Self-Interaction Corrections of a Charge in a Nonrelativistic Particle Formalism*

JOHN E. KRIZAN
Department of Physics, University of Vermont, Burlington, Vermont

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A nonlinear equation for the description of self-interaction corrections is developed by means of stochastic mechanics. It is derived within a nonrelativistic framework in which fields do not appear. The linearized approximation of this equation is also examined, and an alternative, more conventional, interpretation of the resulting effective interaction is given. Calculations involving effects which have a possible interpretation as the Lamb shift and anomalous magnetic moment of the electron, and which are free of cutoffs, are made; these give semiquantitative agreement, offering some encouragement for a possible attempt at a relativistic generalization.

I. INTRODUCTION

THERE have been questions raised recently about the adequacy of the present formulation of quantum electrodynamics (QED). The discontent has both theoretical and experimental roots[1]: The theoretical difficulties arise from the presence of divergences and the use of perturbation methods,[2] while the experimental questions arise in particular in recent Lamb shift determinations (in spite of the latter questions, it would be difficult to detract from the quantitative successes of QED).

This paper in its present form does not pretend to provide any rigorous answers to the problems plaguing quantum electrodynamics. It is intended to suggest a possible direction which represents a different point of view, in that there are no fields with independent degrees of freedom (consequently, there are no analogous divergence problems).

At the outset, it should be emphasized that the treatment is nonrelativistic. It deals with so-called radiative corrections (here, we prefer to call them self-interaction corrections) which in the conventional treatment come from the emission and absorption of the virtual quanta of a single charged particle. Here we derive an effective interaction, taking for a starting point the pattern of recent work in attempts to find alternatives to the usual formulation of quantum mechanics[3,4] (although we also give an interpretation in linear approximation which is more consistent with the usual formulation).

In Sec. II the stochastic equations of motion are derived assuming a random process involving the coordinate as a stochastic variable. Section III deals with the derivation of a nonlinear equation, as well as a linearized approximation of it, which reduce in the limit of no self-interaction to the time-independent Schrödinger equation. An alternative to the stochastic interpretation of the linearized equation, in terms of a complex time transformation, is given. Section IV deals with the generalization to the time-dependent case, while the following section involves the calculation of self-interaction effects in lowest order. Finally, a possible formulation for nonconservative systems is suggested.

II. STOCHASTIC KINEMATICS AND EQUATIONS OF MOTION

In the description which follows, we assume a kinematical description which is identical with the one given by Nelson.[3] The process is one of Brownian motion according to the theory of Einstein and Smoluchowski. The characteristic time involved is that associated with the Zitterbewegung frequency.[5] The trajectory is not differentiable on this time scale[6,7]; however, one can define a mean forward velocity $D\mathbf{x}(t)=\mathbf{b}(\mathbf{x}(t),t)$ and a mean backward velocity $D_*\mathbf{x}(t)=\mathbf{b}_*(\mathbf{x}(t),t)$, and combine them to form a probability current velocity

$$\mathbf{v}=\tfrac{1}{2}(\mathbf{b}+\mathbf{b}_*)\,, \tag{2.1}$$

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[1] See the comments of R. Wilson and S. D. Drell, Comments Nucl. Particle Phys. **1**, 22 (1967).

[2] However, nonrelativistic calculations involving dispersion techniques in calculating the Lamb shift have recently been made by X. Artru, J. L. Basdevant, and R. Omnes, Phys. Rev. **150**, 1387 (1966). Also see H. Abarbanel, Ann. Phys. (N. Y.) **39**, 177 (1966).

[3] E. Nelson, Phys. Rev. **150**, 1079 (1966).

[4] D. Bohm and J. P. Vigier, Phys. Rev. **96**, 208 (1954); N. Wiener and A. Siegel, *ibid.* **91**, 1551 (1953); I. Fenyes, Z. Physik **132**, 81 (1952); W. Weizel, *ibid.* **134**, 264 (1953), **135**, 270 (1953), **136**, 582 (1954); D. Kershaw, Phys. Rev. **136**, B1850 (1964). Also see L. de Broglie, *Etude Critique des Bases de l'Interprétation Actuelle de la Mécanique Ondulatoire* (Gauthier Villars, Paris, 1963) [English transl.: *The Current Interpretation of Wave Mechanics* (Elsevier Publishing Co., New York, 1964)].

[5] It is recognized that ordinarily this effect has a natural setting in the Dirac equation.

[6] The nondifferentiability of the trajectory due to the *Zitterbewegung* here is intuitively suggested by the relativistic explanation (via Dirac) of an electron which oscillates rapidly between the limits of $\pm c$. The mathematical property that the derivative does not exist is to be interpreted physically as a reflection of the highly irregular motion on the given time scale.

[7] For more information on the distribution function of the Wiener process as well as other information on stochastic processes, see the articles in *Selected Papers on Noise and Stochastic Processes*, edited by N. Wax (Dover Publications, Inc., New York, 1954). In particular see the papers of S. Chandrasekhar and J. L. Doob.

where the definition arises from the combination of forward and backward Fokker-Planck equations which the probability density $\rho(\mathbf{x},t)$ must satisfy. Proceeding further one arrives at Eq. (32) of Nelson:

$$\partial\mathbf{v}/\partial t=\mathbf{a}-(\mathbf{v}\cdot\nabla)\mathbf{v}+(\mathbf{u}\cdot\nabla)\mathbf{u}+\nu\Delta\mathbf{u}\,, \qquad (2.2)$$

where $\mathbf{u}=\nu(\nabla\rho/\rho)$ and $\nu=\hbar/2m$.

In defining a Newtonian dynamics with radiation damping, we look for an equation which is a time-symmetric analog of the classical Lorentz equation,

$$\mathbf{a}=\gamma^{-1}d\mathbf{a}/dt+\mathbf{f}. \qquad (2.3)$$

Here $\gamma^{-1}=2\tau_0/3=2e^2/3mc^3$, and $\mathbf{f}$ is the external force per unit mass. It is well known that (2.3) adequately describes nonrelativistic classical radiation damping, although when $\mathbf{f}=0$, the equation gives "self-accelerated" solutions. The reformulation of radiation damping in terms of an integrodifferential equation circumvents this difficulty, although it leads to the requirement that the force must be specified for future times of the order of τ_0.

Rather than using the form of (2.3), we can define a *time-symmetric* equation of motion, using the stochastic process introduced above:

$$\mathbf{a}=(2\gamma)^{-1}[D-D_*]\mathbf{a}+\mathbf{f}. \qquad (2.4)$$

This equation is invariant under time "reversal" ($\mathbf{f}$ is assumed to be time-symmetric); this operation is defined as the inversion $t\rightarrow -t$ combined with the interchange of D and D_*.

Equation (2.4) is now substituted into (2.2), after we make use of the definition of the mean acceleration of the stochastic process:

$$\mathbf{a}=\tfrac{1}{2}[DD_*\mathbf{x}(t)+D_*D\mathbf{x}(t)]. \qquad (2.5)$$

The acceleration defined in this fashion is therefore symmetric in the operations D and D_* and is time-reversal invariant in the above sense. Note that in Nelson's treatment the definition first appears in relation to his derivation of the dynamics by means of an Ornstein-Uhlenbeck process via the Langevin equation, although he later applies the same definition in relation to the Einstein-Smoluchowski process.

Applying Nelson's equations (22) and (23) to (2.4), we obtain

$$\mathbf{a}=\gamma^{-1}[\mathbf{u}\cdot\nabla+\nu\Delta][\partial\mathbf{v}/\partial t+(\mathbf{v}\cdot\nabla)\mathbf{v}-(\mathbf{u}\cdot\nabla)\mathbf{u}-\nu\Delta\mathbf{u}]+\mathbf{f}. \qquad (2.6)$$

Defining the operator $\mathcal{O}\equiv\mathbf{u}\cdot\nabla+(\hbar/2m)\Delta$, $\epsilon\equiv\gamma^{-1}$, and using (2.2), we may write the following set of equations[3]:

$$\partial\mathbf{u}/\partial t=-(\hbar/2m)\nabla(\nabla\cdot\mathbf{v})-\nabla(\mathbf{v}\cdot\mathbf{u})\,, \qquad (2.7\text{a})$$

$$[1-\epsilon\mathcal{O}][\partial\mathbf{v}/\partial t+(\mathbf{v}\cdot\nabla)\mathbf{v}]=\mathbf{f}+[1-\epsilon\mathcal{O}]\mathcal{O}\mathbf{u}. \qquad (2.7\text{b})$$

III. NONLINEAR AND LINEAR EQUATIONS WITH EFFECTIVE INTERACTION

To get a time-independent equation from (2.7), we set the mean velocity $\mathbf{v}=0$. Then from Eq. (2.7a) (this equation is related to the continuity equation for the probability density), $\mathbf{u}$ is explicitly time-independent and the equation of motion (2.7b) can be written

$$[1-\epsilon\mathcal{O}][\nabla(\mathbf{u}^2/2+\hbar\nabla\cdot\mathbf{u}/2m)]-\nabla V/m=0\,, \qquad (3.1)$$

where it has been assumed that $\mathbf{f}=-\nabla V/m$. With the definitions $R\equiv(\ln\rho)/2$, $\psi\equiv e^R$ we get $\mathbf{u}=\hbar\nabla\psi/m\psi$, and (3.1) becomes

$$[1-\epsilon\mathcal{O}][(\hbar^2/2m^2)\nabla(\nabla^2\psi/\psi)]-\nabla V/m=0. \qquad (3.2)$$

By putting in the explicit form for $\mathcal{O}$ and using a vector identity, we may write

$$\nabla\{[1-(\epsilon\hbar/m)(\psi^{-1}\nabla\psi\cdot\nabla+\tfrac{1}{2}\Delta)](\hbar^2\Delta\psi/2m^2\psi)-V/m\}+(\epsilon\hbar/m)[\nabla(\hbar^2\Delta\psi/2m^2\psi)\cdot\nabla](\nabla\psi/\psi)=0. \qquad (3.3)$$

If we neglect the last term in (3.3) (we will attempt to justify this later in the discussion in Sec. IV), we get the time-independent equation

$$-(\hbar^2/2m)\Delta\psi/\psi+V+V_{\text{eff}}=E=\text{const.}, \qquad (3.4)$$

where

$$\begin{aligned}V_{\text{eff}}&=(\epsilon\hbar^3/2m^2)[(\nabla\ln\psi\cdot\nabla)+\tfrac{1}{2}\Delta]\Delta\psi/\psi\\&=(\epsilon\hbar^3/4m^2)\psi^{-1}[\Delta(\Delta\psi)-\psi^{-1}(\Delta\psi)^2]\\&=(\epsilon/\hbar)\psi^{-1}[T^2\psi-\psi^{-1}(T\psi)^2].\end{aligned} \qquad (3.5)$$

Above, $T=-(\hbar^2/2m)\nabla^2$.[8] Finally we may write (3.4) in the form

$$\{T+V-E+(\epsilon/\hbar)[T^2-(\psi^{-1}T\psi)^2]\}\psi=0. \qquad (3.6)$$

Note that (3.6) is nonlinear and so it differs quite radically from the time-independent Schrödinger equation. However, to a first approximation one can say $T\psi\sim(E-V)\psi$, so that to lowest order (3.6) becomes

$$\{T+V+(\epsilon/\hbar)[V,T]\}\psi=E\psi. \qquad (3.7)$$

One could, of course, take (3.7) as a point of departure for the usual formulation of quantum mechanics. Then, the equation is suggestive since an intuitive way of looking at the self-interaction is to consider (3.7) as a first-order result in the expansion of the "time-translated" expression

$$H'=\exp(iS_0\tau)H\exp(-iS_0\tau)\,, \qquad (3.8)$$

where $H\equiv T+V$, $S_0\equiv\tfrac{2}{3}V/\hbar$, and $\tau\equiv i\tau_0$. Thus, if τ were real, the result would leave the energy spectrum unchanged. For times of the order of τ_0, however, self-interaction effects must be examined, and these virtual

[8] Another way of writing this is as $V_{\text{eff}}=-\epsilon\mathcal{O}\varphi_B$, where φ_B is the Bohm potential (Ref. 4); for a recent discussion of this potential in relation to a connection with the Navier-Stokes equation see R. J. Harvey, Phys. Rev. **152**, 1115 (1966). Harvey's reservations about a theory based on Brownian motion do not seem to apply to Nelson's work.

effects enter by means of a complex time transformation of magnitude τ_0, which shifts the energy spectrum.

Perhaps the above interpretation can lead to a linear generalization of quantum mechanics with self-interaction effects; on the other hand, one should keep in mind the caveat involving the extension of series in the parameter τ_0 beyond the first order.[9]

IV. TIME-DEPENDENT CASE

The generalization to the time-dependent case proceeds directly in the manner following Nelson except that the canonical momentum $\hbar\nabla S$ is chosen differently. The basic equations are still (2.7), although now we have

$$\psi=\exp(R+iS), \tag{4.1}$$

where

$$\nabla S=(m/\hbar)(\mathbf{v}+\mathbf{\Gamma}), \tag{4.2}$$

$$\mathbf{\Gamma}=(\epsilon\hbar/m)\int_0^t(\nabla\varphi_B\cdot\nabla)\mathbf{u}dt'. \tag{4.3}$$

The reason for this choice of $\mathbf{\Gamma}$ will become evident later; here[8] $\varphi_B=-(\hbar/2m)(\nabla^2\psi/\psi)$.

We also set

$$\mathbf{f}=-\nabla V/m+\mathbf{v}\times(\nabla\times\mathbf{\Gamma}), \tag{4.4}$$

so that an effective force (analogous in form to a magnetic force) is introduced. In what follows we assume we are looking at situations for which $\mathbf{v}$ is small (or is taken equal to zero) so that the term of order ϵ and involving $\mathbf{v}$ in (2.7b) is neglected. This is consistent with the cases examined in Sec. V, involving the bound electron in a hydrogen atom and an electron at rest in a fixed magnetic field.

Equations (2.7) with (4.2) and (4.4) then lead to the following time-dependent equation:

$$\partial\psi/\partial t=-(i/2m\hbar)[i\hbar\nabla-m\mathbf{\Gamma}]^2\psi -(i/\hbar)(V+V_{\text{eff}})\psi+i\alpha(t)\psi, \tag{4.5}$$

where V_{eff} is given by (3.5), $R\equiv\frac{1}{2}\ln(\psi^*\psi)$, $\mathbf{u}=(\hbar/m)\nabla R$, and $\alpha(t)$ is an arbitrary phase factor (which can be taken equal to zero).

Of course, one can also proceed from Eq. (4.5) and derive Eqs. (2.7) (within the limitation of the small-$\mathbf{v}$ assumption) without regard to the stochastic derivation of the self-interaction potential. In fact, the Eqs. (2.7) (with $\epsilon=0$) are used, with suitable further approximations, for application of the WKBJ method.[10]

Although we will not explore in detail the time-dependent case for more general situations, it can be easily seen that if $\mathbf{v}$ is not small, the structure of (4.5) [as it results exactly from (2.7)] is changed insofar as self-interaction terms appear on both sides of the equation. Also it may be shown that a generalization involving the Lorentz force may be obtained by a suitable definition of $\mathbf{v}$ in terms of S, $\mathbf{\Gamma}$, and the usual vector potential; we feel, however, that such a generalization is more appropriate to the relativistic case.

Since we are concerned with the limit $\mathbf{v}\to 0$, and recalling that $\partial\mathbf{u}/\partial t=0$ for this case, we get from (4.3)

$$\mathbf{\Gamma}=(\epsilon\hbar/m)(\nabla\varphi_B\cdot\nabla)\mathbf{u}t.$$

Now for times t of $O(\epsilon)$, $\mathbf{\Gamma}$ is of $O(\epsilon^2)$. Thus we can say that we effectively have, to order ϵ,

$$\partial\psi/\partial t=i(\hbar/2m)\Delta\psi-(i/\hbar)(V+V_{\text{eff}})\psi. \tag{4.6}$$

In subsequent calculations we assume the above is true. For significantly longer times this is no longer the case and $\mathbf{\Gamma}$ would have to be taken into account. Note that in the $\mathbf{v}\to 0$ limit, (4.4) reduces to $\mathbf{f}=-\nabla V/m$ and so we recover the case in Sec. III; it also then follows that neglect of $\mathbf{\Gamma}$ above leads to (3.4).

If one is dissatisfied with the stochastic interpretation, one can refer to the argument given in connection with the latter part of Sec. III. However, we wish to point out that it is possible that the nonlinear equation (3.6) may give better results quantitatively, and it is not clear how one could obtain such an equation without reference to the stochastic argument.

It must always be borne in mind that the treatment is nonrelativistic. In the following section we examine the case of the Lamb shift in the hydrogen atom to lowest order, and, viewed as a two-body problem, it is seen that the interaction in (3.7) will be invariant under the Galilean group of transformations.[11]

A relativistic generalization would have the relativistic radiation-damping equation[12] as a starting point.[13] However, it is not clear in what sense one should expect the Dirac equation to result since it would appear that the *Zitterbewegung* is already contained in it; but at any rate it may be possible to obtain relativistic self-interaction corrections which compare favorably with experiment by an extension of the present treatment.

V. LAMB SHIFT AND ANOMALOUS MAGNETIC MOMENT

Using (3.7), we evaluate the energy shift, interpreted as the Lamb shift, in hydrogen. In a first-order perturbation calculation we get (note that although con-

[9] This is so because we have assumed only linear terms at the outset. For a discussion of this point see F. Rohrlich, in *Classical Charged Particles* (Addison-Wesley Publishing Co., Cambridge, Mass., 1965), p. 156.

[10] For a recent use of the equations in this form (but without self-interaction) see T. W. Kibble, Phys. Rev. **150**, 1060 (1966).

[11] L. L. Foldy, Phys. Rev. **122**, 275 (1961).

[12] See A. O. Barut, *Electrodynamics and Classical Theory of Fields and Particles* (Macmillan Publishing Co., New York, 1964), p. 185.

[13] A recent covariant generalization of Brownian motion by R. Hakim, J. Math. Phys. **6**, 1482 (1965) may be of interest in a possible relativistic generalization of the stochastic theory here. However, a preprint recently received from Hakim suggests that there may be difficulties in defining relativistically covariant stochastic processes in the manner of Nelson.

ventional notation is used here, the expectation can also be written as an average involving the probability density)

$$\delta E=(\epsilon/\hbar)\langle\psi|[V,T]|\psi\rangle, \qquad (5.1)$$

where the ψ's represent the unperturbed hydrogen levels. Observing that

$$[V,T]\psi=(\hbar^2/2m)[(\nabla^2V)+2\nabla V\cdot\nabla]\psi,$$

we find

$$\delta E=(\epsilon\hbar/2m)Ze^2[\langle\psi|4\pi\delta(\mathbf{r})|\psi\rangle+2\langle\psi|r^{-2}\partial/\partial r|\psi\rangle]. \qquad (5.2)$$

This gives no contribution to the $2P_{1/2}$ state, but both terms in (5.2) contribute equally to a shift in the $2S_{1/2}$ state. Thus the Lamb shift is here

$$\delta E=(\epsilon\hbar/2m)Ze^2 8\pi|\psi_{2S}(0)|^2=(\tfrac{2}{3})Z^4\alpha^3\,\mathrm{Ry}, \qquad (5.3)$$

where $\alpha=e^2/\hbar c$. This gives a value[14] of 855 Mc/sec for the shift and compares with the result of 1600 Mc/sec obtained by Welton[15] and frequently quoted as giving an heuristic picture of this self-interaction effect in quantum electrodynamics. Note, however, that the result given here does not require cutoffs (there is no logarithmic term), because the introduction of independent fields has been avoided. So a desirable feature of the present calculation is the absence of any requirement for a renormalization or cutoff procedure, of any kind. Our calculation compares semiquantitatively with the recent experimental value[1] of 1058 Mc/sec; however, it must again be stressed that the treatment given here is nonrelativistic. Nevertheless we feel that the comparative simplicity of the approach is appealing; also, there is the possibility that the nonlinear equation (3.6) might give better agreement.

One can also make an estimate of the correction to the magnetic moment of an electron by means of (3.7). Consider the interaction of an otherwise free electron in an external magnetic field of constant magnitude. Then we may write

$$\begin{aligned}H'\psi&=(\epsilon/\hbar)[(\boldsymbol{\mu}\cdot\mathbf{H})T-T(\boldsymbol{\mu}\cdot\mathbf{H})]\psi\\&=(\epsilon\hbar/2m)[\psi\Delta(\boldsymbol{\mu}\cdot\mathbf{H})+2\nabla\psi\cdot\nabla(\boldsymbol{\mu}\cdot\mathbf{H})], \qquad (5.4)\end{aligned}$$

where $H'=(\epsilon/\hbar)[V,T]$, and $V=\boldsymbol{\mu}\cdot\mathbf{H}$. We see from (5.4) that the total interaction to first order is

$$\boldsymbol{\mu}\cdot\mathbf{H}+(\epsilon\hbar/2m)[\Delta(\boldsymbol{\mu}\cdot\mathbf{H})+2\nabla(\boldsymbol{\mu}\cdot\mathbf{H})\cdot\nabla]. \qquad (5.5)$$

When we examine the limit as the electron wave vector goes to zero, the second term in brackets in (5.5) gives vanishing contribution and if we introduce the Compton wavelength as a characteristic length, such that

$$\Delta\mu/\mu\sim(\lambda_c)^{-2}. \qquad (5.6)$$

The following correction results:

$$\delta\mu/\mu_0=\epsilon\hbar/2m\lambda_c^2=\alpha/3,$$

where $\mu_0=\epsilon\hbar/2mc$. This compares with the first order QED result of $\alpha/2\pi$. The crudity of the assumption (5.6) is apparent, however.[16]

It is to be pointed out that while the temptation to insert an adjustable constant in place of $\epsilon=(\frac{2}{3})\tau_0$ exists [one could, for instance, multiply τ_0 by a term proportional to $\ln(\alpha^{-1})$], it would be somewhat meaningless in view of our nonrelativistic treatment. Also, such adjustment would run counter to the intended spirit of the argument.[17]

The theory and examples thus far considered have dealt with situations involving "virtual quanta" (in the language of QED). We were led to the equations (2.7) via stochastic mechanics, assuming a formulation which bore resemblance to damping theory in electrodynamics, and coupling the theory with the desire for time-symmetric equations of motion.

If a description involving energy loss is sought, we conjecture that a possible way to proceed is as follows: First we abandon the time-symmetry requirement by writing, instead of (2.4),

$$\mathbf{a}=(2\gamma)^{-1}[D+D_*]\mathbf{a}+\mathbf{f}. \qquad (5.7)$$

Proceeding in the same way as before, we arrive at the equations of motion

$$\partial\mathbf{u}/\partial t=-(\hbar/2m)\nabla(\nabla\cdot\mathbf{v})-\nabla(\mathbf{v}\cdot\mathbf{u}), \qquad (5.8)$$

$$\mathcal{L}\mathbf{v}=\mathbf{f}+\mathcal{O}\mathbf{u}+\epsilon\mathcal{L}[\mathcal{L}\mathbf{v}-\mathcal{O}\mathbf{u}], \qquad (5.9)$$

where $\mathcal{L}=\partial/\partial t-\mathbf{v}\cdot\nabla$, and where $\mathbf{u}$, $\mathbf{v}$, and $\mathcal{O}$ are defined in the same manner as before. Note the time-asymmetry in (5.9) as opposed to (2.7b). Thus while we have qualitatively different equations here (they are identical only if $\epsilon=0$), the attitude here is that the distinction between conservative and nonconservative systems is clear enough to justify it. On the other hand, we are aware that a formalism capable of describing radiation damping in classical electrodynamics, starting from an initially time-symmetric theory, has been developed.[18] Nevertheless, the Wheeler-Feynman formalism is classical and deterministic, and requires

[14] The small "vacuum polarization" effects are not involved in this calculation; this is consistent with the spirit of the nonrelativistic argument.

[15] T. A. Welton, Phys. Rev. **74**, 1157 (1948). See also J. D. Bjorken and S. D. Drell, *Relativistic Quantum Mechanics* (McGraw-Hill Book Co., Inc., New York, 1964), p. 58; E. A. Power, *Introductory Quantum Electrodynamics* (American Elsevier Publishing Co., New York, 1964), p. 35. The numerical value quoted here is that of Power; there is some variation depending on how one chooses the cutoff. For a modification of Welton's ideas, including "vacuum polarization," see Z. Koba, Progr. Theoret. Phys. (Kyoto) **4**, 319 (1949).

[16] An assumption similar to this, involving the Compton wavelength, is made by Welton, Ref. 15.

[17] A phenomenological relativistic treatment of the Lamb shift and the anomalous magnetic moment, which contains three adjustable parameters, has been given by F. J. Belinfante, Phys. Rev. **84**, 949 (1951).

[18] J. Wheeler and R. P. Feynman, Rev. Mod. Phys. **17**, 157 (1945). See also Ref. 12 for a discussion of such action-at-a-distance theories.

additional assumptions which have not been fully amplified with respect to their role on the question of irreversibility.

Finally we observe that one could write a closer, time-asymmetric analog to (2.3):

$$\mathcal{L}\mathbf{v}=\mathbf{f}+\mathcal{O}\mathbf{u}+\epsilon[\mathcal{L}+\mathcal{O}][\mathcal{L}\mathbf{v}-\mathcal{O}\mathbf{u}]. \quad (5.10)$$

In contrast with the situation with (5.9), however, the results for $\mathbf{v}=0$ follow as before.

Note added in proof. There is an error of sign in going from (5.2) to (5.3), so that the equal contributions actually cancel. This also follows from the anti-Hermiticity of the linearized interaction; we are grateful to Professor Richard P. Feynman for pointing this out. Thus the linearized equation does not give an energy shift, unless possibly one examines the case with complex wave functions and re-examines the role of $\mathbf{\Gamma}$, in Sec. IV. The basic set of Eqs. (2.7) and the nonlinear Eq. (3.6) are not affected by the above considerations.

VI. DISCUSSION

We have obtained generalizations of the Schrödinger equation, involving self-interactions, by means of classical stochastic mechanics. While it is possible to give a more conventional interpretation for the genesis of the interaction term (as in Sec. III), this interpretation only appears to apply to the linearized approximation of (3.6). The calculations of effects interpreted as the Lamb shift and anomalous magnetic moment seem to give some encouragement to the hope that a relativistic generalization of the preceding theory might give some interesting results. While the avoidance of the explicit introduction of fields is considered an anathema in some quarters, here we view the divergence difficulties and renormalization procedures of current field theories as offering inducement in attempting to find a different formulation.

Apart from treating self-interactions, we would like to point out that a desirable feature of our treatment vis-à-vis Nelson's, is that a dynamics based on radiation damping phenomena appears more readily justifiable on a microscopic basis than one which proceeds from the Langevin equation (where the interpretation of the frictional term is more obscure).

A difficulty in the present treatment concerns the interpretation of Eq. (4.2). While the form is suggestive, and while it may be possible eventually to understand it physically, the way is not clear at the present time. In a recent article[19] however, it has been argued that in a classical correspondence from QED involving "beam-induced" self-energies it is necessary to "renormalize" the momentum, in carrying out classical calculations. While there is no beam in the examples considered here, perhaps an argument for an analogous procedure can be made.

An interesting point about the linearized equation (3.7) is that if $[V,T]=0$, there is no self-interaction. Thus with this equation (that is, to first order in ϵ) it is meaningless to speak of the self-interaction of an isolated charge. On the other hand, if higher orders in ϵ are examined, then the self-interaction of an isolated charge can appear.

Of course, the formalism describes a spinless, nonrelativistic single particle, and thus it ignores questions involving statistics and incorporation of many-body interparticle-interaction effects. An interesting point can be made relative to a possible relativistic many-body generalization: If one did create a valid rigorous generalization, it might serve to remove a serious objection to the formulation of current direct interaction theories,[20] namely the objection involving noninvariance of world lines. A measure of validity for the stochastic hypothesis would indicate that the requirement that these trajectories be invariant as seen by all Lorentz observers should be relaxed. This relaxation would then follow as a result of a theory involving classical indeterminism (however, the equivalence of this hypothetical relativistic theory to a theory involving quantum indeterminism might not hold).

Finally, we again point out that the stochastic argument in the derivation of the effective interaction may be in a sense replaced by the interpretation in connection with (3.8), especially if we demand that the principle of superposition must hold down to phenomena involving the time constant τ_0.[21]

ACKNOWLEDGMENTS

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[19] P. Stehle and P. G. DeBaryshe, Phys. Rev. **152**, 1135 (1966).

[20] D. G. Currie, J. Math. Phys. **4**, 1470 (1963). For a review of these difficulties see P. Havas, in *Statistical Mechanics of Equilibrium and Nonequilibrium*, edited by J. Meixner (North-Holland Publishing Co., Amsterdam, 1965).

[21] With respect to the equivalence of the stochastic and usual formulations where superposition holds, see the argument of Nelson (Ref. 3).