be deduced from (4.4).

V. ALLOWED STATES

We should like to consider the physical meaning of the Fierz transformation relations, used frequently in Sec. III, and to prove the following statement.

Statement II. The dynamical system with which we are concerned does not have localized eigenstates of the arbitrary i.r. representations of $U^{(+)}(6)\times U^{(-)}(6)$. Only the representations which satisfy

$$\begin{aligned}&\mathbf{G}^{(\pm)2}(6)=-(7/24)(3B\pm N_Q')^2+21/2,\\&\mathbf{G}^{(\pm)2}(3)=-\mathbf{S}^{(\pm)2}-(5/48)(3B\pm N_Q')^2+15/4\end{aligned}\tag{5.1}$$

are allowed.

The relations (5.1) are the Fierz transformation relations (3.15) and (3.16) expressed in terms of the generators. Applying (3.17) to $(q^{(+)*}\sigma_\mu\lambda_\alpha q^{(+)})^2$ and $(q^{(+)*}\sigma_0\lambda_\alpha q^{(+)})^2$, or $(q^{(-)}\sigma_\mu\lambda_\alpha q^{(-)*})^2$ and $(q^{(-)}\sigma_0\lambda_\alpha q^{(-)*})^2$, we get (5.1).[14]

The main reasons why we get such restrictions on the states are (i) the locality of the interactions and (ii) the Pauli principle for the quarks. The relations (5.1), however, prohibit a state which transforms, for example, like **(1,6)** with $B=\frac{1}{3}$ and $N_Q=1$ under $U^{(+)}(6)\times U^{(-)}(6)$. This comes from the fact that (iii) we do not have a creation operator which transforms like **(1,6)** in the quark model.

VI. REMARKS AND CONCLUSION

We have presented a strong-coupling theory of the fermion interactions. Using the theory, we have examined the quark model assuming quark-number conservation in the limit when the quarks are at rest. The assumption we have used seems very natural from the viewpoint of the phenomenological analysis of the quark model. We have shown that the localized eigenstate of H_I belongs to an i.r. representation of $U^{(+)}(6)\times U^{(-)}(6)$ although the mass levels are nondegenerate in the multiplet, and that only a restricted class of the i.r. representations are allowed according to (5.1). The meson mass of **(6,6*)** has been studied in detail, where we have had to introduce three-body forces to explain the experimental situation. This brings a complication to the quark model.

The baryon state of **(56,1)** can not be regarded as a localized state in our approach. However, taking the kinetic-energy term into account as is shown in the Appendix, the baryon mass is also calculable in the quark model, in which case the state vector may be classified by a group $[U^{(+)}(6)\times U^{(-)}(6)]^3\times O(3)$. Each $U^{(+)}(6)\times U^{(-)}(6)$ is the group associated with the

[16] The 3×3 matrix $(\lambda_a{}^b)_{cd}=\delta_{ac}\delta_{bd}$ which is, of course, written as a linear combination of λ_α's.

different space box, and $O(3)$ is the group of the orbital angular momentum. This is another complication in the quark model.

There remains, therefore, the question whether there is a simple model in which the three-body force is not necessary and the baryon is a localized state as well as the boson. We know some possible candidates—the paraquark model,[17] the two-triplet model,[18] the three-triplet model,[10] etc. Though the first model gives the same commutation relations among the bilinear quantities of the field as the orthoquark model, the Fierz transformation can not be performed in a simple way. We shall therefore avoid it. The second model does not give the localized baryon state.

Probably the third model, the three-triplet model, is the best candidate. According to Han and Nambu,[10] we can simply formulate the model by introducing a nine-component field $Q_{ia}(x)$ and a group $U'(3)\times U''(3)$, where the suffix i transforms like a triplet under $U'(3)$ and the suffix a like an antitriplet under $U''(3)$. Here we show the general interaction energy of four-fermion type, assuming the invariance under $U'(3)\times U''(3)$;

$$\begin{aligned} M=m_0N_Q+c_1(N_Q-18)^2&+c_2[\mathbf{G}^{(+)\prime}(3)-\mathbf{G}^{(-)\prime}(3)]^2 \\ &+c_3[\mathbf{G}^{(+)\prime\prime}(3)-\mathbf{G}^{(-)\prime\prime}(3)]^2 \\ &+c_4[\mathbf{G}^{(+)}(9)-\mathbf{G}^{(-)}(9)]^2 \\ &+c_5[(10/9)B^2+4\mathbf{S}^2+\mathbf{G}^2(18)] \\ &+c_6[(2/9)B^2+(\mathbf{G}'(6))^2+(\mathbf{G}''(6))^2], \quad (6.1) \end{aligned}$$

where G's are all generators of a group

$U^{(+)}(18)\times U^{(-)}(18):G(18)\sim\sigma_\mu\lambda_\alpha'\lambda_\beta''$, $G(9)\sim\sigma_0\lambda_\alpha''\lambda_\beta''$, $G'(6)\sim\sigma_\mu\lambda_\alpha'\lambda_0''$, $G''(6)\sim\sigma_\mu\lambda_0'\lambda_\alpha''$, $G'(3)\sim\sigma_0\lambda_\alpha'\lambda_0''$, $G''(3)\sim\sigma_0\lambda_0'\lambda_\alpha''$, and $S_i\sim\sigma_i\lambda_0'\lambda_0''$.

The formula (6.1) is valid both for the baryons and the bosons. Unfortunately, however, the term $[G^{(+)\prime\prime}(3)+G^{(-)\prime\prime}(3)]^2$, whose existence has been assumed by Han and Nambu, and is really necessary to raise the mass of the particle with nonzero triality (the charm number), does not appear in (6.1). We must therefore break either the $U'(3)\times U''(3)$ invariance or the quark and antiquark number conservation in the limit when the quarks are at rest. Details are, however, not discussed in this text.

In conclusion we wnat to make a few remarks about the usual quark model.

(i) If the quark-number conservation in statement I is not assumed, the state vectors are classified by $U(2)\times U(6)$. This will be discussed in a separate paper.

(ii) If the kinetic-energy term in calculating the boson masses is taken into account, the spin-orbit coupling may be obtained, as is shown for a simple model in the Appendix. This is favorable for explaining the energy levels of the higher mass resonances.[2]

(iii) Although we have neglected the $SU(3)$-violating interaction in this paper, the mass splitting in a $SU(3)$ multiplet can also be calculated exactly[19] in our limit.

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APPENDIX: ENERGY LEVEL OF A TWO-PARTICLE BOUND STATE

We show the calculation of the energy level of a two-particle bound state in the model of Sec. II, laying emphasis on the kinetic-energy perturbation.

Defining the creation operator of two-particle state with momentum $\mathbf{p}$,

$$\phi^*(\mathbf{p})=\sum_{\mathbf{l},\mathbf{l}',r,s}\varphi_{rs}(\mathbf{p};\mathbf{l}-\mathbf{l}')\psi_r^{(+)*}(\mathbf{l})\psi_s^{(-)*}(\mathbf{l}')e^{i\mathbf{p}\cdot(\mathbf{l}+\mathbf{l}')/2},$$

we calculate the second-order term of the kinetic energy in the center-of-mass frame to get

$$\begin{aligned} \langle 2|\Delta E|2\rangle &= -\sum_n \frac{\langle 2|K|n\rangle\langle n|K|2\rangle}{E(n)-E(2)} \\ &= \varphi^*\Bigg[-\frac{1}{2\mu}\frac{\partial^2}{\partial\mathbf{r}^2}-\tfrac{2}{3}\sum_i\frac{d}{dr_i^2}\left(\frac{1}{E(4)-E(2)}\right) \\ &\qquad -\tfrac{1}{3}\mathbf{L}\cdot\mathbf{S}\frac{1}{r}\sum_i\frac{d}{dr_i}\left(\frac{1}{E(4)-E(2)}\right)\Bigg]\varphi, \end{aligned}$$

where

$$\varphi_{rs}\equiv\varphi(\mathbf{p};\mathbf{r})[\beta\gamma_5\tfrac{1}{2}(1-\beta)]_{rs},$$

which means a pseudoscalar state when $\mathbf{L}=0$, $\mathbf{r}=\mathbf{l}-\mathbf{l}'$, μ is the reduced mass of m_0, and r_i are the relative coordinates of the four particles in the intermediate state. The summation $\sum_i$ means the sum over the possible relative coordinates r_i.

We remark that the $\mathbf{L}\cdot\mathbf{S}$ force is obtained in the second-order correction term.

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