to have the form

$$E_\sigma = \epsilon^{ab3}\tfrac{1}{2}(q-k)_\nu p_\lambda \int d\xi d\eta\, e^{i(k+q)\xi+il\eta} \times\langle 0|T(V_\nu^3(\xi)A_\lambda^8(0)V_\sigma^3(\eta))|0\rangle$$
$$= \epsilon^{ab3}\tfrac{1}{2}(q-k)_\nu p_\lambda[\epsilon_{\nu\sigma\tau\rho}(k+q)_\rho l_\tau] \times[(k+q)_\lambda G(k+q,l)+l_\lambda G(l,k+q)], \quad (15)$$

where G is a scalar function of the invariants.

We now notice that when $l\cdot p=l\cdot q=l\cdot k=l^2=0$, T_σ must vanish. In this limit, therefore, A_σ and E_σ are equal. (We are now referring to the invariant functions which multiply the usual kinematic ϵ-tensor form in T_σ, A_σ, and E_σ.) However, then E_σ is proportional to p^2; therefore, we divide out the factor of p^2 before we use PCAC to obtain amplitudes for which the η meson is on the mass shell. Apart from this trick, the computation proceeds without difficulty. Note that although the η-decay amplitudes have a common factor of p^2, there is no need to conclude that the amplitudes vanish on the mass shell.[15] All the interesting information is obtained from the coefficient of p^2. We are able to divide out this kinematic factor since we have already eliminated the model-dependent term T_σ. Finally, we conclude that the CA calculations done recently[4,5] give the correct results for the η decays.

The author wishes to acknowledge the hospitality of the Aspen Institute for Humanistic Studies, where this work was done.

[15] D. G. Sutherland, Nucl. Phys. **B2**, 433 (1967); see also A. D. Dolgov *et al.*, Phys. Letters **24B**, 425 (1967).

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Current Algebra and the $\eta \to \pi^+\pi^-\pi^0\gamma$ Decay Mode

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The rate for $\eta\to\pi^+\pi^-\pi^0\gamma$ has been calculated in terms of the rate for $\eta\to\pi^0\gamma\gamma$, using current algebra and the hypothesis of a partially conserved axial-vector current. The results agree with a previous calculation by Singer based on a vector-meson dominance model. The rate for bremsstrahlung emission was found to be smaller than the direct decay rate, but not negligible.

I. INTRODUCTION

RECENTLY, current algebra has been used to study various η-decay modes.[1–7] In particular, the decay mode $\eta\to 2\pi+\gamma$ has been examined[5–7] using current algebra, and a branching ratio of $(\eta\to 2\pi+\gamma)/(\eta\to 2\gamma)=0.19$ has been obtained, in good agreement with the experimental value 0.15.

In this paper we shall be concerned with applying this same method to the decay mode $\eta\to 3\pi+\gamma$. Such a mode has been considered previously by Singer.[8] Singer's calculation, based on a quadrilinear meson-interaction model, predicts the branching ratio $(\eta\to\pi^+\pi^-\pi^0\gamma)/(\eta\to\pi^0\gamma\gamma)=0.23\%$. However, if one adopts the theory of Bronzan and Low[9] in which a new A quantum number is introduced, the above ratio is much larger. The current algebra approach would permit a model-independent estimate of this branching ratio.

There have been two recently reported experimental searches for the $\eta\to\pi^+\pi^-\pi^0\gamma$ mode. These experiments have established upper limits for the ratio $\Gamma(\eta\to\pi^+\pi^-\pi^0\gamma)/\Gamma(\eta\to\pi^+\pi^-\pi^0)$ of 7 and 0.9%.[10,11]

It should also be noted that the $\eta\to 3\pi+\gamma$ mode could offer a new test of a possible C violation in electromagnetic interactions. If a sufficient number of $\eta\to\pi^+\pi^-\pi^0\gamma$ decays are found, one might hope to observe a π^+/π^- asymmetry due to electromagnetic C-violating interactions.

II. DETERMINATION OF THE FORM-FACTOR RELATIONS

In applying the current-commutation relations and the partially conserved axial-vector current (PCAC)

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[8] P. Singer, Phys. Rev. **154**, 1592 (1967).
[9] J. B. Bronzan and F. E. Low, Phys. Rev. Letters **12**, 522 (1964).
[10] S. M. Flatté, Phys. Rev. Letters **18**, 976 (1967).
[11] L. R. Price and F. S. Crawford, Phys. Rev. Letters **18**, 1207 (1967).

hypothesis, we shall adopt the technique developed by Weinberg.[12,13] We will expand decay amplitudes in powers of the pion momenta. The resulting expansions will give us the on-mass-shell decay amplitudes up to lowest nonvanishing order in pion momenta. This expansion technique has been successfully used by Weinberg[13] for K_{e4} decay, by Abarbanel[14] for $K_{3\pi}$ decays, and by Pasupathy and Marshak[5] for the $\eta\to 2\pi+\gamma$ decay mode.

We begin by considering the quantity

$$M_{\mu\nu\lambda}=\int dxdydz\; e^{i(q_a\cdot x+q_b\cdot y+q_c\cdot z)}\langle 0|T(A_\mu{}^a(x)A_\nu{}^b(y)A_\lambda{}^c(z)J_\sigma{}^{\mathrm{EM}}(0))|\eta\rangle\,, \tag{2.1}$$

where the electromagnetic current

$$J_\sigma{}^{\mathrm{EM}}=V_\sigma{}^3+(1/\sqrt{3})V_\sigma{}^8;$$

$V_\sigma{}^3$ is the third component of the $\Delta I=1$ strangeness-conserving current; $V_\sigma{}^8$ is the hypercharge current; $A_\mu{}^a$ is the $\Delta S=0$ axial-vector current; q_a, q_b, q_c are the pion four-momenta; and a, b, c are isospin indices.

Isolating the pion pole terms in Eq. (2.1) in the manner of Weinberg, we write

$$-iq_{a\mu}iq_{b\nu}iq_{c\lambda}N_{\mu\nu\lambda}=-iq_{a\mu}iq_{b\nu}iq_{c\lambda}M_{\mu\nu\lambda}+\frac{F_\pi q_a{}^2iq_{b\nu}iq_{c\lambda}M_{\nu\lambda}(q_b,q_c)}{(q_a{}^2-\mu^2)}+\text{permutations}$$
$$-\frac{F_\pi{}^2q_a{}^2q_b{}^2iq_{c\lambda}M_\lambda(q_c)}{(q_a{}^2-\mu^2)(q_b{}^2-\mu^2)}+\text{permutations}+\frac{F_\pi{}^3q_a{}^2q_b{}^2q_c{}^2M}{(q_a{}^2-\mu^2)(q_b{}^2-\mu^2)(q_c{}^2-\mu^2)}\,, \tag{2.2}$$

with

$$M_{\mu\nu}(q_a,q_b)=(-q_c{}^2+\mu^2)\int dxdydz\; e^{i(q_a\cdot x+q_b\cdot y+q_c\cdot z)}\langle 0|T(A_\mu{}^a(x)A_\nu{}^b(y)\phi_\pi{}^c(z)J_\sigma{}^{\mathrm{EM}}(0))|\eta\rangle\,, \tag{2.3}$$

$$M_\mu(q_a)=(-q_b{}^2+\mu^2)(-q_c{}^2+\mu^2)\int dxdydz\; e^{i(q_a\cdot x+q_b\cdot y+q_c\cdot z)}\langle 0|T(A_\mu{}^a(x)\phi_\pi{}^b(y)\phi_\pi{}^c(z)J_\sigma{}^{\mathrm{EM}}(0))|\eta\rangle\,, \tag{2.4}$$

$$M=(-q_a{}^2+\mu^2)(-q_b{}^2+\mu^2)(-q_c{}^2+\mu^2)\int dxdydz\; e^{i(q_a\cdot x+q_b\cdot y+q_c\cdot z)}\langle 0|T(\phi_\pi{}^a(x)\phi_\pi{}^b(y)\phi_\pi{}^c(z)J_\sigma{}^{\mathrm{EM}}(0))|\eta\rangle\,, \tag{2.5}$$

where F_π is the pion decay amplitude, ϕ_π is the pion field, and μ is the pion mass.

Adopting the notation of Weinberg,[13] the current-commutation relations proposed by Gell-Mann[15] are given by[16]

$$[A_0{}^a(x),A_\nu{}^b(y)]\delta(x_0-y_0)=2i\epsilon_{abc}V_\nu{}^c(x)\delta^4(x-y)\,, \tag{2.6}$$

$$[V_0{}^a(x),V_\nu{}^b(y)]\delta(x_0-y_0)=2i\epsilon_{abc}V_\nu{}^c(x)\delta^4(x-y)\,, \tag{2.7}$$

$$[A_0{}^a(x),V_\nu{}^b(y)]\delta(x_0-y_0)=2i\epsilon_{abc}A_\nu{}^c(x)\delta^4(x-y)\,, \tag{2.8}$$

$$[A_0{}^a(x),V_\nu{}^8(y)]\delta(x_0-y_0)=[V_0{}^a(x),V_\nu{}^8(y)]\delta(x_0-y_0)=0\,, \tag{2.9}$$

where ϵ_{abc} is the totally antisymmetric symbol with $\epsilon_{123}=+1$. We also make use of the conserved vector current and PCAC hypotheses

$$\partial_\mu V_\mu{}^a(x)=0\,, \tag{2.10a}$$

$$\partial_\mu A_\mu{}^a(x)=F_\pi\mu^2\phi_\pi{}^a(x)\,, \tag{2.10b}$$

along with the additional commutation relations[17]

$$[A_0{}^a(x),\partial_\mu A_\mu{}^b(y)]\delta(x_0-y_0)=\sigma_{ab}(x)\delta^4(x-y)\,, \tag{2.11a}$$

$$[A_0{}^a(x),\sigma_{bc}(y)]\delta(x_0-y_0)=\delta_{bc}\partial_\mu A_\mu{}^a(x)\delta^4(x-y)\,. \tag{2.11b}$$

[12] S. Weinberg, Phys. Rev. Letters **16**, 879 (1966).
[13] S. Weinberg, Phys. Rev. Letters **17**, 336 (1966).
[14] H. D. I. Abarbanel, Phys. Rev. **153**, 1547 (1967).
[15] M. Gell-Mann, Phys. Rev. **125**, 1064 (1962).
[16] In these commutation relations we ignore the so-called "Schwinger" terms.
[17] In the σ-model the term $\sigma_{ab}(x)$ would be just $\delta_{ab}\sigma(x)$, with $\sigma(x)$ the σ-meson field. See M. Gell-Mann and M. Lévy, Nuovo Cimento **16**, 705 (1960).

Computing the various terms in Eq. (2.2) by partial integrations,[18] and making use of Eqs. (2.3)–(2.11) as well as the following identity for time-ordered products[14]:

$$\frac{\partial}{\partial x^\mu}T\{A_\mu(x)B_\nu(y)C_\lambda(z)D_\sigma(0)\}=T\{\partial_\mu A_\mu(x)B_\nu(y)C_\lambda(z)D_\sigma(0)\}+\delta(x_0-y_0)T\{[A_0(x),B_\nu(y)]C_\lambda(z)D_\sigma(0)\}$$
$$+\delta(x_0-z_0)T\{[A_0(x),C_\lambda(z)]B_\nu(y)D_\sigma(0)\}+\delta(x_0)T\{[A_0(x),D_\sigma(0)]B_\nu(y)C_\lambda(z)\}\,, \quad (2.12)$$

we find[19]

$$-iq_{a\mu}iq_{b\nu}iq_{c\lambda}N_{\mu\nu\lambda}=F_\pi{}^3(-q_a{}^2+\mu^2)(-q_b{}^2+\mu^2)(-q_c{}^2+\mu^2)$$
$$\times\int dxdydz\; e^{i(q_a\cdot x+q_b\cdot y+q_c\cdot z)}\langle 0|T(\phi_\pi{}^a(x)\phi_\pi{}^b(y)\phi_\pi{}^c(z)J_\sigma{}^{\text{EM}}(0))|\eta\rangle+2iF_\pi{}^2\epsilon_{a3d}(-q_b{}^2+\mu^2)(-q_c{}^2+\mu^2)$$
$$\times\int dydz\; e^{i(q_b\cdot y+q_c\cdot z)}\langle 0|T(\phi_\pi{}^b(y)\phi_\pi{}^c(z)A_\sigma{}^d(0))|\eta\rangle+\text{permutations}-2F_\pi(\epsilon_{a3e}\epsilon_{bcd}+\epsilon_{b3e}\epsilon_{aed})(-q_c{}^2+\mu^2)$$
$$\times\int dx\; e^{iq_c\cdot x}\langle 0|T(\phi_\pi{}^c(x)V_\sigma{}^d(0))|\eta\rangle+\text{permutations}-F_\pi\epsilon_{bad}(q_b-q_a)_\mu(-q_c{}^2+\mu^2)$$
$$\times\int dxdz\; e^{i(q_a+q_b)\cdot x+iq_c\cdot x}\langle 0|T(\phi_\pi{}^d(z)V_\mu{}^e(x)J_\sigma{}^{\text{EM}}(0))|\eta\rangle+\text{permutations}-\tfrac{4}{3}i\epsilon_{cae}\epsilon_{bed}(q_c-q_a)_\mu$$
$$\times\int dx\; e^{i(q_a+q_b+q_c)\cdot x}\langle 0|T(A_\mu{}^d(x)J_\sigma{}^{\text{EM}}(0))|\eta\rangle+\text{permutations}-2i\epsilon_{cae}\epsilon_{b3d}(q_c-q_a)_\mu$$
$$\times\int dz\; e^{i(\alpha_a+q_c)\cdot x}\langle 0|T(V_\mu{}^e(x)A_\sigma{}^d(0))|\eta\rangle+\text{permutations}$$
$$-\tfrac{4}{3}i\epsilon_{a3d}(\epsilon_{bdf}\epsilon_{cfe}+\epsilon_{cdf}\epsilon_{bfe})\langle 0|A_\sigma{}^e|\eta\rangle+\text{permutations}. \quad (2.13)$$

In order to proceed with the calculation it is necessary to neglect a large number of terms in Eq. (2.13). The validity of this procedure will, of course, be ultimately determined by a comparison with conclusive experimental data. However, we are encouraged by the fact that our final result is consistent with recent experiments,[10,11] as well as being in agreement with the model calculation by Singer.[8]

We consider first the terms involving the matrix element $\langle 2\pi|A_\sigma|\eta\rangle$. We neglect terms of this kind on the basis of G-parity considerations and the absence of second-class currents as discussed by Weinberg.[20] Next, we consider terms with the matrix elements $\langle 0|T(V_\mu(x)A_\sigma(0))|\eta\rangle$ and $\langle 0|T(A_\mu(x)J_\sigma{}^{\text{EM}}(0))|\eta\rangle$. If we include only single-particle intermediate states, then these matrix elements will vanish by ordinary parity arguments or the absence of second-class currents. Furthermore, terms of the form $\langle 0|A_\sigma|\eta\rangle$ must vanish since they involve matrix elements of a $\Delta I=1$ current between $I=0$ states.

[18] We will neglect all surface terms arising from the partial integration.

[19] We neglect, as is always done, the "σ terms" which are generated from the commutation relation (2.11a). One assumes that this is a good approximation without any real justification. See H. Abarbanel (Ref. 14).

[20] S. Weinberg, Phys. Rev. **112**, 1375 (1958).

We note that our matrix elements are all taken on the mass shell.[13,14] This means that $(P_\eta-q_a-q_b-q_c)^2=0$. But since we may also consistently take $(P_\eta-q_c)^2=0$ simultaneously, the matrix element $\langle\pi^c|V_\sigma|\eta\rangle$ corresponding to $\eta\to\pi^c+\gamma$ is also dropped as this decay is forbidden.

Finally, we also drop the term $q_{a\mu}q_{b\nu}q_{c\lambda}N_{\mu\nu\lambda}$ which is cubic in the pion momenta. Neglect of such terms is notalways justified, as noted by Rubinstein and Veneziano.[21] This is particularly true in the analysis of the $\eta\to 2\pi+\gamma$ decay carried out by Marshak and Pasupathy.[5]

However, in our particular case the approximation is indeed valid. This may be seen as follows: As indicated below, the $\eta\to 3\pi+\gamma$ amplitude is linear in the momenta of the mesons. We shall also find that the form factors are quadratic in the pion momenta, so it would appear that cubic terms cannot be neglected. However, only g_4, the form factor multiplying the η momentum, is nonvanishing. Thus, to lowest order, the amplitude is actually quadratic in the pion momenta and we may neglect the cubic term.

[21] H. R. Rubinstein and S. Veneziano, Phys. Rev. Letters **18**, 411 (1967).

Thus, Eq. (2.13) becomes

$$F_\pi^3(2\pi)^{9/2}(8q_a^0q_b^0q_c^0)^{1/2}\langle\pi^a\pi^b\pi^c|J_\sigma^{\mathrm{EM}}(0)|\eta\rangle=F_\pi(2\pi)^{3/2}(2q_c^0)^{1/2}\epsilon_{bae}(q_b-q_a)_\mu$$
$$\times\int dx\, e^{i(q_a+q_b)\cdot x}\langle\pi^c|T(V_\mu^e(x)J_\sigma^{\mathrm{EM}}(0))|\eta\rangle+F_\pi(2\pi)^{3/2}(2q_b^0)^{1/2}\epsilon_{cae}(q_c-q_a)_\mu$$
$$\times\int dx\, e^{i(q_a+q_c)\cdot x}\langle\pi^b|T(V_\mu^e(x)J_\sigma^{\mathrm{EM}}(0)|\eta\rangle+F_\pi(2\pi)^{3/2}(2q_a^0)^{1/2}\epsilon_{cbe}(q_c-q_b)_\mu$$
$$\times\int dx\, e^{i(q_b+q_c)\cdot x}\langle\pi^a|T(V_\mu^e(x)J_\sigma^{\mathrm{EM}}(0))|\eta\rangle. \tag{2.14}$$

The $\eta\to 3\pi+\gamma$ form factors may be defined by

$$(2\pi)^{9/2}(8q_a^0q_b^0q_c^0)^{1/2}\langle\pi^a\pi^b\pi^c|J_\sigma^{\mathrm{EM}}|\eta\rangle\equiv -i(2\pi)^{-3/2}\times(2P^0)^{-1/2}\frac{1}{M_\eta^2}\{g_1q_{a\sigma}+g_2q_{b\sigma}+g_3q_{c\sigma}+g_4P_\sigma\}, \tag{2.15}$$

where P_σ is the four-momentum of the η, and M_η is the η mass.

We write the amplitude for the decay of $\eta\to\pi^0+2\gamma$, where one photon is an *isovector photon*, as

$$(2\pi)^{3/2}(2q_c^0)^{1/2}\int dx\, e^{i(q_a+q_b)\cdot x}\langle\pi^c|T(V_\mu^e(x)J_\sigma^{\mathrm{EM}}(0))|\eta\rangle\equiv(2\pi)^{-3/2}(2P^0)^{-1/2}T_{\mu\sigma}^e(P,\,q_a+q_b,\,q_c). \tag{2.16}$$

On grounds of covariance, the most general expression for $T_{\mu\sigma}$, associated with $\eta\to\pi^0+2\gamma$ (real), is

$$T_{\mu\sigma}=A_1P_\mu(k_{1\sigma}+k_{2\sigma})+A_2(k_{1\mu}+k_{2\mu})P_\sigma+A_3(k_{1\mu}k_{2\sigma}+k_{2\mu}k_{1\sigma})+A_4P_\mu P_\sigma+A_5(k_{1\mu}k_{1\sigma}+k_{2\mu}k_{2\sigma})+A_6\delta_{\mu\sigma}, \tag{2.17}$$

where we have demanded symmetry in the photon momenta k_1, k_2.

Gauge invariance demands that

$$k_{1\mu}T_{\mu\sigma}=0. \tag{2.18}$$

Thus,

$$A_1(P\cdot k_1)+A_3(k_1\cdot k_2)+A_6=0, \tag{2.19a}$$
$$A_1(P\cdot k_1)+A_5(k_1\cdot k_2)=0, \tag{2.19b}$$
$$A_2(k_1\cdot k_2)+A_4(P\cdot k_1)=0. \tag{2.19c}$$

We shall assume that the form factor A_6 is slowly varying, so that it will remain essentially constant for arbitrary photon momentum. Then we note from Eq. (2.19a) that in the soft-photon limits, $P\cdot k_1\to 0$, $k_1\cdot k_2\to 0$, A_6 vanishes. This condition, of course, rests on the fact that there are no pole terms in A_1 and A_3 whose residues would contribute in this soft-photon limit. Thus, we take

$$A_6\simeq 0$$

and obtain

$$A_5=A_3, \tag{2.20a}$$
$$A_1=-\beta A_5, \tag{2.20b}$$
$$A_4=-\beta A_2, \tag{2.20c}$$

where

$$\beta=k_1\cdot k_2/P\cdot k_1. \tag{2.21}$$

Let us write $\beta=1-\epsilon$, where

$$\epsilon=q\cdot k_1/P\cdot k_1. \tag{2.22}$$

Then,

$$T_{\mu\sigma}=A_5(\epsilon P_\mu-q_\mu)(P_\sigma-q_\sigma)+A_2(\epsilon P_\mu-q_\mu)P_\sigma, \tag{2.23}$$

where q_μ is the pion momentum.

We retain only the lowest-order terms in the pion momenta. We then have

$$T_{\mu\sigma}=F(\epsilon P_\mu-q_\mu)P_\sigma, \tag{2.24}$$

with $F=A_2+A_5$. Thus, although there are actually two form factors in the amplitude for the decay mode $\eta\to\pi+2\gamma$, for the case of "soft" pions, i.e., neglecting higher-order terms in the pion momenta, $T_{\mu\sigma}$ is given in terms of a single form factor F.

We note, however, that the preceding result is not symmetric in photon momenta since $\epsilon=q\cdot k_1/P\cdot k_1$. We have not demanded that the form factors be symmetric in k_1 and k_2. We may now impose this symmetry by symmetrizing $T_{\mu\sigma}$ from the start, in the obvious way:

$$T_{\mu\sigma}=\frac{1}{2}\left\{\int e^{ik_1\cdot x}dx\langle\pi|T(J_\mu^{\mathrm{EM}}(x)J_\sigma^{\mathrm{EM}}(0))|\eta\rangle+\int e^{ik_2\cdot x}dx\langle\pi|T(J_\mu^{\mathrm{EM}}(x)J_\sigma^{\mathrm{EM}}(0))|\eta\rangle\right\}. \tag{2.25}$$

Then one finds

$$T_{\mu\sigma}=F(\epsilon P_\mu-q_\mu)P_\sigma, \tag{2.24$'$}$$

where now

$$\epsilon=\frac{1}{2}\left(\frac{q\cdot k_1}{P\cdot k_1}+\frac{q\cdot k_2}{P\cdot k_2}\right) \tag{2.22$'$}$$

and F must be symmetric in k_1 and k_2.

For the isovector photon decay, we define

$$T_{\mu\sigma}^e(P,\,q_a+q_b,\,q_c)=F^e[\epsilon(a,b)P_\mu-q_{c\mu}]P_\sigma, \tag{2.26}$$

where

$$\epsilon(a,b)=\frac{1}{2}\left(\frac{q_c\cdot(q_a+q_b)}{P\cdot(q_a+q_b)}+\frac{q_c\cdot(P-q_a-q_b-q_c)}{P\cdot(P-q_a-q_b-q_c)}\right). \tag{2.27}$$

If we assume that the photon is a U-spin scalar, then by SU_3 invariance these form factors can be related to those associated with the real photon process by

$$F^e=\tfrac{3}{2}F\delta_{e3}. \tag{2.28}$$

Combining Eqs. (2.14)–(2.16), (2.26), and (2.28) we obtain

$$\begin{aligned}\frac{i}{M_\eta^2}\{g_1q_{a\sigma}+g_2g_{b\sigma}+g_3q_{c\sigma}+g_4P_\sigma\} &= -\tfrac{3}{2}\frac{F}{F_\pi^2}\epsilon_{ba3}(q_b-q_a)_\mu\{\epsilon(a,b)P_\mu-q_{c\mu}\}P_\sigma \\ &\quad -\tfrac{3}{2}\frac{F}{F_\pi^2}\epsilon_{ca3}(q_c-q_a)_\mu\{\epsilon(a,c)P_\mu-q_{b\mu}\}P_\sigma \\ &\quad -\tfrac{3}{2}\frac{F}{F_\pi^2}\epsilon_{cb3}(q_c-q_b)_\mu\{\epsilon(b,c)P_\mu-q_{a\mu}\}P_\sigma. \end{aligned}\tag{2.29}$$

From Eq. (2.29) we obtain

$$g_1=g_2=g_3=0\,, \tag{2.30a}$$

and

$$\begin{aligned}g_4=\frac{3iFM_\eta^2}{2F_\pi^2}\big[&\epsilon_{ba3}\{\epsilon(a,b)P\cdot(q_b-q_a)-q_c\cdot(q_b-q_a)\} \\ &+\epsilon_{ca3}\{\epsilon(a,c)P\cdot(q_c-q_a)-q_b\cdot(q_c-q_a)\} \\ &+\epsilon_{cb3}\{\epsilon(b,c)P\cdot(q_c-q_b)-q_a\cdot(q_c-q_b)\}\big],\end{aligned}\tag{2.30b}$$

where $\epsilon(a,c)$ and $\epsilon(b,c)$ are defined analogously to $\epsilon(a,b)$.

Neglecting terms quadratic in the pion momenta, Eq. (2.27) reduces to

$$\epsilon(a,b)=\frac{1}{2}\left(\frac{q_c\cdot(q_a+q_b)}{P\cdot(q_a+q_b)}+\frac{P\cdot q_c}{M_\eta^2}\right), \tag{2.31}$$

and Eq. (2.30b) becomes

$$\begin{aligned}g_4=\frac{3iFM_\eta^2}{4F_\pi^2}\Bigg[&\epsilon_{ba3}\Bigg\{P\cdot(q_b-q_a)\left(\frac{q_c\cdot(q_a+q_b)}{P\cdot(q_a+q_b)}+\frac{q_c\cdot P}{M_\eta^2}\right) \\ &-2q_c\cdot(q_b-q_a)\Bigg\}+\epsilon_{ca3}\Bigg\{P\cdot(q_c-q_a)\left(\frac{q_b\cdot(q_a+q_c)}{P\cdot(q_a+q_c)}\right. \\ &\left.+\frac{q_b\cdot P}{M_\eta^2}\right)-2q_b\cdot(q_c-q_a)\Bigg\}+\epsilon_{cb3}\Bigg\{P\cdot(q_c-q_b) \\ &\times\left(\frac{q_a\cdot(q_b+q_c)}{P\cdot(q_b+q_c)}+\frac{q_a\cdot P}{M_\eta^2}\right)-2q_a\cdot(q_c-q_b)\Bigg\}\Bigg].\end{aligned}\tag{2.32}$$

For the $\eta\to 3\pi^0+\gamma$ mode, we obtain

$$g_1=g_2=g_3=g_4=0\,.$$

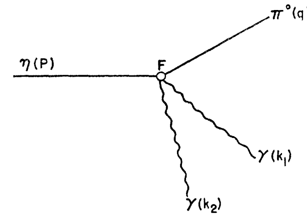

FIG. 1. Feynman diagram for the decay $\eta\to\pi^0\gamma\gamma$.

Thus, $\Gamma(\eta\to 3\pi^0+\gamma)=0$, a result which we know to be true from the fact that such a mode corresponds to a forbidden 0–0 transition.

For the $\eta\to\pi^+\pi^-\pi^0\gamma$ mode, we obtain

$$\begin{aligned}g_4=\frac{3FM_\eta^2}{4F_\pi^2}\Bigg[&P\cdot(q_--q_+)\left(\frac{q_0\cdot(q_++q_-)}{P\cdot(q_++q_-)}\right. \\ &\left.+\frac{P\cdot q_0}{M_\eta^2}\right)2q_0\cdot(q_--q_+)\Bigg],\end{aligned}\tag{2.33}$$

where q_+, q_-, q_0 are the π^+, π^-, π^0 momenta, respectively.

We have thus related the form factor for $\eta\to\pi^+\pi^-\pi^0\gamma$ to the form factor for $\eta\to\pi^0\gamma\gamma$. We observe, however, that the form factor g_4 is, to lowest order, a function *quadratic* in the pion momenta. This strong momentum dependence suppresses the rate for the mode $\eta\to\pi^+\pi^-\pi^0\gamma$ and accounts for the small branching ratio for this mode.

III. CALCULATION OF THE $\eta\to\pi^0+2\gamma$ DECAY RATE

The S-matrix element for the decay $\eta\to\pi^0+2\gamma$ obtained from Fig. 1 is

$$\begin{aligned}\langle\pi^0\gamma\gamma|\eta\rangle=-&\frac{e^2(2\pi)^4\delta^4(P-q-k_1-k_2)}{2(2\pi)^6(16M_\eta\omega_1\omega_2E_\pi)^{1/2}} \\ &\times(\epsilon_{1\nu}\epsilon_{2\lambda}+\epsilon_{2\nu}\epsilon_{1\lambda})T_{\nu\lambda}\,,\end{aligned}\tag{3.1}$$

where we have symmetrized with respect to the photon momenta, and

$$T_{\nu\lambda}=F(\epsilon P_\nu-q_\nu)P_\lambda\,, \tag{3.2}$$

and

$$\epsilon=\frac{1}{2}\left(\frac{q\cdot k_1}{P\cdot k_1}+\frac{q\cdot k_2}{P\cdot k_2}\right). \tag{3.3}$$

Then

$$\begin{aligned}\langle\pi^0\gamma\gamma|\eta\rangle=-&\frac{e^2}{2(2\pi)^6}(2\pi)^4\delta^4(P-q-k_1-k_2) \\ &\times\frac{F}{(16M_\eta\omega_1\omega_2E_\pi)^{1/2}}[(\epsilon\epsilon_1\cdot P-\epsilon_1\cdot q)(\epsilon_2\cdot P) \\ &+(\epsilon\epsilon_2\cdot P-\epsilon_2\cdot q)(\epsilon_1\cdot P)].\end{aligned}\tag{3.4}$$

The decay rate is given by

$$\Gamma=\frac{\alpha^2F^2}{16M_\eta(2\pi)^3}\int\frac{d^3q}{E_\pi}\frac{d^3k_1}{\omega_1}\frac{d^3k_2}{\omega_2}\delta^4(P-q-k_1-k_2)$$
$$\times\sum_{\text{pol.}}|(\epsilon\epsilon_1\cdot P-\epsilon_1\cdot q)(\epsilon_2\cdot P)+(\epsilon\epsilon_2\cdot P-\epsilon_2\cdot q)(\epsilon_1\cdot P)|^2, \quad (3.5)$$

where α is the fine-structure constant. We have taken F out of the integral, assuming that it may be approximated by a constant. This approximation is certainly legitimate to lowest order in the pion momentum.

We have

$$\sum_{\text{pol.}}(\epsilon_{1,2}\cdot P)^2=P^2=M_\eta^2, \quad (3.6)$$

$$\sum_{\text{pol.}}[\epsilon_{1,2}\cdot(\epsilon P-q)]^2=\epsilon^2M_\eta^2+\mu_0^2-2\epsilon P\cdot q, \quad (3.7)$$

$$\sum_{\text{pol.}}(\epsilon_{1,2}\cdot P)[\epsilon_{1,2}\cdot(\epsilon P-q)]=\epsilon M_\eta^2-P\cdot q, \quad (3.8)$$

where μ_0 is the π^0 mass. Thus,

$$\Gamma=\frac{\alpha^2f^2}{2M_\eta^3(2\pi)^3}\int\frac{d^3q}{E_\pi}\frac{d^3k_1}{2\omega_1}\frac{d^3k_2}{2\omega_2}\left[2\epsilon^2M_\eta^2+\mu_0^2-4\epsilon P\cdot q+\frac{(P\cdot q)^2}{M_\eta^2}\right]\delta^4(P-q-k_1-k_2), \quad (3.9)$$

where we have defined the dimensionless form factor f by

$$F=f/M_\eta^2. \quad (3.10)$$

The detailed calculation of the $\eta\to\pi^0+2\gamma$ decay rate is carried out in Appendix A. We obtain the result

$$\Gamma(\eta\to\pi^0\gamma\gamma)=\frac{\alpha^2f^2\mu_0}{4\lambda^3(2\pi)}\left[\frac{(\lambda^4-1)(2\lambda^2+1)}{8\lambda^2}+\frac{(\lambda^4-1)(\lambda^2-1)^2}{64\lambda^3}+\frac{(\lambda^4-1)}{32\lambda}-\tfrac{5}{8}\ln\lambda-\lambda^2\ln\lambda-\frac{1}{6\lambda^2}(\lambda^2-1)^3+4B_1+B_2-4\lambda B_3\right], \quad (3.11)$$

where

$$B_1=\int_1^{x_{\max}}x\Psi(x)dx, \quad (3.12a)$$

$$B_2=\int_1^{x_{\max}}(x^2-1)^{1/2}(\lambda^2-2\lambda x+1)\times\left[1+\frac{\Psi(x)}{(\lambda-x)(x^2-1)^{1/2}}\right]dx, \quad (3.12b)$$

$$B_3=\int_1^{x_{\max}}\Psi(x)dx, \quad (3.12c)$$

$$\Psi(x)=(\lambda^2-2\lambda x+1)\tanh^{-1}\left[\frac{x^2-1}{(\lambda-x)^2}\right]^{1/2}, \quad (3.13)$$

$$\lambda=M_\eta/\mu_0, \quad (3.14)$$

$$x_{\max}=(\lambda^2+1)/2\lambda. \quad (3.15)$$

The integrals B_1, B_2, B_3 can be evaluated numerically. From the experimental value of the rate,[22]

$$\Gamma(\eta\to\pi^0\gamma\gamma)=2.05\text{ keV}, \quad (3.16)$$

where we have taken $\Gamma(\eta\to\text{all modes})=10$ keV, we find from Eq. (3.11) the effective coupling constant for the mode $\eta\to\pi^0\gamma\gamma$ to be

$$f=6.2. \quad (3.17)$$

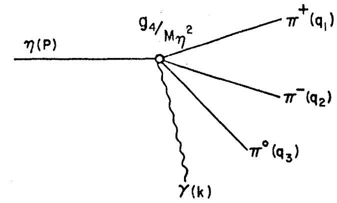

FIG. 2. Feynman diagram for the decay $\eta\to\pi^+\pi^-\pi^0\gamma$.

IV. CALCULATION OF THE $\eta\to\pi^+\pi^-\pi^0\gamma$ DECAY RATE

The S-matrix element for the decay $\eta\to\pi^+\pi^-\pi^0\gamma$ obtained from Fig. 2 is

$$\langle\pi^+\pi^-\pi^0\gamma|\eta\rangle=\frac{e}{(2\pi)^{5/2}}(2\pi)^4\delta^4(P-q_1-q_2-q_3-k)\times\frac{1}{(32M_\eta\omega_1\omega_2\omega_3k)^{1/2}}\epsilon_\nu M_\nu, \quad (4.1)$$

where

$$M_\nu=(g_4/M_\eta^2)P_\nu. \quad (4.2)$$

Using the results obtained in Sec. II, we have for the decay rate for this mode

$$\Gamma=\frac{9\alpha f^2}{32(2\pi)^7F_\pi^4M_\eta^3}\int\frac{d^3k}{k}\frac{d^3q_1}{2\omega_1}\frac{d^3q_2}{2\omega_2}\frac{d^3q_3}{2\omega_3}\times\delta^4(P-q_1-q_2-q_3-k)\left\{P\cdot(q_2-q_1)\left[\frac{q_3\cdot(q_1+q_2)}{P\cdot(q_1+q_2)}+\frac{P\cdot q_3}{M_\eta^2}\right]-2q_3\cdot(q_2-q_1)\right\}^2, \quad (4.3)$$

where once again we approximate f to be a constant.

[22] A. Rosenfeld *et al.*, Rev. Mod. Phys. 39, 1 (1967).

The detailed calculation of the $\eta \to \pi^+\pi^-\pi^0\gamma$ decay rate is carried out in Appendix B. We obtain the result

$$\Gamma(\eta \to \pi^+\pi^-\pi^0\gamma) = \frac{3\alpha f^2}{64(2\pi)^4 F_\pi^{\,4} M_\eta^{\,3}} \int_0^{k_{\max}} k dk \int_{\mu_0}^{(\omega_3)_{\max}} q_3 \left(\frac{Q^2-4\mu^2}{Q^2}\right)^{3/2} \left\{ (Q\cdot q_3)^2 - 4\mu_0^2 Q^2 + \frac{Q^2}{M_\eta^{\,2}}(3c^2+b^2) - 2\frac{(Q\cdot q_3)}{M_\eta^{\,2}} \right.$$
$$\left. \times (ac - \tfrac{1}{3}b^2) - \frac{Q^2 M_\eta^{\,2}(Q\cdot q_3)^2}{(a^2-b^2)} + 2Q^2(Q\cdot q_3)\left[\frac{(a+c)}{2b}\ln\left(\frac{a+b}{a-b}\right) - 1\right] + \frac{1}{M_\eta^{\,4}}\left[a^2c^2 + \tfrac{1}{5}b^4 + \tfrac{1}{3}b^2(a^2-4ac+c^2)\right] \right\} d\omega_3, \quad (4.4)$$

where

$$a = M_\eta(M_\eta - k)\left(\frac{Q^2 + Q\cdot q_3}{Q^2 + 2Q\cdot q_3 + \mu_0^2}\right), \quad (4.5a)$$

$$b = \frac{kM_\eta}{(Q^2+2Q\cdot q_3+\mu_0^2)}\left[(Q\cdot q_3)^2 - \mu_0^2 Q^2\right]^{1/2}, \quad (4.5b)$$

$$c = \frac{M_\eta(M_\eta - k)}{Q^2 + 2(Q\cdot q_3) + \mu_0^2}\left[(Q\cdot q_3) + \mu_0^2\right], \quad (4.5c)$$

and

$$k_{\max} = \frac{M_\eta^{\,2} - (2\mu+\mu_0)^2}{2M_\eta}, \quad (4.6)$$

$$(\omega_3)_{\max} = \frac{E_k^{\,2} + \mu_0^2 - 4\mu^2}{2E_k}, \quad (4.7)$$

$$E_k = (M_\eta^{\,2} - 2M_\eta k)^{1/2}. \quad (4.8)$$

The remaining two integrals in Eq. (4.4) were carried out numerically. Using the results of Sec. III, namely, $f = 6.2$, we obtain for the rate[23]

$$\Gamma(\eta \to \pi^+\pi^-\pi^0\gamma) = 7.5\times 10^{15}\ \mathrm{sec}^{-1}, \quad (4.9)$$

which gives us the branching ratio

$$\Gamma(\eta \to \pi^+\pi^-\pi^0\gamma)/\Gamma(\eta \to \pi^0\gamma\gamma) = 2.4\times 10^{-3}. \quad (4.10)$$

V. CALCULATION OF THE INTERNAL BREMSSTRAHLUNG CONTRIBUTION

The calculation of the rate for the decay mode $\eta \to \pi^+\pi^-\pi^0\gamma$ in Sec. IV only contains the direct electromagnetic decay. It is also of interest to estimate the contribution to the rate from internal bremsstrahlung. We therefore consider in this section the contribution from the bremsstrahlung emission alone which may be computed using perturbation theory.[24]

The diagrams for the bremsstrahlung process are shown in Fig. 3. The matrix element has the form

$$\mathfrak{M} = \left(\frac{q_1\cdot\epsilon}{q_1\cdot k} - \frac{q_2\cdot\epsilon}{q_2\cdot k}\right), \quad (5.1)$$

where ϵ is the photon polarization. Squaring and summing over photon polarizations gives

$$\sum_{\mathrm{pol.}} |\mathfrak{M}|^2 = \frac{2(q_1\cdot q_2)}{(q_1\cdot k)(q_2\cdot k)} - \frac{\mu^2}{(q_1\cdot k)^2} - \frac{\mu^2}{(q_2\cdot k)^2}, \quad (5.2)$$

and the rate becomes

$$\Gamma(\eta \to 3\pi+\gamma) = \frac{g^2\alpha}{2(2\pi)^7 M_\eta}\int \frac{d^3k}{k}\frac{d^3q_1}{2\omega_1}\frac{d^3q_2}{2\omega_2}\frac{d^3q_3}{2\omega_3} \times \sum_{\mathrm{pol.}} |\mathfrak{M}|^2 \delta^4(P-q_1-q_2-q_3-k), \quad (5.3)$$

where we assume that g is constant.

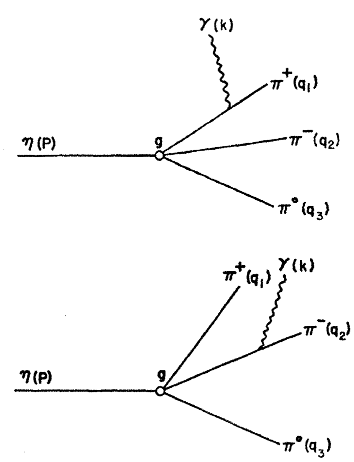

FIG. 3. Feynman diagrams for the internal bremsstrahlung in the decay $\eta \to \pi^+\pi^-\pi^0\gamma$.

[23] We obtain the value of F_π from the Goldberger-Treiman relation, $F_\pi = \sqrt{2} g_A M_N / g_r$.
[24] R. H. Dalitz, Phys. Rev. 99, 915 (1955).

We wish to compare this rate with that for the $\eta \to 3\pi$ mode. The diagram for the latter mode is shown in Fig. 4, where g is the same effective coupling constant as in the $\eta \to 3\pi+\gamma$ mode.

The rate is then given by

$$\Gamma(\eta \to 3\pi) = \frac{g^2}{2(2\pi)^5 M_\eta} \int \frac{d^3q_1}{2\omega_1} \frac{d^3q_2}{2\omega_2} \frac{d^3q_3}{2\omega_3} \times \delta^4(P-q_1-q_2-q_3). \quad (5.4)$$

The branching ratio for the two modes will be independent of the coupling constant g. As shown in Appendix C, we obtain the rate

$$\Gamma(\eta \to 3\pi) = 43.7\ g^2\ \mathrm{keV}. \quad (5.5)$$

In Appendix C we also find the rate

$$\Gamma(\eta \to 3\pi+\gamma) = \frac{g^2\alpha}{2(2\pi)^4 M_\eta} \times \int_{k_{\min}}^{k_{\max}} \left(1-\frac{2k}{M_\eta}\right) \frac{dk}{k} \Phi_2(k), \quad (5.6)$$

where

$$\Phi_2(k) = \int_{\mu_0}^{\omega_{\max}} \{q[1+x^2(k)] \tanh^{-1}x(k) - qx(k)\} d\omega, \quad (5.7)$$

and

$$x(k) = \left(\frac{Q^2-4\mu^2}{Q^2}\right)^{1/2}, \quad (5.8)$$

$$\omega_{\max} = \frac{E_k{}^2+\mu_0{}^2-4\mu^2}{2E_k}, \quad (5.9)$$

and E_k is defined by Eq. (4.8). Let

$$P(3\pi) \equiv \Gamma(\eta \to 3\pi), \quad (5.10)$$

and

$$P(3\pi+\gamma)dk = \frac{g^2\alpha}{2(2\pi)^4 M_\eta} \left(1-\frac{2k}{M_\eta}\right) \frac{dk}{k} \Phi_2(k). \quad (5.11)$$

Then the integral

$$R(k) = \int_k^{k_{\max}} \frac{P(3\pi+\gamma)}{P(3\pi)} dk \quad (5.12)$$

is the branching ratio of $\eta \to \pi^+\pi^-\pi^0\gamma$ compared to $\eta \to \pi^+\pi^-\pi^0$ as a function of the photon cutoff energy for internal bremsstrahlung. A plot of $R(k)$ versus k is shown in Fig. 5.

VI. CONCLUSIONS

We have shown in this paper that we may determine the rate for $\eta \to \pi^+\pi^-\pi^0\gamma$ in terms of the rate for $\eta \to \pi^0\gamma\gamma$ using the method of current algebra and PCAC.

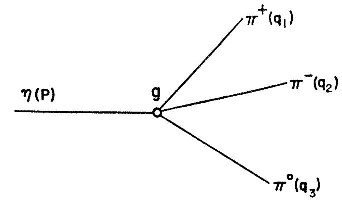

FIG. 4. Feynman diagram for the decay $\eta \to \pi^+\pi^-\pi^0$.

The result obtained is $\Gamma(\eta \to \pi^+\pi^-\pi^0\gamma)/\Gamma(\eta \to \pi^0\gamma\gamma) = 0.24\%$, which agrees with Singer's model calculation[8] assuming vector-meson dominance.

Aside from phase-space considerations, the current-algebra calculation affords a convenient way in which to understand the suppression of the $\eta \to \pi^+\pi^-\pi^0\gamma$ mode. The presence of a strong momentum dependence in the direct-emission form factor is largely responsible for the resulting small branching ratio.

We note that the photon spectrum shown in Fig. 6 differs from that of Singer (see his Fig. 1) in that our spectrum tends to be skewed toward lower photon energies. However, the total rates are in agreement.

We have also calculated the rate for bremsstrahlung emission. The branching ratio for bremsstrahlung emission as a function of minimum photon energy is plotted in Fig. 5, and compared to the similar ratio for the direct decay. We note that the contribution from the

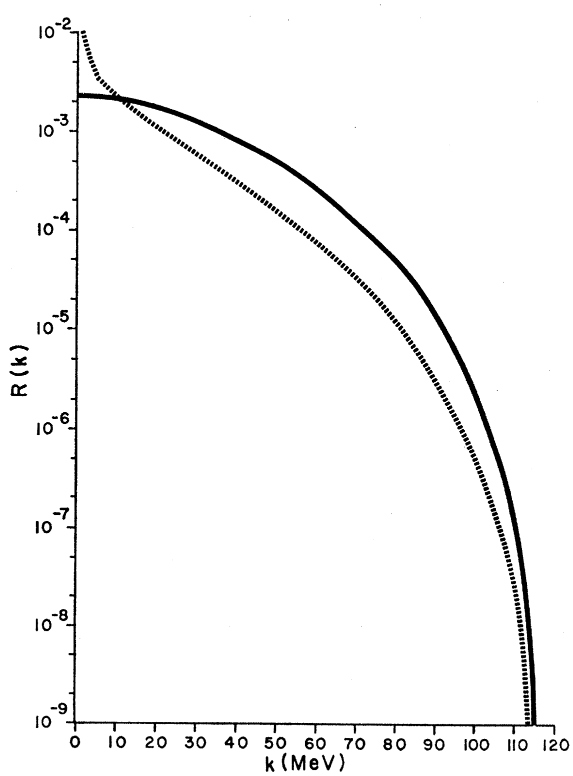

FIG. 5. Plot of the branching ratio $R(k)$ as a function of the photon energy. The solid (dashed) curve represents the direct emission (internal bremsstrahlung) contribution.

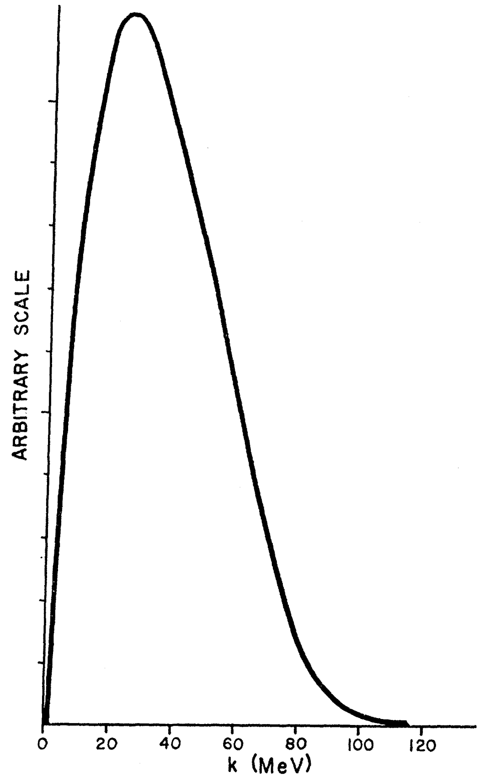

FIG. 6. Plot of the photon energy spectrum for the decay $\eta\to\pi^+\pi^-\pi^0\gamma$.

direct decay is about five times the bremsstrahlung contribution except for the very low-energy photons where, of course, the bremsstrahlung spectrum diverges.

This result contradicts an assertion of Singer[8] that we may neglect the bremsstrahlung since it is of order α^2 compared to the direct emission, and illustrates the pitfalls involved in making order-of-magnitude estimates of η-decay modes by counting powers of α.

The agreement of our calculation using current algebra with Singer's vector-meson-dominance model is an additional confirmation of the equivalence of the two methods. The vector-meson-dominance model has been used in several papers.[25–28] These results also agree quite well with the corresponding predictions of current algebra. This agreement suggests that such model calculations do yield excellent estimates of actual rates.

As Singer has noted, the $\eta\to\pi^+\pi^-\pi^0\gamma$ decay may offer an opportunity to observe a C violation directly through an asymmetry in the π^+-versus-π^- spectrum at moderate photon energies where the direct emission exceeds the bremsstrahlung. While the branching ratio for $\eta\to 3\pi+\gamma$ is small, it would be of great interest if such decays could be observed.

ACKNOWLEDGMENT

We wish to thank the staff at the Stevens Computer Center for their assistance in carrying out the numerical calculations.

APPENDIX A

In this Appendix we carry out the detailed calculation of the $\eta\to\pi^0+2\gamma$ decay rate. Equation (3.9) is

$$\Gamma=\frac{\alpha^2f^2}{2M_\eta^3(2\pi)^3}\int\frac{d^3q}{E_\pi}\frac{d^3k_1}{2\omega_1}\frac{d^3k_2}{2\omega_2}\left[2\epsilon^2M_\eta^2+\mu_0^2-4\epsilon P\cdot q+\frac{(P\cdot q)^2}{M_\eta^2}\right]\delta^4(P-q-k_1-k_2).\quad\text{(A1)}$$

Let us write

$$\Gamma=\frac{\alpha^2f^2}{2M_\eta^2(2\pi)^3}\int\frac{d^3q}{E_\pi}(I_1+I_2+I_3),\quad\text{(A2)}$$

where

$$I_1(M_\eta^2,\mu_0^2,P\cdot q)=\left[\mu_0^2+\frac{(P\cdot q)^2}{M_\eta^2}\right]\int\frac{d^3k_1}{2\omega_1}\frac{d^3k_2}{2\omega_2}\times\delta^4(P-q-k_1-k_2),\quad\text{(A3)}$$

$$I_2(M_\eta^2,\mu_0^2,P\cdot q)=-4(P\cdot q)\int\frac{d^3k_1}{2\omega_1}\frac{d^3k_2}{2\omega_2}\times\epsilon\delta^4(P-q-k_1-k_2),\quad\text{(A4)}$$

$$I_3(M_\eta^2,\mu_0^2,P\cdot q)=2M_\eta^2\int\frac{d^3k_1}{2\omega_1}\frac{d^3k_2}{2\omega_2}\times\epsilon^2\delta^4(P-q-k_1-k_2).\quad\text{(A5)}$$

We shall evaluate I_1, I_2, I_3 in the center-of-mass system of the photons. We introduce the relative coordinates Q and R where

$$Q=k_1+k_2,$$
$$R=k_1-k_2.$$

Thus,

$$I_1=\left(\mu_0^2+\frac{(P\cdot q)^2}{M_\eta^2}\right)\int d^4k_1d^4k_2\times\delta(k_1^2)\delta(k_2^2)\delta^4(P-q-k_1-k_2)$$
$$=\frac{1}{4}\left(\mu_0^2+\frac{(P\cdot q)^2}{M_\eta^2}\right)\int d^4Qd^4R\times\delta(Q\cdot R)\delta(Q^2-R^2)\delta^4(P-q-Q),$$

or

$$I_1=\frac{\pi}{2M_\eta^2}[\mu_0^2M_\eta^2+(P\cdot q)^2].\quad\text{(A6)}$$

[25] L. J. Clavelli, Phys. Rev. **154**, 1509 (1967).
[26] L. J. Clavelli, Phys. Rev. **160**, 1384 (1967).
[27] K. Kawarabayashi and M. Suzuki, Phys. Rev. Letters **16**, 255, 384 (1966).
[28] W. W. Wada, Phys. Rev. Letters **16**, 956 (1966).

Similarly,

$$I_2=-2(P\cdot q)\int d^4k_1d^4k_2\delta(k_1^2)\delta(k_2^2)\left(\frac{q\cdot k_1}{P\cdot k_1}+\frac{q\cdot k_2}{P\cdot k_2}\right)\times\delta^4(P-q-k_1-k_2)$$

$$=-\tfrac{1}{2}(P\cdot q)\int d^4Qd^4R\;\delta(Q\cdot R)\delta(Q^2+R^2)\times\left(\frac{q\cdot Q+q\cdot R}{P\cdot Q+P\cdot R}+\frac{q\cdot Q-q\cdot R}{P\cdot Q-P\cdot R}\right)\delta^4(P-q-Q),$$

or

$$I_2=-2\pi(P\cdot q)+(P\cdot q)(P\cdot Q)J,\tag{A7}$$

where

$$J=Q^2\int d^4R\frac{\delta(Q\cdot R)\delta(Q^2+R^2)}{(P\cdot Q)^2-(P\cdot R)^2},\tag{A8}$$

with

$$Q=P-q.\tag{A9}$$

Also,

$$I_3=\tfrac{1}{2}M_\eta^2\int d^4k_1d^4k_2\delta(k_1^2)\delta(k_2^2)\left(\frac{q\cdot k_1}{P\cdot k_1}+\frac{q\cdot k_2}{P\cdot k_2}\right)^2\times\delta^4(P-q-k_1-k_2)$$

$$=\tfrac{1}{2}M_\eta^2\int d^4Qd^4R\;\delta(Q\cdot R)\delta(Q^2+R^2)\times\left\{1-\frac{2Q^2(Q\cdot P)}{(P\cdot Q)^2-(P\cdot R)^2}+\frac{Q^4(P\cdot Q)^2}{[(P\cdot Q^2)-(P\cdot R)^2]^2}\right\}\times\delta^4(P-q-Q),$$

or

$$I_3=\pi M_\eta^2-M_\eta^4(Q\cdot P)J+\tfrac{1}{2}M_\eta^2(Q\cdot P)^2K,\tag{A10}$$

where

$$K=Q^4\int d^4R\frac{\delta(Q\cdot R)\delta(Q^2+R^2)}{[(P\cdot Q)^2-(P\cdot R)^2]^2}.\tag{A11}$$

J and K may be evaluated explicitly and we obtain

$$J=\frac{2\pi Q^2}{(P\cdot Q)^2(1-M_\eta^2Q^2/(P\cdot Q)^2)^{1/2}}\times\tanh^{-1}\left[1-\frac{M_\eta^2Q^2}{(P\cdot Q)^2}\right]^{1/2},\tag{A12}$$

$$K=\pi Q^2\left\{\frac{1}{M_\eta^2(P\cdot Q)^2}+\frac{Q^2}{(P\cdot Q)^4[1-M_\eta^2Q^2/(P\cdot Q)^2]^{1/2}}\times\tanh^{-1}\left(1-\frac{M_\eta^2Q^2}{(P\cdot Q)^2}\right)^{1/2}\right\}.\tag{A13}$$

Combining Eqs. (A6), (A7) and (A10) gives

$$I_1+I_2+I_3=\frac{\pi}{2M_\eta^2}[\mu_0^2M_\eta^2+(P\cdot q)^2]-2\pi(P\cdot q)+(P\cdot q-M_\eta^2)(P\cdot Q)J+\pi M_\eta^2+\tfrac{1}{2}M_\eta^2(P\cdot Q)^2K.\tag{A14}$$

Thus, the decay rate is given by

$$\Gamma=\frac{\alpha^2f^2}{M_\eta^3(2\pi)^3}\int\frac{d^3q}{E_\pi}\left\{\frac{\pi}{4M_\eta^2}\times[2M_\eta^4+\mu_0^2M_\eta^2+(P\cdot q)^2]-\pi(P\cdot q)+\tfrac{1}{2}(P\cdot Q)[(P\cdot q)-M_\eta^2]J+\tfrac{1}{4}M_\eta^2(P\cdot Q)^2K\right\}.\tag{A15}$$

We have for the various integrations

$$\tfrac{1}{4}\pi(\mu_0^2+2M_\eta^2)\int\frac{d^3q}{E_\pi}=\tfrac{1}{2}\pi^2(\mu_0^4+2\mu_0^2M_\eta^2)\times\left\{\frac{\lambda^4-1}{4\lambda^2}-\ln\lambda\right\},\tag{A16}$$

$$\frac{\pi}{4M_\eta^2}\int\frac{(P\cdot q)^2}{E_\pi}d^3q=\pi^2\mu_0^4\left\{\left(\frac{\lambda^4-1}{16\lambda}\right)\left(\frac{\lambda^2-1}{2\lambda}\right)^2+\frac{1}{8}\left(\frac{\lambda^4-1}{4\lambda}\right)-\tfrac{1}{8}\ln\lambda\right\},\tag{A17}$$

$$-\pi\int\frac{d^3q}{E_\pi}(P\cdot q)=-\tfrac{1}{6}\pi^2\mu_0^4\frac{(\lambda^2-1)^3}{\lambda^2},\tag{A18}$$

$$\frac{1}{2}\int P\cdot(P-q)(P\cdot q)J\frac{d^3q}{E_\pi}=4\pi^2\mu_0^4\int_1^{x_{\max}}x(\lambda^2+1-2\lambda x)\times\tanh^{-1}\left[\frac{x^2-1}{(\lambda-x)^2}\right]^{1/2}dx,\tag{A19}$$

$$-\tfrac{1}{2}M_\eta^2\int\frac{d^3q}{E_\pi}P\cdot(P-q)J=-4\pi^2\mu_0^3M_\eta\times\int_1^{x_{\max}}(\lambda^2-2\lambda x+1)\tanh^{-1}\left[\frac{x^2-1}{(\lambda-x)^2}\right]^{1/2}dx,\tag{A20}$$

$$\tfrac{1}{4}M_\eta^2\int\frac{d^3q}{E_\pi}[P\cdot(P-q)]^2K=\pi^2\mu_0^4\times\int_1^{x_{\max}}(x^2-1)^{1/2}(\lambda^2-2\lambda x+1)\times\left\{1+\frac{(\lambda^2-2\lambda x+1)}{(\lambda-x)(x^2-1)^{1/2}}\tanh^{-1}\left[\frac{x^2-1}{(\lambda-x)^2}\right]^{1/2}\right\}dx.\tag{A21}$$

Combining Eqs. (A16)–(A21) and substituting in Eq.

(A15) yields

$$\Gamma(\eta\to\pi^0\gamma\gamma)=\frac{\alpha^2 f^2\mu_0}{4\lambda^3(2\pi)}\left\{\frac{(\lambda^4-1)(2\lambda^2+1)}{8\lambda^2}+\frac{(\lambda^4-1)(\lambda^2-1)^2}{64\lambda^3}+\frac{(\lambda^4-1)}{32\lambda}-\tfrac{5}{8}\ln\lambda\right.$$
$$\left.-\lambda^2\ln\lambda-\frac{1}{6\lambda^2}(\lambda^2-1)^3+4B_1+B_2-4\lambda B_3\right\}. \quad \text{(A22)}$$

APPENDIX B

In this Appendix we carry out the detailed calculation of the $\eta\to\pi^+\pi^-\pi^0\gamma$ decay rate. Equation (4.3) is

$$\Gamma=\frac{9\alpha f^2}{32(2\pi)^7F_\pi{}^4M_\eta{}^3}\int\frac{d^3k}{k}\frac{d^3q_1}{2\omega_1}\frac{d^3q_2}{2\omega_2}\frac{d^3q_3}{2\omega_3}$$
$$\times\delta^4(P-q_1-q_2-q_3-k)\left\{P\cdot(q_2-q_1)\left[\frac{q_3\cdot(q_1+q_2)}{P\cdot(q_1+q_2)}+\frac{P\cdot q_3}{M_\eta{}^2}\right]-2q_3\cdot(q_2-q_1)\right\}^2. \quad \text{(B1)}$$

We shall carry out the integrations over the pion momenta in the center-of-mass system of the three pions. In this system $P=(P_0,\mathbf{K})$, $k=(K_0,\mathbf{K})$, $K_0=K$, where

$$\mathbf{K}=\mathbf{k}\left[\frac{M_\eta}{M_\eta-2k}\right]^{1/2}, \quad \text{(B2)}$$

$$P_0=(K^2+M_\eta{}^2)^{1/2}. \quad \text{(B3)}$$

Also, $P-k=(E_k,0)$, where

$$E_k=(M_\eta{}^2-2M_\eta k)^{1/2}. \quad \text{(B4)}$$

For the q_1 and q_2 integrations, we again introduce the relative coordinates Q and R. Then

$$\Gamma=\frac{9\alpha f^2}{128(2\pi)^7F_\pi{}^4M_\eta{}^3}\int\frac{d^3k}{k}\frac{d^3q_3}{2\omega_3}\int d^4Qd^4R$$
$$\times\delta(Q\cdot R)\delta(Q^2+R^2-4\mu^2)\delta^4(P-Q-q_3-k)$$
$$\times\left[(P\cdot R)\left(\frac{q_3\cdot Q}{P\cdot Q}+\frac{P\cdot q_3}{M_\eta{}^2}\right)-2q_3\cdot R\right]^2, \quad \text{(B5)}$$

where μ is the $\pi^\pm$ mass. Expanding the bracketed expression in Eq. (B5) gives

$$\int d^4Qd^4R\ \delta(Q\cdot R)\delta(Q^2+R^2-4\mu^2)\left[(P\cdot R)\left(\frac{q_3\cdot Q}{P\cdot Q}+\frac{P\cdot q_3}{M_\eta{}^2}\right)-2q_3\cdot R\right]^2\delta^4(P-Q-q_3-k)$$
$$=\int d^4Qd^4R\ \delta(Q\cdot R)\delta(Q^2+R^2-4\mu^2)\left\{\frac{(P\cdot R)^2(Q\cdot q_3)^2}{(P\cdot Q)^2}+\frac{(P\cdot R)^2(P\cdot q_3)^2}{M_\eta{}^4}+\frac{2(P\cdot R)^2(Q\cdot q_3)(P\cdot q_3)}{(P\cdot Q)M_\eta{}^2}\right.$$
$$\left.+4(q_3\cdot R)^2-\frac{4(P\cdot R)(q_3\cdot R)(q_3\cdot Q)}{(P\cdot Q)}-\frac{4(P\cdot R)(q_3\cdot R)(P\cdot q_3)}{M_\eta{}^2}\right\}\delta^4(P-Q-q_3-k)=L_1+L_2+L_3+L_4+L_5+L_6=L, \quad \text{(B6)}$$

where

$$L_1=\int d^4Qd^4R\delta(Q\cdot R)\delta(Q^2+R^2-4\mu^2)\frac{(P\cdot Q)^2(Q\cdot q_3)^2}{(P\cdot Q)^2}\delta^4(P-Q-q_3-k)=\frac{2\pi}{3}\frac{(Q\cdot q_3)^2}{(Q\cdot P)^2}[(Q\cdot P)^2-Q^2M_\eta{}^2]\left(\frac{Q^2-4\mu^2}{Q^2}\right)^{3/2}, \quad \text{(B7a)}$$

$$L_2=\frac{(P\cdot q_3)^2}{M_\eta{}^4}\int d^4Qd^4R\delta(Q\cdot R)\delta(Q^2+R^2-4\mu^2)(P\cdot R)^2\delta^4(P-Q-q_3-k)$$
$$=\frac{2\pi}{3}\frac{(P\cdot q_3)^2}{M_\eta{}^4}[(Q\cdot P)^2-Q^2M_\eta{}^2]\left(\frac{Q^2-4\mu^2}{Q^2}\right)^{3/2}, \quad \text{(B7b)}$$

$$L_3=2\frac{(P\cdot q_3)}{M_\eta{}^2}\int d^4Qd^4R\delta(Q\cdot R)\delta(Q^2+R^2-4\mu^2)\frac{(P\cdot R)^2(Q\cdot q_3)}{(P\cdot Q)}\delta^4(P-Q-q_3-k)$$
$$=\tfrac{4}{3}\pi\frac{(P\cdot q_3)(Q\cdot q_3)}{(Q\cdot P)M_\eta{}^2}[(Q\cdot P)^2-Q^2M_\eta{}^2]\left(\frac{Q^2-4\mu^2}{Q^2}\right)^{3/2}, \quad \text{(B7c)}$$

$$L_4=4\int d^4Qd^4R\delta(Q\cdot R)\delta(Q^2+R^2-4\mu^2)(q_3\cdot R)^2\delta^4(P-Q-q_3-k)=\frac{8\pi}{3}[(q_3\cdot Q)^2-\mu_0{}^2Q^2]\left(\frac{Q^2-4\mu^2}{Q^2}\right)^{3/2}, \quad \text{(B7d)}$$

$$L_5=-4\int d^4Qd^4R\delta(Q\cdot R)\delta(Q^2+R^2-4\mu^2)\frac{(P\cdot R)(q_3\cdot R)(q_3\cdot Q)}{(P\cdot Q)}\delta^4(P-Q-q_3-k)$$

$$=-\frac{8\pi}{3}\frac{(Q\cdot q_3)}{(Q\cdot P)}[(Q\cdot P)(Q\cdot q_3)-Q^2(P\cdot q_3)]\left(\frac{Q^2-4\mu^2}{Q^2}\right)^{3/2}, \quad \text{(B7e)}$$

$$L_6=-4\frac{(P\cdot q_3)}{M_\eta^2}\int d^4Qd^4R\delta(Q\cdot R)\delta(Q^2+R^2-4\mu^2)(P\cdot R)(q_3\cdot R)\delta^4(P-Q-q_3-k)$$

$$=-\frac{8\pi}{3}\frac{(P\cdot q_3)}{M_\eta^2}[(Q\cdot P)(Q\cdot q_3)-Q^2(P\cdot q_3)]\left(\frac{Q^2-4\mu^2}{Q^2}\right)^{3/2}. \quad \text{(B7f)}$$

Thus,

$$L=\tfrac{2}{3}\pi\left(\frac{Q^2-4\mu^2}{Q^2}\right)^{3/2}\Bigg\{\left[\frac{(Q\cdot q_3)^2}{(Q\cdot P)^2}+\frac{(P\cdot q_3)}{M_\eta^4}+2\frac{(P\cdot q_3)(Q\cdot q_3)}{(Q\cdot P)M_\eta^2}\right][(Q\cdot P)^2-Q^2M_\eta^2]$$

$$+4[(q_3\cdot Q)^2-\mu_0^2Q^2]-4\left[\frac{(Q\cdot q_3)}{(Q\cdot P)}+\frac{(P\cdot q_3)}{M_\eta^2}\right][(Q\cdot P)(Q\cdot q_3)-Q^2(P\cdot q_3)]\Bigg\}, \quad \text{(B8)}$$

where $Q=P-q_3-k$. Equation (B8) reduces to

$$L=\tfrac{2}{3}\pi\left(\frac{Q^2-4\mu^2}{Q^2}\right)^{3/2}\Bigg\{(Q\cdot q_3)^2-4\mu_0^2Q^2+\frac{3Q^2}{M_\eta^2}(P\cdot q_3)^2+2Q^2\frac{(Q\cdot q_3)(P\cdot q_3)}{(Q\cdot P)}$$

$$-2\frac{(Q\cdot q_3)}{M_\eta^2}(Q\cdot P)(P\cdot q_3)+\frac{(Q\cdot P)^2(P\cdot q_3)^2}{M_\eta^4}-\frac{Q^2M_\eta^2(Q\cdot q_3)^2}{(Q\cdot P)^2}\Bigg\}. \quad \text{(B9)}$$

Then, performing the angular integrations, we obtain from Eq. (B5)

$$\Gamma(\eta\to\pi^+\pi^-\pi^0\gamma)=\frac{3\alpha f^2}{64(2\pi)^4F_\pi^4M_\eta^3}\int_0^{k_{\max}}kdk\int_{\mu_0}^{(\omega_3)_{\max}}q_3\left(\frac{Q^2-4\mu^2}{Q^2}\right)^{3/2}\Bigg\{(Q\cdot q_3)^2-4\mu_0^2Q^2+\frac{Q^2}{M_\eta^2}(3c^2+b^2)$$

$$-2\frac{(Q\cdot q_3)}{M_\eta^2}(ac-\tfrac{1}{3}b^2)-\frac{Q^2M_\eta^2(Q\cdot q_3)^2}{(a^2-b^2)}+2Q^2(Q\cdot q_3)\left[\frac{(a+c)}{2b}\ln\left(\frac{a+b}{a-b}\right)-1\right]$$

$$+\frac{1}{M_\eta^2}[a^2c^2+\tfrac{1}{5}b^4+\tfrac{1}{3}b^2(a^2-4ac+c^2)]\Bigg\}d\omega_3, \quad \text{(B10)}$$

where a, b, c are defined by Eqs. (4.5).

APPENDIX C

In this Appendix we carry out the detailed calculation of the contribution from internal bremsstrahlung to the $\eta\to\pi^+\pi^-\pi^0\gamma$ decay rate.

1. $\eta\to\pi^+\pi^-\pi^0$

Introducing once again the relative coordinates Q, R, Eq. (5.4) becomes

$$\Gamma(\eta\to 3\pi)=\frac{g^2}{8(2\pi)^5M_\eta}\int\frac{d^3q_3}{2\omega_3}\int d^4Qd^4R\;\delta(Q\cdot R)\delta(Q^2+R^2-4\mu^2)\delta^4(P-q_3-Q)=\frac{g^2}{16(2\pi)^4M_\eta}\int\frac{d^3q_3}{\omega_3}$$

$$\times\left(\frac{Q^2-4\mu^2}{Q^2}\right)^{1/2}=\frac{g^2}{8(2\pi)^3M_\eta}\int_{\mu_0}^{(\omega_3)_{\max}}(\omega_3^2-\mu_0^2)^{1/2}\left(\frac{M_\eta^2-2M_\eta\omega_3+\mu_0^2-4\mu^2}{M_\eta^2-2M_\eta\omega_3+\mu_0^2}\right)^{1/2}d\omega_3, \quad \text{(C1)}$$

where $Q=P-q_3$, $Q^2=M_\eta{}^2-2M_\eta\omega_3+\mu_0{}^2$ in the rest frame of the η, and

$$(\omega_3)_{\max}=\frac{1}{2M_\eta}(M_\eta{}^2+\mu_0{}^2-4\mu^2). \qquad \text{(C2)}$$

We may write Eq. (C1) as

$$\Gamma(\eta\to 3\pi)=\frac{g^2\mu_0}{8(2\pi)^3\lambda}\int_1^{x_{\max}}\Phi_1(x)dx, \qquad \text{(C3)}$$

where

$$\Phi_1(x)=(x^2-1)^{1/2}\left(\frac{\lambda^2-2\lambda x+1-4\rho^2}{\lambda^2-2\lambda x+1}\right)^{1/2}, \qquad \text{(C4)}$$

$$x_{\max}=\frac{1}{2\lambda}(\lambda^2+1-4\rho^2), \qquad \text{(C5)}$$

$$\lambda=M_\eta/\mu_0, \quad \rho=\mu/\mu_0. \qquad \text{(C6)}$$

Carrying out the integration, we obtain

$$\Gamma(\eta\to 3\pi)=34.7\ g^2\ \text{keV}. \qquad \text{(C7)}$$

2. $\eta\to\pi^+\pi^-\pi^0\gamma$

Introducing Q and R, the q_1 and q_2 integrations in Eq. (5.3) become

$$\int\frac{d^3q_1}{2\omega_1}\frac{d^3q_2}{2\omega_2}\sum_{\text{pol.}}|\mathfrak{M}|^2\delta^4(P-q_1-q_2-q_3-k)$$
$$=\frac{1}{4}\int d^4Qd^4R\ \delta(Q\cdot R)\delta(Q^2+R^2-4\mu^2)$$
$$\times\delta^4(P-q_3-Q-k)\left[-\frac{4\mu^2}{(Q\cdot k+R\cdot k)^2}-\frac{4\mu^2}{(Q\cdot k-R\cdot k)^2}+\frac{2(Q^2-R^2)}{(Q\cdot k)^2-(R\cdot k)^2}\right]. \qquad \text{(C8)}$$

The first two terms in the bracket of Eq. (C8) contribute equally since the integration is symmetric.

$$2\int\frac{d^4Qd^4R}{(Q\cdot k+R\cdot k)^2}\delta(Q\cdot R)\delta(Q^2+R^2-4\mu^2)$$
$$\times\delta^4(P-k-q_3-Q)=\frac{\pi Q^2}{\mu^2(Q\cdot k)^2}\left(\frac{Q^2-4\mu^2}{Q^2}\right)^{1/2}. \qquad \text{(C9)}$$

The third term gives[24]

$$\int\frac{Q^2-R^2}{(Q\cdot k)^2-(R\cdot k)^2}\delta(Q\cdot R)\delta(Q^2+R^2-4\mu^2)d^4R$$
$$=4\pi\frac{(Q^2-2\mu^2)}{(Q\cdot k)^2}\tanh^{-1}\left(\frac{Q^2-4\mu^2}{Q^2}\right)^{1/2}. \qquad \text{(C10)}$$

Thus, Eq. (C8) becomes

$$\int\frac{d^3q_1}{2\omega_1}\frac{d^3q_1}{2\omega_2}\sum_{\text{pol.}}|\mathfrak{M}|^2\delta^4(P-q_1-q_2-q_3-k)=-\frac{\pi Q^2}{(Q\cdot k)^2}$$
$$\times\left(\frac{Q^2-4\mu^2}{Q^2}\right)^{1/2}+2\pi\frac{(Q^2-2\mu^2)}{(Q\cdot k)^2}\tanh^{-1}\left(\frac{Q^2-4\mu^2}{Q^2}\right)^{1/2},$$

and

$$\Gamma(\eta\to 3\pi+\gamma)=\frac{g^2\alpha}{2(2\pi)^5M_\eta}\int kdk\int\frac{q_3}{K^2}$$
$$\times\left\{2\pi(Q^2-2\mu^2)\tanh^{-1}\left(\frac{Q^2-4\mu^2}{Q^2}\right)^{1/2}-\pi Q^2\left(\frac{Q^2-4\mu^2}{Q^2}\right)^{1/2}\right\}\int_{-1}^{1}\frac{dx}{[(E_k-\omega_3)+q_3\cdot x]^2},$$

where K and E_k are defined in Eqs. (B2) and (B4). The above reduces to

$$\Gamma(\eta\to 3\pi+\gamma)=\frac{g^2\alpha}{(2\pi)^5M_\eta}\int kdk\int\frac{q_3d\omega_3}{K^2Q^2}\left\{2\pi(Q^2-2\mu^2)\right.$$
$$\left.\times\tanh^{-1}\left(\frac{Q^2-4\mu^2}{Q^2}\right)^{1/2}-\pi Q^2\left(\frac{Q^2-4\mu^2}{Q^2}\right)^{1/2}\right\}. \qquad \text{(C11)}$$

It is convenient to define

$$x(k)=\left(\frac{Q^2-4\mu^2}{Q^2}\right)^{1/2}. \qquad \text{(C12)}$$

The rate in Eq. (C11) is then given by

$$\Gamma(\eta\to 3\pi+\gamma)=\frac{g^2\alpha}{2(2\pi)^4M_\eta}$$
$$\times\int_{k_{\min}}^{k_{\max}}\left(1-\frac{2k}{M_\eta}\right)\frac{dk}{k}\Phi_2(k), \qquad \text{(C13)}$$

where $\Phi_2(k)$ is defined by Eq. (5.7).