The $\omega\rightarrow\eta\gamma$ width obtained from (3.8) requires the correction

$$C(\omega\rightarrow\eta\gamma)=\left[\cos\lambda_1\cos\lambda_2-(\sqrt{8})\frac{(4m_V m_{\Pi^0})^{1/2}G_{\gamma V\Pi^0}{}^0}{(4m_\Pi m_{V^0})^{1/2}G_{\gamma V^0\Pi^0}}\sin\lambda_1\sin\lambda_2-\left(\frac{8}{5}\right)^{1/2}\frac{(4m_\Pi m_V)^{1/2}G_{\gamma V\Pi}{}^D}{(4m_\Pi m_{V^0})^{1/2}G_{\gamma V^0\Pi^0}}\sin\lambda_2\cos\lambda_1\right]^2.$$

Here again our choice of relative sign in (A19) and (A20) determines this correction uniquely as[12,25]

$$C(\omega\rightarrow\eta\gamma)=[\cos\lambda_1\cos\lambda_2+\sin|\lambda_1|\,\sin|\lambda_2|-\tfrac{1}{2}\sqrt{2}\,\sin|\lambda_1|\,\cos\lambda_2]^2.$$

For the $V\rightarrow l^+l^-$ decays the ω-ϕ mixing presents no complications because the singlet component does not contribute. We find

$$\Lambda_4(\omega^0\rightarrow l^+l^-)=-\tfrac{1}{3}\sqrt{3}\,\sin\lambda_2$$

and

$$\Lambda_4(\phi\rightarrow l^+l^-)=\tfrac{1}{3}\sqrt{3}\,\cos\lambda_2\,,$$

where Λ_4 is defined in (3.10).

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Algebra of Current Components and the Hypothesis of Partially Conserved Axial-Vector Current Applied at High Energies

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Using the hypothesis of partially conserved axial-vector currents, the algebra of current components, and the assumption that the pion-hadron total cross section $\sigma(s)$ approaches its asymptotic value rapidly, a method is developed which allows a calculation of the elastic amplitude at high energies and small momentum transfers. This method uses the fact that asymptotically the dynamics is given by the commutator on the light cone. The results are $\sigma_{\pi p}(\infty)=25.7\pm4.2$ mb and $d\sigma/dt=(d\sigma/dt)_{t=0}[G_E{}^2-(t/4M^2)G_M{}^2](1-t/4M^2)^{-1}$ (for small values of the momentum transfer t), where $G_E(t)$ and $G_M(t)$ are the electric and magnetic form factors of the proton. It is shown that possible Schwinger terms in the equal-time commutators are without importance for our results. An important feature of our calculation is that the energy and the momentum are allowed to go to infinity simultaneously; our method therefore deviates essentially from the Bjorken limit, which in general involves a continuation of the amplitude infinitely off the mass shell.

1. INTRODUCTION

In the present paper we shall present a calculation of the high-energy total cross sections $\sigma(\infty)$ for pion-hadron scattering which gives good agreement with the value of $\sigma_{\pi p}(\infty)$ obtained by fitting forward dispersion relations. The main tools in our derivation are the following three assumptions.

(i) The partially conserved axial-vector current (PCAC) hypothesis: The divergence of the $\Delta S=0$ axial-vector current $j_\mu{}^\pm(x)$ is proportional to the pion field[1,2] in the $SU(3)$ limit,[3,4]

$$\langle\alpha|\,\partial^\mu j_\mu{}^\pm(0)\,|\beta\rangle_{SU(3)}=-if_\pi m^2\langle\alpha|\,\varphi^\pm(0)\,|\beta\rangle_{SU(3)}\,,\quad(1)$$

$$f_\pi=\sqrt{2}Mg_A/g\,,\quad(1')$$

where $\varphi^\pm(x)$ is the renormalized Heisenberg field of the charged pions, m is the pion mass, M is the nucleon mass, g is the renormalized pion-nucleon coupling constant, and g_A is the renormalization (by the strong interactions) of the axial-vector coupling constant in β-decay. The index "$SU(3)$" indicates that the matrix elements are evaluated in the mass-degenerate $SU(3)$ limit. As pointed out in Ref. 4, it is reasonable to expect that the $SU(3)$ limit is achieved at high energies, i.e., when the energy difference between the states $|\alpha\rangle$ and $|\beta\rangle$ becomes very large; at the same time the invariant momentum transfer between $|\alpha\rangle$ and $|\beta\rangle$ approaches zero.

(ii) The equal-time commutators ($x_0=0$):

$$[j_\mu{}^+(x),\,j_0{}^-(0)]=2j_\mu{}^{V3}(0)\delta(\mathbf{x})+\text{S.T.}\,,\quad(2)$$

$$[j_l{}^+(x),\,j_k{}^-(0)]=2\delta_{kl}j_0{}^{V3}(0)\delta(\mathbf{x})+\text{tensor term antisymmetric in } k \text{ and } l+\text{S.T.}\quad(3)$$

are assumed. Here $j_\mu{}^{V3}(x)$ is the third component of the isovector current, and S.T. stands for possible Schwinger terms. The commutators (2) and (3) are obtained from a quark model for the currents. We assume that these commutators can be abstracted from the model and postulated as true for the physical

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currents.[3] It has been shown by Okubo and co-workers[5] that Eq. (3) contains operator Schwinger terms, and in general this will cause severe troubles in the application of Eq. (3); however, in our use of Eqs. (2) and (3) the S.T. shall be shown not to play any role. It is also well known[6] that Eq. (3) leads to difficulties if one applies sum-rule technique. However, we use Eq. (3) directly without introducing intermediates states and saturation.

(iii) We assume that the high-energy cross section $\sigma(s)$ ($\sqrt{s}$= center-of-mass energy) approaches a constant asymptotic limit $\sigma(\infty)$. We furthermore make the assumption that the dynamics determining $\sigma(\infty)$ at high energies is "smooth," which in x space can be formulated precisely by assuming that the commutator

$$\langle\alpha|[j_\mu^+(x),j_\nu^-(0)]|\beta\rangle$$

does not contain terms more singular than $\delta(x^2)$ and derivatives of $\delta(x^2)$. In addition to these singular terms the commutator is allowed to contain arbitrarily complicated regular terms.[7,8]

The program of the paper is as follows: In Sec. 2 we discuss the applications of PCAC at high energies, not only from the point of view expressed in assumption (i) above, but also from the point of view of the pion pole dominance (PDDAC). At high energies these points of view are equivalent. In Sec. 3 we discuss the role played by PCAC [assumption (i)] in eliminating the Schwinger terms. After these preliminaries, we derive in Sec. 4 an expression for the scattering amplitude at high energies and small momentum transfers. The final expression for the scattering amplitude contains a factor N, which is of the order 1. In Sec. 5 we calculate N. In Sec. 6 we compare our results with the existing experimental and phenomenological information. We also discuss the connection between our results and the results obtained by Domokos and Karplus[9] for the scattering amplitude at high energies and large momentum transfers.

2. PCAC AT HIGH ENERGIES

The form of PCAC given in Eq. (1) was based on the observation that the baryon mass splitting destroys the exact validity of PCAC.[4] We therefore assumed the validity of PCAC in cases where one can neglect the mass splitting, and this is just achieved using PCAC at high energies, i.e., when the energy E_α of the state $|\alpha\rangle$ is much larger than the energy of the state $|\beta\rangle$ (or, more generally, when $|E_\alpha-E_\beta|\to\infty$). In this section we shall show that this point of view is equivalent to PDDAC (pion pole dominance in matrix elements of the divergence of the axial-vector current).

Consider the matrix element $\langle f|\partial^\mu j_\mu^\pm(0)|1\rangle$, where $|1\rangle$ is a one-particle state. The contribution from graphs with a single pion line connecting $\partial^\mu j_\mu^\pm(0)$ with the other lines is given by

$$m^4 f_\pi \frac{1}{m^2-q^2}\langle f|\varphi^\pm(0)|1\rangle, \tag{4}$$

where q is the four-momentum difference between the states $|1\rangle$ and $|f\rangle$. Now, when the energy difference between the states $|1\rangle$ and $|f\rangle$ goes to infinity, the three-momentum difference between these states cancels the energy difference and $q^2\to 0$. However, since PDDAC tells us that the contribution (4) to $\langle f|\partial^\mu j_\mu^\pm(0)|1\rangle$ dominates for $q^2\to 0$, it follows that PDDAC is equivalent to PCAC (in the sense of Ref. 4). Thus the main point in our application of PCAC at high energies is that the zero-momentum-transfer limit $q^2=0$ is achieved in a "natural" way at high energies (where one also has mass degeneration in a "natural" way). In this connection we note that the index $SU(3)$ in Eq. (1) only refers to mass degeneration; it is, e.g., not necessary to use $SU(3)$ coupling constants in the evaluation of the matrix elements entering in Eq. (1).

For our application of PCAC it is of interest to have an estimate of the degree of validity of the pion pole dominance. We can obtain a rough estimate by assuming superconvergence. Consider, e.g., the matrix element $\langle 0|\partial^\mu j_\mu^\pm(0)|\pi\rangle$ for which we have the dispersion relation

$$\langle 0|\partial^\mu j_\mu^\pm(0)|\pi\rangle=\frac{m^4 f_\pi}{m^2-q^2}+\frac{1}{\pi}\int_{9m^2}^\infty \frac{\sigma(q'^2)dq'^2}{q'^2-q^2}, \tag{5}$$

where $\sigma(q^2)$ is a spectral function. Defining the mean value

$$\langle 1/q^2\rangle=\int_{9m^2}^\infty \frac{\sigma(q^2)}{q^2}dq^2 \Big/ \int_{9m^2}^\infty \sigma(q^2)dq^2, \tag{6}$$

we have

$$\langle 0|\partial^\mu j_\mu^\pm(0)|\pi\rangle=m^2 f_\pi+\langle 1/q^2\rangle\int_{9m^2}^\infty \sigma(q^2)dq^2 \tag{7}$$

for $q^2=0$. From superconvergence we get

$$m^4 f_\pi=-\int_{9m^2}^\infty \sigma(q^2)dq^2, \tag{8}$$

so that

$$\langle 0|\partial^\mu j_\mu^\pm(0)|\pi\rangle=m^2 f_\pi(1-m^2\langle 1/q^2\rangle) \tag{9}$$

for $q^2=0$. Now it is reasonable to expect that $\langle 1/q^2\rangle \lesssim 1/(9m^2)$. Hence the pion pole dominates, i.e., $\langle 0|\partial^\mu j_\mu^\pm(0)|\pi\rangle=f_\pi m^2$, with an accuracy of $\frac{1}{9}$, i.e., 11%. This is consistent with the numerical evaluation of the Goldberger-Treiman relation, assuming that the pion-

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nucleon vertex function varies a very small amount from $q^2=m^2$ to $q^2=0$.

3. ASYMPTOTIC BOUNDS AND SCHWINGER TERMS

In the application of PCAC and the algebra of current components an essential tool is the use of differentiations of retarded commutators (or time-ordered products). We therefore consider the matrix element of the retarded commutator

$$R_{\mu\nu}(q_1^2,q_2^2,(q_1)_0) = \int dx\, e^{iq_2x}\langle p'|[j_\mu^-(x),j_\nu^+(0)]|p\rangle\theta(x_0)\,, \quad (10)$$

where $q_2=p+q_1-p'$. The states $|p\rangle$ and $|p'\rangle$ are single-particle states. From Eq. (10) we obtain the identity

$$q_1^\mu q_2^\nu R_{\mu\nu}(q_1^2,q_2^2,(q_1)_0) = \int dx\, e^{-iq_1x}\{\theta(x_0)\langle p'|[\partial^\mu j_\mu^-(0),\partial^\nu j_\nu^+(0)]|p\rangle - \delta(x_0)\langle p'|[\partial^\mu j_\mu^-(0),j_0^+(x)]|p\rangle + i\delta(x_0)q_1^\nu\langle p'|[j_0^-(0),j_\nu^+(x)]|p\rangle\}\,. \quad (11)$$

In this equation we have two equal-time commutators, and Schwinger terms give rise to higher powers of q than the canonical values of the commutators. Hence we can identify the Schwinger terms by looking at the asymptotic behavior of the expressions involving the retarded commutators, as pointed out by Bjorken.[10]

At this stage it is convenient to introduce PCAC in order to identify the first term on the right-hand side of Eq. (11) with the pion-nucleon amplitude (the zero-mass approximation should be completely harmless at high energies)

$$q_1^\mu q_2^\nu R_{\mu\nu}(q_1^2=0,\ q_2^2=0,\ (q_1)_0) = if_\pi^2\langle p'\pi^+(q_2)|T|p\pi^+(q_1)\rangle - \int dx\,\delta(x_0)\,e^{-iq_1x}\{\langle p'|[\partial^\mu j_\mu^-(0),j_0^+(x)]|p\rangle - iq_1^\nu\langle p'|[j_0^-(0),j_\nu^+(x)]|p\rangle\}\,. \quad (12)$$

Following Bjorken[10] the number of necessary Schwinger terms can be obtained from the asymptotic behavior of $R_{\mu\nu}((q_1)_0\to\infty)$ $[\neq\tilde{R}_{\mu\nu}((q_1)_0\to\infty)]$ as compared to the asymptotic behavior of the *covariant amplitude* $\tilde{R}_{\mu\nu}((q_1)_0\to\infty)$. The difference in the asymptotic behavior of $\tilde{R}_{\mu\nu}$ and $R_{\mu\nu}$ (constructed from the retarded commutator) is a polynomial in q_0, which can be identified with the polynomial in q_0 arising from the possible Schwinger terms.

The device given by Bjorken is usually not of much practical value because the asymptotic behavior of $\tilde{R}_{\mu\nu}$ is unknown. However, PCAC is of great help here because the forward pion-nucleon amplitude

$$\langle p\pi^+(q)|T|p\pi^+(q)\rangle$$

is bounded from analyticity and unitarity. Hence we know that the first term (which we now replace by the covariant pion-nucleon amplitude) on the right-hand side of Eq. (12) is bounded by a certain power of q_0, q_0^m, say (we neglect logarithms). The asymptotic behavior of $R_{\mu\nu}$ will be shown in Sec. 4 to be given by $1/q_0$, and hence $q_\mu q_\nu R^{\mu\nu}$ behaves like q_0. Hence the maximal number of Schwinger terms is $m-1$.

The actual value of the maximal power m has recently been obtained by Martin,[11] who established the Foissart[12] bound

$$|\langle p\pi^+(q)|T|p\pi^+(q)\rangle\leq Cs(\ln s)^2 \quad (13)$$

(if s is sufficiently large) from axiomatic field theory and polynomial boundedness for $s\to\infty$. It then follows that $m=1$, and hence the Schwinger terms do not, in general, contribute. Thus, in spite of what one would expect immediately, Schwinger terms do not become important at high energies; in this respect, high-energy theorems are similar to low-energy theorems.

The result that S.T. do not contribute does not contradict the results of Ref. 5, where it is shown that the commutator (3) contains operator S.T. The point is that operator S.T. can have certain vanishing matrix elements and certain nonvanishing matrix elements. Our conclusion is then that the baryon matrix elements of the operator S.T. vanish.

We finally mention that above we have only been able to establish the absence of Schwinger terms in the forward direction. However, from "continuity" we expect that these terms are absent also for small angles.

4. CALCULATION OF THE ASYMPTOTIC PION-HADRON AMPLITUDE AT SMALL MOMENTUM TRANSFERS

We shall now calculate the asymptotic behavior of the pion-hadron amplitude from Eq. (12), which can be written

$$if_\pi^2[T_+(p,p';\,(q_1)_0)-T_+(p,p';\,(q_1)_0=0)] = iq_1^\nu\int dx\, e^{iq_2x}\delta(x_0)\langle p'|[j_0^-(x),j_\nu^+(0)]|p\rangle - q_1^\mu q_2^\nu R_{\mu\nu}(q_1^2=0,\ q_2^2=0;\ (q_1)_0)+\text{S.T.}\,, \quad (14)$$

where T_+ is the pion-hadron amplitude. Taking the

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real parts of both sides of Eq. (14) we obtain

$$f_\pi{}^2 \,\mathrm{Im}T_+(p,p';(q_1)_0)=\tfrac{1}{2}q_1{}^\mu q_2{}^\nu\int dx\,\theta(x_0)$$
$$\times(\cos(q_2x)\{\langle p'|[j_\mu{}^-(x),j_\nu{}^+(0)]|p\rangle$$
$$+\langle p|[j_\nu{}^-(0),j_\mu{}^+(x)]|p'\rangle\}+i\sin(q_2x)$$
$$\times\{\langle p'|[j_\mu{}^-(x),j_\nu{}^+(0)]|p\rangle$$
$$-\langle p|[j_\nu{}^-(0),j_\mu{}^+(x)]|p'\rangle\}). \quad (15)$$

Now it has been shown[7,8] that at high energies the light-cone singularities give the dominating contribution to the expression (15). The important question, however, is which type of singularity we have on the light cone. In order to study this problem we shall use the current algebra (it is rather unlikely that local field theory contains any specific information on the type of the light-cone singularities). For simplicity we only consider the forward direction.

For the relevant commutator

$$F_{\mu\nu}(x)=\langle p|[j_\mu{}^-(x),j_\nu{}^+(0)]|p\rangle, \quad (16)$$

we have the Dyson representation[7] (for simplicity p is considered to be scalar)

$$F_{\mu\nu}(x)=\int_0^\infty dm^2 f_{\mu\nu}(m^2,x,p)\Delta(x,m) \quad (17)$$

$$f_{\mu\nu}(m^2,x,p)=\rho_1(x,p,m^2)\partial_\mu\partial_\nu+\rho_2(x,p,m^2)p_\mu\partial_\nu$$
$$+\rho_3(x,p,m^2)p_\mu p_\nu+\rho_4(x,p,m^2)g_{\mu\nu}, \quad (17')$$

where we only consider the symmetric part of $F_{\mu\nu}$ [since we are multiplying $F_{\mu\nu}$ by $q_\mu q_\nu$ in Eq. (15)]. Considering the various equal-time commutators, we find the relations $\mu=0$, $\nu=0$:

$$\int\left[\rho_2 M\frac{\partial}{\partial x_0}+\rho_3 M^2+\rho_4\right]\Delta(x,m)|_{x_0=0}dm^2$$
$$=2\langle p|j_0{}^{V3}(0)|p\rangle\delta(\mathbf{x}), \quad (18a)$$

$\mu=k$, $\nu=l$:

$$\int\rho_4\Delta(x,m)|_{x_0=0}dm^2=-2\langle p|j_0{}^{V3}(0)|p\rangle\delta(\mathbf{x}). \quad (18b)$$

In the space-space case we have used the fact that any possible S.T. finally turns out to be unimportant, so that

$$\rho_1(x,p,M^2)=0. \quad (18c)$$

The *simplest* solution consistent with Eqs. (18) is (other derivatives with respect to x_0 do not change the arguments)

$$\rho_4(x,p,m^2)=\rho_4(m^2,xp,x^2)M\partial/\partial x_0, \quad (18d)$$

$$M\int_0^\infty dm^2\rho_4(m^2,\,xp=0,\,x^2=0)$$
$$=-2\langle p|j_0{}^{V3}(0)|p\rangle, \quad (18e)$$

$$\rho_3(x,p,m^2)=0, \quad (18f)$$

$$\rho_2(x,p,m^2)=\rho_2(m^2,xp,x^2), \quad (18g)$$

$$M\int_0^\infty \rho_2(m^2,\,xp=0,\,x^2=0)dm^2=4\langle p|j_0{}^{V3}(0)|p\rangle. \quad (18h)$$

We cannot exclude that other solutions exist, but the solution exhibited in Eqs. (18) is consistent with local field theory as well as the commutators (2) and (3). Collecting our results we get

$$\langle p|[j_\mu{}^-(x),j_\nu{}^+(0)]|p\rangle=\int dm^2\{\rho_2(m^2,xp,x^2)p_\mu\partial_\nu$$
$$+g_{\mu\nu}\rho_4(m^2,xp,x^2)p\partial/\partial x\}\Delta(x,m), \quad (19)$$

i.e., the relevant commutator satisfies a Dyson representation with only two spectral functions. From (19) it is obvious that we have two types of contributions to the commutator, namely, those coming from the singular and from the nonsingular part of $\Delta(x,m)$.

We shall first investigate the nonsingular contributions to the commutator and we shall show that this part of the commutator only contributes on the light cone for $q_\mu\to\infty$. To see this we use local commutativity,

$$F_{\mu\nu}(x)=0 \quad\text{for}\quad x_0<|\mathbf{x}|,$$

and a partial integration then gives the following result for the nonsingular part $\tilde{F}_{\mu\nu}(x)$ of $F_{\mu\nu}(x)$:

$$\int dx\,\theta(x_0)\begin{Bmatrix}\cos(qx)\\ \sin(qx)\end{Bmatrix}\tilde{F}_{\mu\nu}(x_0,\mathbf{x})$$
$$=\int d\mathbf{x}\int_{|\mathbf{x}|}^\infty dx_0\begin{Bmatrix}\cos(qx)\\ \sin(qx)\end{Bmatrix}\tilde{F}_{\mu\nu}(x_0,\mathbf{x})$$
$$=\frac{1}{q_0}\int d\mathbf{x}\begin{Bmatrix}-\sin(q_0|\mathbf{x}|-\mathbf{qx})\\ \cos(q_0|\mathbf{x}|-\mathbf{qx})\end{Bmatrix}$$
$$\times\tilde{F}_{\mu\nu}(|\mathbf{x}|,\mathbf{x})+O(1/q_0{}^2). \quad (20)$$

Hence, in the asymptotic limit $q_\mu\to\infty$ the only contribution comes from the surface of the light cone. This result as well as Eq. (19) shows that asymptotically the light-cone behavior of $F_{\mu\nu}(x)$ determines the dynamics. Equation (20) is actually only valid if $\tilde{F}_{\mu\nu}(x)$ is a smooth function, i.e., if the dynamics is smooth asymptotically (this turns out to be the case for π^+p scattering, but not for π^-p scattering).

Let us for the moment concentrate on the nonsingular contributions in Eq. (20). From Eqs. (15) and (20) we find (using invariance under parity)

$$f_\pi^2\,\mathrm{Im}T_+(p,p',(q_1)_0)=-[q_1{}^\mu q_2{}^\nu/(q_2)_0]$$
$$\times\int d\mathbf{x}\,\sin((q_2)_0|\mathbf{x}|)\,\cos\mathbf{q}_2\mathbf{x}$$
$$\times\langle p'|[j_\mu{}^-(x),j_\nu{}^+(0)]|p\rangle_{x_0=|\mathbf{x}|}{}^{\text{non-sing.}}+O(1)\rightarrow$$
$$-[q_1{}^\mu q_2{}^\nu/(q_2)_0]\int d\mathbf{x}\,\sin((q_2)_0|\mathbf{x}|)$$
$$\times\langle p'|[j_\mu{}^-(x),j_\nu{}^+(0)]|p\rangle_{x_0\sim1/(q_2)_0}{}^{\text{non-sing.}}+O(1)\quad(20')$$

for $q_\mu\rightarrow\infty$. We have used the fact that as $(q_2)_0\rightarrow\infty$ the main contribution to the integral comes from $|\mathbf{x}|\sim1/(q_2)_0$ because of the oscillating trigonometric function. The commutator with $x_0\sim1/(q_2)_0$ is almost an equal-time commutator, but we cannot perform the limit $x_0\sim1/(q_2)_0\rightarrow0$ directly since the argument of the sine then becomes undetermined ($\infty\times0$). Nevertheless we expect that the commutator is given roughly by its value at equal times, which actually means that the nonsingular part of the commutator does not contribute.

The singular contributions to the commutator are of the type

$$S(x)=\int_0^\infty dm^2\rho(m^2,Mx_0,x^2)(\partial/\partial x_0)\Delta(x,m)\,,\quad(21\text{a})$$

where we consider only the singular part of $\partial\Delta(x,m)/\partial x_0$. Since the most singular part of $\partial\Delta(x,m)/\partial x_0$ is independent of the mass variable m we actually have

$$S(x)=\frac{\partial}{\partial x_0}\Delta(x,\,m=0)\int_0^\infty dm^2\rho(m^2,\,Mx_0,\,x^2=0)\,,\quad(21\text{b})$$

where $\partial\Delta(x,\,m=0)/\partial x_0$ again means only the most singular part, and where we have used the fact that this part vanishes outside the light cone. It now follows by an argument of the same type as used in Eq. (20′) that provided ρ is a smooth function of x_0, the contribution to $\mathrm{Im}T_+$ of $S(x)$ is given by

$$\mathrm{Im}T_+\sim\int dx\,\cos(q_2x)\frac{\partial}{\partial x_0}\Delta(x,\,m^2=0)$$
$$\times\int_0^\infty dm^2\rho(m^2,\,x_0\sim1/(q_2)_0,\,x^2=0)\,,\quad(21\text{c})$$

so that for $(q_2)_0\rightarrow\infty$ the integral is dominated by

$$\int_0^\infty dm^2\rho(m^2,\,x_0\approx0,\,x^2=0)\,,\quad(21\text{d})$$

which is given by the sum rules (18), i.e., the integral (21d) is given by the current algebra. This procedure is discussed in detail in Appendix D. We find from the sum rules (18),

$$f_\pi^2\,\mathrm{Im}T_+(p,p'p(q_1)_0)=4(q_1)_0\langle p'|j_0{}^{V3}(0)|p\rangle N+\text{S.T.}$$
$$+O(1)=4q_1{}^\nu\langle p'|j_\nu{}^{V3}(0)|p\rangle N+\text{S.T.}\,,\quad(22)$$

where

$$N=\lim_{(q_2)_0\rightarrow\infty}\int d\mathbf{x}\,\sin((q_2)_0|\mathbf{x}|)\,\delta_{(q_2)_0}(\mathbf{x})\,,\quad(23)$$

$$\delta_{q_0}(\mathbf{x})\rightarrow\delta(\mathbf{x})\quad\text{for}\quad q_0\rightarrow\infty\,.\quad(24)$$

The result above can be understood intuitively from Eq. (20a), if we insert the total commutator in Eq. (20a). Equations (22)–(24) then state that the high-energy limit is dominated by the equal-time commutator if we insert instead of an exact δ function an approximate δ function $\delta_{q_0}(\mathbf{x})$ which for $q_0\rightarrow\infty$ becomes an exact δ function. It is very important that the function $\delta_{q_0}(\mathbf{x})$ occurs in Eq. (23); if instead we had an exact δ function the total cross section would vanish. The form of the function δ_{q_0} is suggested by the representation (19), and we shall discuss this in the next section and in Appendix A.

In calculating the equal-time commutator [which enters because of the sum rules (18)], we use Eqs. (2) and (3). The fact that the antisymmetric tensor terms in Eq. (3) do not contribute follows from the multiplication with the symmetric tensor $q_1{}^\mu q_2{}^\nu\rightarrow q_1{}^\mu q_1{}^\nu$ for $q_1{}^\mu\rightarrow\infty$. In Eq. (22) we have also used the fact that asymptotically $\mathbf{q}_1\mathbf{q}_2=(q_1)_0{}^2$ and $\langle p'|j_k{}^{V3}(0)|p\rangle=0$ (for small momentum transfers). Because of the asymptotic bound (13), possible S.T. in Eq. (22) are in fact absent.

In the next section we shall calculate N. Here we only mention that N is less than 1. This follows from the fact that $\delta_{q_0}(\mathbf{x})$ is practically vanishing for $|\mathbf{x}|>1/q_0$, and hence $\sin(q_0|\mathbf{x}|)$ is potisive. Replacing $\sin(q_0|\mathbf{x}|)$ by its upper limit 1, we obtain

$$N<\lim_{q_0\rightarrow\infty}\int d\mathbf{x}\,\delta_{q_0}(\mathbf{x})=1\,.\quad(25)$$

Hence we expect N to be some number between 0 and 1.

5. CALCULATION OF THE CONSTANT N

In this section we shall calculate the constant N defined in Eq. (23). It is therefore necessary to introduce some explicit expression for the function $\delta_{q_0}(\mathbf{x})$. The Källén representation suggests that the equal-time commutator behaves like

$$[\partial\Delta^{(+)}(x,m^2)/\partial x_0)/\partial x_0]_{x_0=0}\,,\quad(26)$$

and in Appendix A it is shown that this behavior leads to the following representation of the δ_{q_0} function:

$$\delta_{q_0}(\mathbf{x})=\frac{1}{\pi^2}\frac{\mu(q_0)}{[\mathbf{x}^2+\mu^2(q_0)]^2}\,,\quad(27)$$

$$\mu(q_0) \to 0 \quad \text{for} \quad q_0 \to \infty. \tag{28}$$

Equation (27) follows from (26) by replacing x_0 by $x_0 - i\mu$; instead of an exact δ function one then obtains Eq. (27). Equation (28) follows from Eq. (24). From Eq. (23) we obtain

$$N(q_0) = \frac{4}{\pi} \int_0^\infty \frac{\mu(q_0)|\mathbf{x}|^2 \sin(q_0|\mathbf{x}|)}{[|\mathbf{x}|^2 + \mu^2(q_0)]^2} d|\mathbf{x}|. \tag{29}$$

At high energies we must have scale invariance, i.e.,

$$N(\lambda q_0) = N(q_0). \tag{30}$$

From Eq. (29) we therefore obtain

$$\lambda \mu(\lambda q_0) = \mu(q_0), \tag{31}$$

with the solution

$$\mu(q_0) = \alpha / q_0, \tag{32}$$

where α is some constant. From Eq. (33) we then obtain (by combination of various formulas in Ref. 13)

$$N(\alpha) = \frac{1}{\pi} [(1-\alpha) e^{-\alpha} \overline{\mathrm{Ei}}(\alpha) - (1+\alpha) e^{\alpha} \mathrm{Ei}(-\alpha)], \tag{33}$$

where

$$\overline{\mathrm{Ei}}x \equiv \mathrm{li}\, e^x, \quad \mathrm{li}x \equiv \int_0^x \frac{dt}{\ln t}, \tag{34a}$$

$$-\mathrm{Ei}(-x) \equiv \int_x^\infty \frac{e^{-t}}{t} dt. \tag{34b}$$

In Table I we have calculated some values of $N(\alpha)$, and it is seen that $N(\alpha)$ is practically stationary for $0.4 \leq \alpha \leq 0.6$. Since $\mu > 0$ it follows that α is restricted to be in the range $0 \leq \alpha < 4$.

So far we have expressed the asymptotic amplitude (22) in terms of the parameter α. However, from assumption (iii) $\sigma(\infty)$ is nonvanishing. This implies, according to well-known results,[14] that the diffraction picture is satisfied, i.e., $\mathrm{Re}T_- / \mathrm{Im}T_- \to 0$ for $s \to \infty$. In order to avoid $\mathrm{Re}T_- / \mathrm{Im}T_-$ to go to a constant it follows by a calculation quite analogous to the work in Sec. 4 that

$$\alpha = 0.32. \tag{35}$$

This result is shown explicitly in Appendix B. From Eq. (33) we find

$$N(0.32) = 0.459. \tag{36}$$

Finally, we mention that $N(\alpha)$ has one (and only one) maximum in the allowed range of α values. One finds

$$\mathrm{Max}\, N(\alpha) = 0.489 \quad \text{for} \quad \alpha = 0.48. \tag{37}$$

Table I. Numerical variation of $N(\alpha)$ and $\sigma_{\pi p}(\infty) = 56\, N(\alpha)$ mb as functions of the parameter α.

α	$N(\alpha)$	$\sigma_{\pi p}(\infty)$ (mb)
0.10	0.294	16.4
0.20	0.398	22.2
0.30	0.455	25.4
0.40	0.479	26.7
0.42	0.485	27.1
0.44	0.485	27.1
0.46	0.485	27.1
0.48	0.489	27.4
0.50	0.486	27.2
0.52	0.486	27.2
0.54	0.485	27.1
0.56	0.476	26.6
0.58	0.476	26.6
0.60	0.471	26.4
0.70	0.459	26.2
0.80	0.440	24.6
0.90	0.408	22.8
1.00	0.382	21.4
1.50	0.242	13.5
2.00	0.140	7.8
3.00	0.022	1.2
4.00	−0.006	−0.3

It is interesting (see the next section) to note that the two values of N given in Eqs. (36) and (37) only deviate by 6%.

6. DISCUSSION

A. Total Cross Sections

Specializing Eq. (22) to the forward direction, we obtain

$$\sigma_{\pi p}(\infty) = N\sigma_0, \tag{38}$$

where σ_0 is a characteristic quantity with the dimension of a cross section

$$\sigma_0 = 16\pi \frac{f^2}{g_A{}^2} \frac{1}{m^2} = 56 \text{ mb}. \tag{39}$$

In Eq. (39)

$$f^2 = \frac{g^2}{4\pi} \left(\frac{m}{2M}\right)^2 = 0.082, \tag{40}$$

according to Ref. 15.

Recent fits to forward pion-nucleon dispersion relations give[16]

$$\sigma_{\pi^+ p}{}^{\mathrm{fit}}(s) = \sigma_{\pi p}{}^{\mathrm{fit}}(\infty) + a_+ / s^{1.128}, \tag{41a}$$

$$\sigma_{\pi^- p}{}^{\mathrm{fit}}(s) = \sigma_{\pi p}{}^{\mathrm{fit}}(\infty) + a_- / s^{0.697}, \tag{41b}$$

where a_+ and a_- are constants and

$$\sigma_{\pi p}{}^{\mathrm{fit}}(\infty) = 22.72 \text{ mb}. \tag{42}$$

To compare the phenomenological information in Eqs.

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(41) and (42) we insert the value of N determined in Eq. (36), and we get

$$\sigma_{\pi p}(\infty)=(25.7\pm4.2)\text{mb}, \qquad (43)$$

where the uncertainty comes from the uncertainty in PCAC (which we have applied twice) estimated in Sec. 2. In Appendix C we show that the result (43) is independent of the special representation of the δ_{q_0} function [given in Eqs. (27) and (28)], at least as far as a large class of δ_{q_0} functions is concerned. The reason for this model independence is that the requirement $\text{Re}T_+/\text{Im}T_+\rightarrow 0$ for $s\rightarrow\infty$ puts strong restrictions on the δ_{q_0} function.

Within the uncertainty limits, our theoretical prediction (43) is seen to be in complete agreement with the phenomenological value (43). The deviation between the mean value in Eq. (43) and the phenomenological value (42) is 13%. This deviation is not unreasonable in view of the accuracy of the usual sum-rule predictions of current algebra. It is evident that $\sigma_{\pi p}{}^{\text{fit}}(\infty)$ is also somewhat uncertain,[16–18] and there is therefore an excellent agreement between the results (42) and (43).

One interesting feature should be pointed out. If, instead of the value of N in Eq. (36), we had used the maximum value of N, given by Eq. (37), we would obtain

$$\sigma_{\pi p}{}^{\max}(\infty)=27.4\pm4.3\text{ mb}. \qquad (44)$$

This value is almost equal to the correct value in Eq. (43). Our result is therefore in agreement with the principle of maximum strength of the strong interactions proposed by Chew and Frautschi.[19]

In Secs. 4 and 5 we calculated the cross section for π^+p scattering. One can of course repeat all the arguments to obtain the cross section for π^-p scattering. It turns out, however, that the results conflict with unitarity since $\sigma_{\pi^-p}(\infty)$ is less than zero. This means that in the π^-p case one of our three basic assumptions (i)–(iii) must be wrong, and it seems natural to assume that the second half of assumption (iii) is wrong, i.e., that the π^-p light-cone singularities are stronger than $\delta(x^2)$ or derivatives of $\delta(x^2)$. In terms of the spectral functions $\rho(m^2,xp,x^2)$ it means that ρ is singular in the π^-p case. Physically this implies that π^-p scattering has a more complicated high-energy behavior than π^+p scattering, something which is also indicated by the phenomenological Eqs. (41). It is also well known at machine energies that the π^+p system has a behavior which is rather different from the π^-p system. Hence it is perhaps not too surprising that the two systems behave in different ways asymptotically. A possible intuitive explanation would be that the π^+ does not see the low-energy resonances at high energies, whereas the π^- continues to be influenced by the resonant structure at low energies. In any case the structure difference indicates that very interesting phenomena might show up above the present machine energies.[17,18] From our point of view we know, however, that because of the Pomeranchuk theorem the prediction (43) also applies to π^-p scattering, although the dynamics producing $\sigma(\infty)$ is different in the π^+p and π^-p case. Finally we mention that a possible way to change the situation mentioned above would be to introduce additional terms in the space-space commutator (3), since this commutator has not been checked in low-energy sum rules. Such additional terms would unfortunately be very complicated in x space. The models studied by Okubo[8] and Domokos and Karplus[9] have the serious disadvantage that they do not reproduce the ordinary successful current algebra (in these models the equal-time commutator either vanishes or does not exist). It might be, however, that the ordinary current algebra is some sort of low-energy approximation, although it appears to be hard to formulate this statement in a more precise way. In view of the arguments in Ref. 4 as well as in Sec. 2 we believe that PCAC is a very reasonable assumption for high-energy matrix elements; in this connection we would like to emphasize that PCAC is of vital importance in order to obtain a constant cross section.

We now turn to the pion-pion case where Eq. (22) gives

$$\sigma_{\pi\pi}(\infty)=2\sigma_0N=(51.4\pm8.4)\text{ mb}. \qquad (45)$$

Since no experimental fit exists for $\sigma_{\pi\pi}(s)$ we cannot exclude this value, although it appears to be too large [in any case Eq. (45) has the right order of magnitude]. An explanation of this might be that the commutators between pion states behave in a more singular way than the proton-proton matrix elements of the commutators. Because of the lack of information on $\sigma_{\pi\pi}(s)$ we shall not discuss this point further. Similar remarks apply to $\sigma_{\pi\Sigma}(\infty)$ and $\sigma_{\pi\Xi}(\infty)$.

One could also try to calculate the kaon-nucleon cross sections by extending PCAC to $\Delta s=1$ currents. Since σ_{K^+n} and σ_{K^+p} are constants from 6 to 20 GeV (this is presumably due to the large number of open channels), condition (iii) is probably satisfied. However, it has been demonstrated that PCAC does not apply to the $\Delta s=1$ current[20] and we therefore think that such an extension is not reasonable.

B. Small-Angle Scattering

We now investigate the nonforward direction. Using Eq. (22) and conserved vector current (CVC), we have

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for pion-proton scattering

$$\langle p'\pi^+(q_2)|T|p\pi^+(q_1)\rangle = \frac{2N}{f_\pi^2}\bar{u}(p')\bigg[\mathbf{q}_1F_1{}^V(t) + \frac{i}{2M}(p_\nu'-p_\nu)(q_1)_\mu\sigma^{\mu\nu}F_2{}^V(t)\bigg]u(p)\,, \quad (46)$$

where $t=(p'-p)^2$ and $F_k{}^V(t)$, $k=1$, 2, are the usual isovector form factors. The unpolarized pion-nucleon differential cross section then becomes

$$\frac{d\sigma}{dt}=\frac{16\pi}{m^4}\frac{f^4}{g_A{}^4}\bigg\{\frac{1+t/(s-M^2)}{1-t/4M^2}\bigg[(G_E{}^V)^2-\frac{t}{4M^2}(G_M{}^V)^2\bigg] - \frac{t^2}{4(s-M^2)^2}\frac{(G_M{}^V)^2-(G_E{}^V)^2}{1-t/4M^2}\bigg\}N^2. \quad (47)$$

Since our derivation only holds for small momentum transfers, we have for $s\gg M^2$, $|t|\ll s-M^2$,

$$\frac{d\sigma}{dt}=\left(\frac{d\sigma}{dt}\right)_{t=0}\bigg[(G_E{}^V)^2-\frac{t}{4M^2}(G_M{}^V)^2\bigg]\frac{1+t/(s-M^2)}{1-t/4M^2} \approx\left(\frac{d\sigma}{dt}\right)_{t=0}\bigg[G_E{}^2-\frac{t}{4M^2}G_M{}^2\bigg]\left(1-\frac{t}{4M^2}\right)^{-1}, \quad (48)$$

where we have neglected the neutron form factor and where $G_E(t)$ is the proton's electromagnetic form factor.

From Eq. (48) one can calculate the quantity

$$R(t)\equiv\left(\frac{d\sigma}{dt}\right)\bigg/\left(\frac{d\sigma}{dt}\right)_{t=0} =\bigg[G_E{}^2-\frac{t}{4M^2}G_M{}^2\bigg](1-t/4M^2)^{-1}, \quad (49)$$

and compare $R(t)$ with experiments. It turns out that Eq. (49) gives reasonable predictions (within 20%) for very small momentum transfers (up to $t\approx 0.7$ GeV/c^2 around 15 GeV). For larger momentum transfers the formula gives too high values for $R(t)$. This is to be expected, since the derivation of Eq. (49) is valid asymptotically and for very small momentum transfers. The last condition is necessary to get rid of the tensor terms in Eq. (3). If these terms contribute appreciably, it is obvious that Eq. (49) is wrong; hence we must restrict ourselves to very small momentum transfers.

It is possible formally to include the tensor terms in Eq. (44) by replacing $F_2{}^V(t)$ by some function $H(t)$, where $H(t)$ is then an unknown function, and by fitting the data one can obtain information about $H(t)$.

In the pion-pion case we obtain

$$\left(\frac{d\sigma}{dt}\right)_{\pi\pi}=\left(\frac{d\sigma}{dt}\right)_{t=0}F(t)^2(1+t/s)\,, \quad (50)$$

where $F(t)$ is the charged pion form factor,

$$\langle\pi^+(p')|j_\mu{}^{V3}(0)|\pi^+(p)\rangle=(p_\mu+p'_\mu)F(t)\,. \quad (51)$$

Since $F(t)$ and $G_E(t)$ are probably very similar (both form factors are expected to be dominated by the ρ meson between the relevant particle lines and the photon propagator) it follows that the diffraction peak should be similar in the pion-proton and in the pion-pion cases.

It is interesting to compare our result (48) with a result recently obtained by Domokos and Karplus.[9] They used an asymptotic expansion which is due to Bjorken[10]; the Bjorken expansion does not take into account the light-cone singularities and leads directly to equal-time commutators with exact δ functions. One can obtain the Bjorken limit from Eq. (18) if one does not replace the lower limit of the x_0 integral by $|\mathbf{x}|$, i.e., if one does not use the condition of local commutativity. In momentum space this means that the Bjorken limit corresponds to taking $q_0\to\infty$, $\mathbf{q}$ finite, and the pion is therefore infinitely off the mass shell. It is clear that this limit is not the proper physical limit, but Domokos and Karplus argue that the Bjorken limit can be valid for large energies and momentum transfers if the Regge poles retreat to $l=-1$. Under these assumptions they find the following differential cross section:

$$\frac{d\sigma}{dt}=\tfrac{3}{2}(\tfrac{1}{4}Z)^2\left(\frac{g^2}{4\pi}\right)^2\frac{[G_M{}^s(t)]^2}{[2(s-M^2)+t]^2} \times\left(1+\frac{M^4}{s}+\frac{3}{2}\frac{M^2+t}{s}\right), \quad (52)$$

where Z is a factor of order 1 and where $G_M{}^s(t)$ is the scalar magnetic form factor. Karplus and Kroll did not use PCAC to derive (52), but used the Bjorken limit directly to the pion-nucleon amplitude (written as a retarded commutator). From the equal-time commutator for the pion current, the result (52) follows. Equation (52) shows that the large angle $d\sigma/dt$ is of the order $1/s^2$, whereas the forward $d\sigma/dt$ is or the order 1 from Eq. (47). The difference between Eqs. (47) and (52) enters because we have used PCAC to derive Eq. (47) and also because of the use of different asymptotic limits.

We finally mention that the Bjorken limit gives $\sigma(\infty)=0$. This means that the cross section for the scattering of a pion infinitely off the mass shell on a particle with finite mass vanishes. In order to obtain a finite cross section it is absolutely necessary to let all components of the energy-momentum vector go to infinity, keeping the pion mass fixed. Fortunately this is also the physical asymptotic limit.

7. CONCLUSIONS

The results discussed in Sec. 6 are reasonable in the sense that they do not contradict existing experimental information. The comparison of $\sigma_{\pi p}(\infty)$ with fits to the forward πp dispersion relation gives an astonishing agreement; one can explain this as an accident or as a support for the basic assumptions (i)–(iii). At present we have no means of distinguishing between the optimistic and the pessimistic points of view.

We wish to emphasize that in our approach the asymptotic limit is approached by letting the energy and the three-momentum go to infinity *simultaneously*; if, e.g., one lets first the energy and then the momentum go to infinity, the result is $\sigma(\infty)=0$. The explanation of this is probably that the special order of limits $q_0\to\infty$ and then $|\mathbf{q}|\to\infty$ implies that the amplitude is continued from the mass shell to $q^2=\infty$. Such a continuation can very well introduce unpleasant features.

Whether the method to determine $\sigma(\infty)$ proposed here can be regarded as satisfactory in principle, or whether this method, in spite of the good numerical agreement for $\sigma_{\pi p}(\infty)$, represents a far too rough and unsatisfactory approach to the problem of calculating the physical high-energy cross sections, cannot be decided without further investigations of the method. However, even if one agrees on the method, many unsolved problems remain, e.g., how to calculate the kaon-baryon and the baryon-baryon cross sections. In this respect the Regge model might be of some help.

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APPENDIX A

In this Appendix we shall show that the Källén representation[21] suggests the representation (27) of the δ function. We consider, for simplicity, a scalar field $\varphi(x)$ for which[21]

$$\langle 0|\varphi(x)\varphi(0)|0\rangle=-i\int_0^\infty dm^2\rho(m^2)\Delta^{(+)}(x,M^2)\,,\quad \text{(A1)}$$

where

$$\Delta^{(+)}(x,m^2)=\frac{i}{(2\pi)^3}\int dq\, e^{-iqx}\delta(q^2-m^2)\theta(q_0)\,.\quad \text{(A2)}$$

The distribution $\Delta^{(+)}(x,m^2)$ is the boundary value of a function analytic in the lower-half x_0 plane,

$$\Delta^{(+)}(x,m^2)=\lim_{\mu\to 0}\Delta^{(+)}(\mathbf{x},\,x_0-i\mu;\,m^2)\,.\quad \text{(A3)}$$

From Eq. (A1) we then obtain

$$\langle 0|\dot\varphi(x)\varphi(0)|0\rangle=\tfrac{1}{2}\lim_{\mu\to 0}\int_0^\infty dm^2\rho(m^2)\delta_{\mu,m}(\mathbf{x})\,,\quad \text{(A4)}$$

where

$$\delta_{\mu,m}=\frac{1}{(2\pi)^2}\int_0^\infty |\mathbf{q}|^2 d|\mathbf{q}|\; e^{-\mu(|\mathbf{q}|^2+m^2)^{1/2}}\int_{-1}^{+1}d\zeta$$

$$\times e^{i|\mathbf{q}||\mathbf{x}|\xi}=\frac{2}{(2\pi)^2}\frac{\mu}{\mathbf{x}^2+\mu^2}m^2K_2(m(\mathbf{x}^2+\mu^2)^{1/2})\,,\quad \text{(A5)}$$

where K_2 is a generalized Bessel function, and the well-known asymptotic expansion of K_2 yields[22] (for $m\gg 1/\mu$)

$$\delta_{\mu,m}(\mathbf{x})\approx\text{const}\frac{m^{3/2}}{(\mathbf{x}^2+\mu^2)^{5/4}}\exp[-m(\mathbf{x}^2+\mu^2)^{1/2}]\,.\quad \text{(A6)}$$

Hence $\delta_{\mu,m}(\mathbf{x})$ is exponentially damped above $m\sim 1/\mu$. The contribution to the integral (A4) over the spectral function is therefore negligible for $m>1/\mu$, and Eq. (A4) can be written

$$\langle 0|\dot\varphi(x)\varphi(0)|0\rangle=\tfrac{1}{2}\lim_{\mu\to 0}\int_0^{1/\mu^2} dm^2\rho(m^2)\delta_{\mu,m}(\mathbf{x})\,.\quad \text{(A7)}$$

For $m\ll 1/\mu$ we have $m(\mathbf{x}^2+\mu^2)^{1/2}\ll 1$, and expanding K_2 in Eq. (A5), we obtain

$$\delta_{\mu,m}(\mathbf{x})=\frac{1}{\pi^2}\frac{\mu}{(\mathbf{x}^2+\mu^2)^2}-\text{const}\mu\ln[\tfrac{1}{2}\mu(\mathbf{x}^2+\mu^2)^{1/2}]\,.\quad \text{(A8)}$$

The term containing the logarithm vanishes for $\mu\to 0$. Hence the only singular term is the first term on the right-hand side of Eq. (A8). Equation (A7) then gives

$$\langle 0|\dot\varphi(x)\varphi(0)|0\rangle=\frac{1}{2\pi^2}\lim_{\mu\to 0}\frac{\mu}{(\mathbf{x}^2+\mu^2)^2}\int_0^{1/\mu} dm^2\rho(m^2)\,,\quad \text{(A9)}$$

from which we easily get

$$\langle 0|[\dot\varphi(x),\varphi(0)]|0\rangle$$

$$=\frac{1}{\pi^2}\lim_{\mu\to 0}\frac{\mu}{(\mathbf{x}^2+\mu^2)^2}\int_0^{1/\mu^2} dm^2\rho(m^2)\,.\quad \text{(A10)}$$

It is trivial to check that

$$\delta_\mu(\mathbf{x})=\frac{1}{\pi^2}\frac{\mu}{(\mathbf{x}^2+\mu^2)^2}\quad \text{(A11)}$$

has the δ function property

$$\int d\mathbf{x} f(\mathbf{x})\delta_\mu(\mathbf{x}-\mathbf{a})=f(\mathbf{a})\quad \text{(A12)}$$

for $\mu\to 0$. Hence we have shown that the Källén representation (A1) suggests the form (A11) for the δ func-

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tion. For $\mu \to 0$, Eq. (A10) yields the usual result

$$\langle 0|[\dot{\varphi}(x),\varphi(0)]|0\rangle=\delta(\mathbf{x})\int_0^\infty dm^2\rho(m^2). \quad \text{(A13)}$$

Note that, strictly speaking, we are only allowed to perform the transition between Eqs. (A10) and (A13) if the integral over the spectral function converges, which is very unlikely. It is therefore reasonable to keep μ finite and let $\mu \to 0$ only at the end of the calculation.

APPENDIX B

In this Appendix we shall discuss the behavior of the real part. The details are very similar to the calculations in Secs. 4 and 5 so we shall only give a very brief account. We specialize the calculations to the forward direction in order to simplify the formulas. From Eq. (14) we obtain

$$\begin{aligned}-if_\pi{}^2[\mathrm{Re}T_+(p,q_0)-T_+(p,\,q_0=0)]\\ =iq^\nu\int dx\; e^{iqx}\delta(x_0)\langle p|[j_0{}^-(x),j_\nu{}^+(0)]|p\rangle\\ -iq^\mu q^\nu\int dx\;\theta(x_0)\sin(qx)\;\langle p|[j_\mu{}^-(x),j_\nu{}^+(0)]|p\rangle,\quad \text{(B1)}\end{aligned}$$

where we have used the fact that asymptotically

$$\begin{aligned}e^{iqx}\langle p|[j_\mu{}^-(x),j_\nu{}^-(0)]|p\rangle-e^{-iqx}\langle p|[j_\nu{}^-(0),j_\mu{}^-(x)]|p\rangle\\ =2i\sin(qx)\langle p|[j_\mu{}^-(x),j_\nu{}^-(0)]|p\rangle. \quad \text{(B2)}\end{aligned}$$

Using the asymptotic Fourier theorem and the commutators (2) and (3), Eq. (B1) yields

$$\begin{aligned}f_\pi{}^2[\mathrm{Re}T_+(p,q_0)-T_+(p,0)]\\ =-2q_0\langle p|j_0{}^{3V}(0)|p\rangle(1-2M), \quad \text{(B3)}\end{aligned}$$

where

$$M=\int d\mathbf{x}\cos(q_0|\mathbf{x}|)\delta_{q_0}(\mathbf{x}), \quad \text{(B4)}$$

i.e.,

$$M=\frac{4}{\pi}\int_0^\infty \frac{x^2\cos(\alpha x)}{(1+x^2)^2}dx=(1-\alpha)\,e^{-\alpha}. \quad \text{(B5)}$$

Again we have used the fact that $\cos(\mathbf{q}_0|\mathbf{x}|)$ acts as a function of the type discussed in Eqs. (20) and (21) [note that $\cos(\mathbf{q},\mathbf{x})$ is *not* a function of this type since $\mathbf{q}\cdot\mathbf{x}=0$ for $\mathbf{q}$ and $\mathbf{x}$ orthogonal]. Now the assumption $\sigma(\infty)\neq 0$ requires[14] $\mathrm{Re}T_+/\mathrm{Im}T_+\to 0$ for $s\to\infty$, and this is only true if $2M=1$, i.e.,

$$e^\alpha=2-2\alpha, \quad \text{(B6)}$$

with the solution (accurate enough for our purpose)

$$\alpha=0.32, \quad \text{(B7)}$$

as we already used in Eq. (35). With this value of α, $\mathrm{Re}T_+$ approaches a constant at high energies, and $\mathrm{Re}T_+/\mathrm{Im}T_+\to 0$ asymptotically.

So far we have assumed that the equal-time commutator

$$[j_0{}^+(x),\partial^\mu j_\mu{}^-(0)]_{x_0=0} \quad \text{(B8)}$$

does not contain Schwinger terms. If such terms exist the quantity $T_+(p,\,q_0=0)$ will be replaced by

$$T_+(p,\,q_0=0)+O(q_0).$$

The term $O(q_0)$ should then be added to the $O(q_0)$ terms on the right-hand side of Eq. (B3). If we keep the value (B7) for α, local field theory requires that these Schwinger terms be absent because of the Martin-bound (13).

APPENDIX C

In this Appendix we shall show that for a large class of δ_{q_0} functions the cross section turns out to be between 20 and 30 mb. The main point is that the quantity M introduced in Appendix B for general reasons is equal to $\frac{1}{2}$. Thus

$$M=4\pi\int_0^\infty \cos(q_0|\mathbf{x}|)\;\delta_{q_0}(\mathbf{x})\,|\mathbf{x}|^2d|\mathbf{x}|=\tfrac{1}{2}. \quad \text{(C1)}$$

Scale invariance $M(q_0)=M(\lambda q_0)$ yields

$$\lambda^3\delta_{q_0}(\mathbf{x})=\delta_{\lambda q_0}(\mathbf{x}|\lambda), \quad \text{(C2)}$$

and for $\mathbf{x}=0$ we therefore obtain

$$\lambda^3\delta_{q_0}(\mathbf{0})=\delta_{\lambda q_0}(\mathbf{0}), \quad \text{(C3)}$$

with the solution

$$\delta_{q_0}(\mathbf{0})=Aq_0{}^3, \quad \text{(C4)}$$

where A is a constant. Now we assume that

$$\int_{|\mathbf{x}|>|\mathbf{x}_1|} d\mathbf{x}\begin{Bmatrix}\sin(q_0|\mathbf{x}|)\\ \cos(q_0|\mathbf{x}|)\end{Bmatrix}\delta_{q_0}(\mathbf{x})\approx 0 \quad \text{(C5)}$$

for

$$|\mathbf{x}_1|=1/q_0,$$

which is our restriction on the δ_{q_0} function.[23] We therefore obtain from Eq. (C1)

$$M\approx 4\pi\int_0^{1/q_0} x^2Aq_0{}^3dx=\tfrac{4}{3}\pi A, \quad \text{(C6)}$$

from which we find using $M=\frac{1}{2}$

$$A=3/8\pi. \quad \text{(C7)}$$

Similarly, we obtain

$$N=4\pi\int_0^\infty x^2\sin(q_0x)\;\delta_{q_0}(x)dx\approx\pi A\approx 0.4, \quad \text{(C8)}$$

[23] The statement in Eq. (C5) can of course be made more rigorous by saying that for a given $\epsilon>0$ there exists a q_0 such that $\delta_{q_0}(\mathbf{x})<\epsilon$. In the same way the arguments in Eqs. (C6)–(C8) can be made more rigorous.

where we used Eq. (C7). The value (C8) leads to

$$\sigma_{\pi p}(\infty)=22.4 \text{ mb}, \tag{C9}$$

and in view of the neglected higher-order terms this value agrees very well with the result in the main text. Inside the uncertainty (from PCAC) Eq. (C9) agrees with Eq. (43). Hence our conclusion is that the condition $M=\frac{1}{2}$ insures that the physical result for $\sigma(\infty)$ is independent of the particular representation of the δ_{q_0} function [provided that (C5) is satisfied].

APPENDIX D

The representation (19) of the axial-vector commutator shows that, provided the spectral functions are not too singular, the most singular part of the commutator is given by $\partial\Delta(x, m=0)/\partial x_0$ times integrals over spectral functions, which are known from current algebra in the limit $q_0\to\infty$ [see Eqs. (21)]. The axial-vector commutator is given by Eq. (19) which we simplify to

$$F(x)=\langle p|[j_0^-(x),j_0^+(0)]|p\rangle$$
$$=\int_0^\infty dm^2\rho(m^2,xp,x^2)\frac{\partial}{\partial x_0}\Delta(x,m), \tag{D1}$$

where $\rho=M\rho_2+M\rho_4$. From current algebra we then know that

$$\int_0^\infty dm^2\rho(m^2,0,0)=2\langle p|j_0^{V3}(0)|p\rangle. \tag{D2}$$

To study the structure of (D1) we shall start by a very simple free field model where (c is a constant)

$$\rho(m^2,xp,x^2)=c\delta(m^2), \tag{D3}$$

giving

$$F(x)=\frac{c}{2\pi}\frac{\partial}{\partial x_0}[\epsilon(x_0)\delta(x^2)]=\frac{c}{2\pi}\epsilon(x_0)\frac{\partial}{\partial x_0}\delta(x^2), \tag{D4}$$

where we have used $\delta(\mathbf{x}^2)=0$. Now we have

$$\frac{\partial}{\partial x_0}\delta(x^2)=\frac{1}{2|\mathbf{x}|}[\delta'(x_0-|\mathbf{x}|)+\delta'(x_0+|\mathbf{x}|)], \tag{D5}$$

and for $x_0=0$ (equal times) we get an exact δ function,

$$F(x)|_{x_0=0}=\frac{-c}{4\pi|\mathbf{x}|}\delta'(|\mathbf{x}|)=\delta(\mathbf{x}). \tag{D6}$$

Now, in our simple model the amplitude is given by (apart from unimportant factors) a sum of expressions of the type

$$T=\int dx\,\theta(x_0)\cos(qx)\,F(x). \tag{D7}$$

Now, from Eqs. (D4) and (D5),

$$\int_0^\infty dx_0\cos(qx)F(x)=\frac{q_0c}{4\pi|\mathbf{x}|}\sin(q_0|\mathbf{x}|-\mathbf{q}\cdot\mathbf{x}), \tag{D8}$$

and we get

$$T=\frac{c}{q_0}\int d\mathbf{x}\sin(q_0|\mathbf{x}|)\,\delta_{q_0}(\mathbf{x}), \tag{D9}$$

where

$$\delta_{q_0}(\mathbf{x})=\frac{q_0^2}{4\pi|\mathbf{x}|}\cos(\mathbf{q}\cdot\mathbf{x}). \tag{D10}$$

It is easily verified that $\delta_{q_0}(\mathbf{x})$ acts as a δ function,

$$\int d\mathbf{x}\,f(|\mathbf{x}|)\delta_{q_0}(\mathbf{x})\to f(0)\quad\text{for}\quad q_0\to\infty. \tag{D11}$$

Reintroducing the spectral function (D3) we find that

$$T=\frac{1}{q_0}\int d\mathbf{x}\sin(q_0|\mathbf{x}|)\,\delta_{q_0}(\mathbf{x})\int_0^\infty dm^2\rho(m^2,0,0), \tag{D12}$$

clearly showing that in the free field model we get the result stated in Eqs. (22)–(24). In other words, at very high energies current algebra determines the amplitude *if* one replaces the exact δ function in the equal-time commutator with an approximate δ function.

If one calculates N in the simple model discussed above it turns out that N diverges [like $\delta(q^2)$]. Introducing a cutoff factor $e^{-\alpha|\mathbf{x}|}$ in the integral (D12) gives a finite, α-dependent result. Requiring the ratio $\mathrm{Re}T_+/\mathrm{Im}T_+$ to vanish asymptotically, does, however, not work since in this model M can never be equal to $\frac{1}{2}$. Since the model is not realistic from a physical point of view it is of course also not to be expected that any reasonable value of $\sigma(\infty)$ should come out of the model.

The mathematical structure of the above model is, however, essentially correct also in the realistic case where the spectral function is nontrivial. To see this we consider the integral

$$\int_0^\infty dx_0\cos(qx)\,\rho(m^2,xp,x^2)\frac{\partial}{\partial x_0}\Delta(x,m=0)$$
$$=\frac{q_0}{4\pi|\mathbf{x}|}\sin(q_0|\mathbf{x}|-\mathbf{q}\mathbf{x})\,\rho(m^2,M|\mathbf{x}|,x^2=0), \tag{D13}$$

where we assume

$$|q_0\rho(m^2,xp,x^2)|\gg\left|\frac{\partial}{\partial x_0}\rho(m^2,xp,x^2)\right|,$$

for q_0 very large. This condition is certainly satisfied if ρ does not oscillate too rapidly on the light cone. In writing Eq. (D12) we have also assumed that ρ is nonsingular on the light cone. We can then calculate the contribution to the amplitude T,

$$T=\int dx\, \theta(x_0)\cos(qx)\, F(x)=\frac{1}{q_0}\int d\mathbf{x}\sin(q_0|\mathbf{x}|)$$
$$\times\int_0^\infty dm^2\rho(m^2, M|\mathbf{x}|, x^2=0)\frac{q_0^2}{4\pi|\mathbf{x}|}\cos(\mathbf{q}\cdot\mathbf{x}). \quad \text{(D14)}$$

For q_0 very large the function multiplying ρ is a δ_{q_0} function [Eqs. (D10) and (D11)], i.e., when multiplied by a function ρ which does *not* depend on q_0 it acts like an ordinary δ function. Thus, if we write

$$\delta_{q_0}(\mathbf{x})\int_0^\infty dm^2\rho(m^2,0,0)\equiv\int_0^\infty dm^2\rho(m^2, M|\mathbf{x}|, x^2=0)$$
$$\times\frac{q_0^2}{4\pi|\mathbf{x}|}\cos(\mathbf{q}\mathbf{x}), \quad \text{(D15)}$$

then we know that the function δ_{q_0} *defined* by this equation becomes an exact δ function for $q_0\to\infty$,

$$\delta_{q_0}(\mathbf{x})\to\delta(\mathbf{x}) \quad \text{for} \quad q_0\to\infty. \quad \text{(D16)}$$

For large, but finite, values of q_0 it is clear that $\delta_{q_0}(\mathbf{x})$ is not given by

$$\frac{q_0^2}{4\pi|\mathbf{x}|}\cos(\mathbf{q}\cdot\mathbf{x}),$$

since ρ depends on $|\mathbf{x}|$. The detailed form of $\delta_{q_0}(\mathbf{x})$ depends clearly on the dynamics. However, as we have shown in Appendix C the *dynamical* condition $\mathrm{Re}T/\mathrm{Im}T\to 0$ for $q_0\to\infty$ fixes (at least to a very high degree of accuracy) the value of the constant N defined by

$$N=\int d\mathbf{x}\,\delta_{q_0}(\mathbf{x})\sin(q_0|\mathbf{x}|). \quad \text{(D17)}$$

In Appendix A we have considered a special type of δ_{q_0} function suggested by the Källén representation. It is obvious that this function does not necessarily correspond to reality; however, the condition $\mathrm{Re}T/\mathrm{Im}T\to 0$ again fixes the value of N [and thereby of $\sigma(\infty)$], and this value of N (which perhaps is calculated in an unrealistic model) turns out to agree very well with the general model-independent conditions in Appendix C. Hence we believe that the condition $\mathrm{Re}T/\mathrm{Im}T\to 0$ makes the numerical result for $\sigma(\infty)$ independent of the special representation of the δ_{q_0} function. The diffraction picture $\mathrm{Re}T/\mathrm{Im}T\to 0$ is then a substitute for our lacking dynamical information about $\rho(m^2, xp, x^2=0)$.

Finally we would like to make a remark on the Bjorken limit[10] which corresponds to taking $q_0\to\infty$, $\mathbf{q}$ fixed and finite and $q^2\to\infty$. In this limit the free field theory gives

$$\int dx\,\theta(x_0)\cos(qx)\frac{\epsilon(x_0)}{2\pi}\frac{\partial}{\partial x_0}\delta(x^2)$$
$$=\frac{1}{q_0}\int d\mathbf{x}\left(\frac{q_0^2}{4\pi|\mathbf{x}|}\sin(q_0|\mathbf{x}|)\right)\cos(\mathbf{q}\mathbf{x})\to 0$$
$$\text{for} \quad q_0\to\infty,\ \mathbf{q}\ \text{finite} \quad \text{(D18)}$$

and

$$\int dx\,\theta(x_0)\sin(qx)\frac{\epsilon(x_0)}{2\pi}\frac{\partial}{\partial x_0}\delta(x^2)$$
$$=\frac{-1}{q_0}\int d\mathbf{x}\left(\frac{q_0^2}{4\pi|\mathbf{x}|}\cos(q_0|\mathbf{x}|)\right)\cos(\mathbf{q}\mathbf{x})\to -1/q_0$$
$$\text{for} \quad q_0\to\infty,\ \mathbf{q}\ \text{finite}. \quad \text{(D19)}$$

In general the Bjorken limit is therefore essentially different from the physical limit $q_\mu\to\infty$, q^2 fixed. The above Eqs. (D18) and (D19) can easily be generalized in analogy with Eqs. (D13)–(D15).