Covariant Propagators and Vertex Functions for Any Spin*

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High-spin wave functions, propagator numerators, and vertex functions are developed in a covariant, on-shell manner. General formulas are given for massive bosons and fermions and for photons. The application to dispersion theory is discussed.

I. INTRODUCTION

CONSIDERING the growing importance of high-spin particles in elementary particle physics, we wish to present a systematic analysis of high-spin propagators and vertex functions with a view towards application to dispersion theory.[1] We shall stress "covariance" instead of "helicity," and the notion of "on-shell" instead of "off-shell."

We begin by unifying previous treatments of high-spin wave functions (Sec. II) and show that they give rise to covariant on-shell propagators which are simply related to rest frame rotation group tensors. Then we use the $O(3)$ tensor analysis of Zemach,[2–4] in covariant form to obtain general formulas for high-spin propagators (Sec. III). This analysis differs from the general projection operator approach of Fronsdal[5] in that we start by contracting all covariant spin labels with momenta, and then remove the momenta one at a time. In addition, Fronsdal's projection operator is off shell, and we shall find in Sec. III that staying on shell and we shall find in Sec. III that staying on shell leads to great simplification for fermion propagators.

In Sec. IV, we discuss the coupling of high-spin to lower-spin particles and systematically list general coupling formulas. In Sec. V we attempt to go off shell to analyze photon couplings. Finally, in Sec. VI we point out some of the areas to which our results can be applied. Since our formalism is written in terms of the usual s-channel variables, we devote an appendix to the transformation of the formalism to the crossed t channel.

II. HIGH-SPIN WAVE FUNCTIONS

Consider a free particle of spin s, and let J be the largest integer in s. First, let us study a boson (with $s=J$) in its rest frame. Its wave function is then a rotation group tensor of rank J; we form such a tensor out of J spin-1 polarization vector wave functions $\epsilon_i^{(\lambda)}$, where λ is the spin projection along some axis $\lambda=1, 0, -1$ and index $i=1, 2, 3$. The index traceless and symmetric tensor spin-J wave function, $\epsilon_{i_1\cdots i_J}^{(\Lambda)}$, with spin state Λ, can then be obtained by a series of Clebsch-Gordan couplings as[4]

$$\epsilon_{i_1\cdots i_J}^{(\Lambda)}=\sum_{\lambda_1\cdots\lambda_J}\langle\lambda_1\cdots\lambda_J|J\Lambda\rangle\epsilon_{i_1}^{(\lambda_1)}\cdots\epsilon_{i_J}^{(\lambda_J)},\quad(1)$$

where $\langle\lambda_1\cdots\lambda_J|J\Lambda\rangle$ is the "parallel coupling coefficient" defined by[4,6]

$$\langle\lambda_1\cdots\lambda_J|J\Lambda\rangle=\left[2^{J-\Sigma|\lambda_i|}\frac{(J+\Lambda)!(J-\Lambda)!}{(2J)!}\right]^{1/2}\delta_{\Sigma\lambda_i,\Lambda}.\quad(2)$$

The traceless condition, $\delta_{i_1i_2}\epsilon_{i_1i_2\cdots i_J}^{(\Lambda)}=0$ easily follows from Eq. (2).

We boost this spin-J particle into a general frame with momentum p and mass m by using covariant polarization vectors $\epsilon_\mu^{(\lambda)}(p)$ (with index $\mu=0, 1, 2, 3$, and λ can be taken as helicity) along with the subsidiary conditions $p^\mu\epsilon_\mu^{(\lambda)}(p)=0$ and obtain the covariant spin-J wave function

$$\epsilon_{\mu_1\cdots\mu_J}^{(\Lambda)}(p)=\sum_{\lambda_1\cdots\lambda_J}\langle\lambda_1\cdots\lambda_J|J\Lambda\rangle\epsilon_{\mu_1}^{(\lambda_1)}(p)\cdots\epsilon_{\mu_J}^{(\lambda_J)}(p)\quad(3)$$

which clearly satisfies the subsidiary conditions

$$p^{\mu_1}\epsilon_{\mu_1\cdots\mu_J}^{(\lambda)}(p)=0.$$

The traceless conditions

$$g_{\mu_1\mu_2}\epsilon_{\mu_1\mu_2\cdots\mu_J}^{(\Lambda)}(p)=0$$

follow from the boost prescription

$$\delta_{ij}\to-(g_{\mu\nu}-p_\mu p_\nu/m^2),$$

where $p^2=m^2$.

Next we consider a high-spin fermion with $s=J+\frac{1}{2}$ and use the Rarita-Schwinger-Kusaka[7–9] prescription for coupling the above integer spin-J wave function to a spin-$\frac{1}{2}$ Dirac bispinor, $u^{(\sigma)}(p)$:

$$u_{\mu_1\cdots\mu_J}^{(\Lambda)}(p)=\sum_{\lambda,\sigma}\langle J,\tfrac{1}{2}\lambda,\sigma|J+\tfrac{1}{2},\Lambda\rangle\epsilon_{\mu_1\cdots\mu_J}^{(\lambda)}(p)u^{(\sigma)}(p).\,(4)$$

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It is easy to show that $u_{\mu_1\cdots\mu_J}{}^{(\Lambda)}(p)$ can also be expressed as[10]

$$u_{\mu_1\cdots\mu_J}{}^{(\Lambda)}(p)=\sum_{\lambda_1\cdots\lambda_J}\langle\lambda_1\cdots\lambda_J\sigma|J+\tfrac{1}{2},\Lambda\rangle \times\epsilon_{\mu_1}{}^{(\lambda_1)}(p)\cdots\epsilon_{\mu_J}{}^{(\lambda_J)}(p)u^{(\sigma)}(p)\,, \quad (5)$$

where $\langle\lambda_1\cdots\lambda_J\sigma|J+\frac{1}{2},\Lambda\rangle$ is the parallel coupling coefficient defined by Eq. (2) with $J\rightarrow J+\frac{1}{2}$. This satisfies the usual subsidiary conditions[11] $(p-m)u_{\mu_1\cdots\mu_J}(p)=\gamma_{\mu_1}u_{\mu_1\cdots\mu_J}(p)=p_{\mu_1}u_{\mu_1\cdots\mu_J}(p)=0$.

The normalization and orthogonality properties of $\epsilon_{\mu_1\cdots\mu_J}(p)$ and $u_{\mu_1\cdots\mu_J}(p)$ are embedded in the orthogonality properties of the parallel coupling coefficients.[4,6] In particular, the traceless symmetric projection operator on the $O(3)$ helicity labels is

$$\mathcal{P}_{\lambda_1'\cdots\lambda_J';\,\lambda_1\cdots\lambda_J}=\sum_{\Lambda}\langle\lambda_1'\cdots\lambda_J'|J\Lambda\rangle \times\langle\lambda_1\cdots\lambda_J|J\Lambda\rangle. \quad (6)$$

III. CONTRACTED PROPAGATORS—ON-SHELL

Now we proceed to calculate the numerator of a high-spin propagator defined as the spin sum

$$\mathcal{P}_{\mu_1\cdots\mu_J;\nu_1\cdots\nu_J}{}^{s}(K)\equiv\sum_{\Lambda}\psi_{\mu_1\cdots\mu_J}{}^{(\Lambda)}(K)\bar{\psi}_{\nu_1\cdots\nu_J}{}^{(\Lambda)}(K)\,, \quad (7)$$

where $\bar{\psi}_{\mu_1\cdots\mu_J}$ is either a boson $\epsilon_{\mu_1\cdots\mu_J}$ or fermion $\mu_{\mu_1\cdots\mu_J}$ wave function, $\bar{\psi}$ is ϵ^* or u, and K is the momentum of the propagated particle of mass M. We contract $\mathcal{P}_{\mu_1\cdots\mu_J;\nu_1\cdots\nu_J}{}^{s}(K)$ with initial momenta p_ν and final momenta p_μ', which produces the contracted propagator $\mathcal{P}^s(p',p;K)$, with $s=J$ or $J+\frac{1}{2}$,

$$\mathcal{P}^s(p',p;K)\equiv p'^{\mu_1}\cdots p'^{\mu_J}\mathcal{P}_{\mu_1\cdots\mu_J;\nu_1\cdots\nu_J}{}^{s}(K)p^{\nu_1}\cdots p^{\nu_J}. \quad (8)$$

If we take a virtual boson of spin J to its rest frame, $\epsilon^{(\lambda)}(M)\cdot p=p_\lambda$ $(\lambda=1,0,-1)$. Then from Eqs. (3) and (6),

$$\mathcal{P}^J(p',p;K)=\sum_{\lambda',\lambda}p_{\lambda_1'}'\cdots p_{\lambda_J'}'\mathcal{P}_{\lambda_1'\cdots\lambda_J';\lambda_1\cdots\lambda_J}p_{\lambda_1}\cdots p_{\lambda_J} =T_J(\mathbf{p}'):T_J(\mathbf{p})\,, \quad (9)$$

the contraction of two $O(3)$ tensors of rank J.[2,4] This is easily worked out by aligning $\mathbf{p}$ along $\hat{e}_3$ so $\Lambda=0$ in Eq. (2), and one can show that

$$\mathcal{P}^J(p',p;M)=c_J\mathcal{P}_J(\mathbf{p}'\cdot\mathbf{p}) \quad (10)$$

where $\mathcal{P}_J(\mathbf{p}'\cdot\mathbf{p})$ is the "solid" Legendre polynomial

$$\mathcal{P}_J(\mathbf{p}'\cdot\mathbf{p})=|\mathbf{p}'|^J|\mathbf{p}|^JP_J(\hat{p}'\cdot\hat{p}) \quad (11)$$

and

$$c_J=\frac{2^JJ!J!}{(2J)!}. \quad (12)$$

To boost this result up to momentum K, providing $K^2=M^2$ (on-shell), we use the prescriptions $\delta_{ij}\rightarrow -(g_{\mu\nu}-K_\mu K_\nu)\equiv -g_{\mu\nu}(K)$ and $p_i\rightarrow -p\mu(K)$, where

$$p_\mu(K)\equiv p_\mu-(p\cdot K/M^2)K_\mu\,, \quad (13)$$

so that

$$\mathbf{p}'\cdot\mathbf{p}\rightarrow -p'(K)\cdot p(K) =-p'\cdot p(K)=-[p'\cdot p-(p'\cdot Kp\cdot K/M^2)]. \quad (14)$$

Thus, the covariant on-shell result for the contracted propagator (numerator) is[12]

$$\mathcal{P}^J(p',p;K)=c_J\mathcal{P}_J\,, \quad (15)$$

where we delete the dependence of the solid harmonic $\mathcal{P}_J$ on its invariant argument $-p'(K)\cdot p(K)$ (see Appendix). This completely specifies the spin-J boson propagator when it is coupled to spin-0 particles.

However, when particles with spin other than zero are at either end of the propagator, covariant labels μ or ν must be "freed" from $\mathcal{P}^J(p',p;K)$. This can be accomplished by realizing that the solid harmonic $\mathcal{P}_J$ is a homogeneous polynomial of degree J in either p' or p. Then we use a covariant version of Zemach's $O(3)$ differential technique[2]; namely, we let initial momenta in Eq. (15) become

$$p_\nu\rightarrow p_\nu+\epsilon g_{\nu\alpha}\,, \quad (16)$$

where ϵ is a small number, so that

$$p_\nu(K)\rightarrow p_\nu(K)+\epsilon g_{\nu\alpha}(K)$$
$$\mathcal{P}^J(p',p;K)\rightarrow\mathcal{P}^J(p',p;K)+\epsilon J\mathcal{P}_{;\alpha}{}^J(p',p;K) \quad (17)$$
$$\mathcal{P}_J\rightarrow\mathcal{P}_J-\epsilon[p_\alpha'(K)\mathcal{P}_J'+p'^2(K)p_\alpha(K)\mathcal{P}_{J-1}']\,,$$

and equate the coefficients of ϵ, where

$$\mathcal{P}_{;\alpha}{}^J(p,p;K) \equiv p'^{\mu_1}\cdots p'^{\mu_J}\mathcal{P}_{\mu_1\cdots\mu_J;\alpha\nu_2\cdots\nu_J}(K)p^{\nu_2}\cdots p^{\nu_J} \quad (18)$$

and $\mathcal{P}_J'$ is a derivative solid harmonic of degree $J-1$, becoming

$$\mathcal{P}_J'\rightarrow|\mathbf{p}'|^{J-1}|\mathbf{p}|^{J-1}P_J'(\hat{p}'\cdot\hat{p}) \quad (19)$$

in the K rest frame and satisfying the covariant recursion relation,

$$p'^2(K)p^2(K)\mathcal{P}_{J-1}'=\mathcal{P}_{J+1}'-(2J+1)\mathcal{P}_J. \quad (20)$$

This same technique can be used to free final labels β by letting $p_\mu'\rightarrow p_\mu+\epsilon g_{\mu\beta}$ with formulas similar to Eqs. (17), where $p'\leftrightarrow p$, $\nu\rightarrow\mu$, $\alpha\rightarrow\beta$, and

$$\mathcal{P}_{\beta;}{}^J(p',p;K)\equiv p'^{\mu_2}\cdots p'^{\mu_J}\mathcal{P}_{\beta\mu_2\cdots\mu_J;\nu_1\cdots\nu_J}p^{\nu_1}\cdots p^{\nu_J}. \quad (21)$$

Successive initial (or final) labels can be freed by again using the above technique along with

$$p^2(K)\rightarrow p^2(K)+2\epsilon p_\alpha(K)$$

and

$$\mathcal{P}_J'\rightarrow\mathcal{P}_J'-\epsilon[p_\alpha'(K)\mathcal{P}_J''+p'^2(K)p_\alpha(K)\mathcal{P}_{J-1}'']\,,$$

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[10] D. Brunoy, Phys. Rev. **145**, 1229 (1966).

[11] We use the γ in *Proceedings of the 1965, Trieste Seminar on Elementary Particles and High Energy Physics* (International Atomic Energy Agency, Vienna, 1965). Our metric is $g_{00}=-g_{ii}=1$.

[12] This result has been shown to hold off shell as well; c.f. S. Frautschi, M. Gell-Mann, and F. Zachariasen, Phys. Rev. **126**, 2204 (1962).

etc. The resulting boson propagator (numerator) formulas are:

$$
\begin{aligned}
\mathcal{P}^J(p',p;K)&=c_J\Delta(J)\,,\\
\mathcal{P}_{;\alpha}{}^J(p',p;K)&=\frac{c_J}{J}\Delta_{;\alpha}(J)\,,\\
\mathcal{P}_{\beta;}{}^J(p',p;K)&=\frac{c_J}{J}\Delta_{\beta;}(J)\,,\\
\mathcal{P}_{\beta;\alpha}{}^J(p',p;K)&=\frac{c_J}{J^2}\Delta_{\beta;\alpha}(J)\,,\\
\mathcal{P}_{;\alpha_1\alpha_2}{}^J(p',p;K)&=\frac{c_J}{J(J-1)}\Delta_{;\alpha_1\alpha_2}(J)\,,\\
\mathcal{P}_{\beta;\alpha_1\alpha_2}{}^J(p',p;K)&=\frac{c_J}{J^2(J-1)}\Delta_{\beta;\alpha_1\alpha_2}(J)\,,
\end{aligned}
\tag{22}
$$

where

$$
\begin{aligned}
\Delta(J)&\equiv\mathcal{P}_J\,,\\
\Delta_{;\alpha}(J)&\equiv-[p_\alpha'(K)\mathcal{P}_J'+p'^2(K)p_\alpha(K)\mathcal{P}_{J-1}']\,,\\
\Delta_{\beta;}(J)&\equiv-[p_\beta(K)\mathcal{P}_J'+p^2(K)p_\beta'(K)\mathcal{P}_{J-1}']\,,\\
\Delta_{\beta;\alpha}(J)&\equiv[p_\beta'(K)p_\alpha(K)+p_\beta(K)p_\alpha'(K)]\mathcal{P}_J''\\
&\quad+[p^2(K)p_\beta'(K)p_\alpha'(K)\\
&\quad+p'^2(K)p_\beta(K)p_\alpha(K)]\mathcal{P}_{J-1}''-g_{\beta\alpha}(K)\mathcal{P}_J'\\
&\qquad-(2J+1)p_\beta'(K)p_\alpha(K)\mathcal{P}_{J-1}'\,,\\
\Delta_{;\alpha_1\alpha_2}(J)&\equiv p_{\alpha_1}'(K)p_{\alpha_2}'(K)\mathcal{P}_J''+p'^2(K)[p_{\alpha_1}'(K)p_{\alpha_2}(K)\\
&\quad+p_{\alpha_1}(K)p_{\alpha_2}'(K)]\mathcal{P}_{J-1}''+p'^4(K)p_{\alpha_1}(K)\\
&\qquad\times p_{\alpha_2}(K)\mathcal{P}_{J-2}''-p'^2(K)g_{\alpha_1\alpha_2}(K)\mathcal{P}_{J-1}'\,,\\
-\Delta_{\beta;\alpha_1\alpha_2}(J)&\equiv\{p'p'p\}_{\beta\alpha_1\alpha_2}\mathcal{P}_J'''+\{p'p'p';p^2\}_{\beta\alpha_1\alpha_2}\mathcal{P}_{J-1}'''\\
&\quad+p'^4(K)p_\beta(K)p_{\alpha_1}(K)p_{\alpha_2}(K)\mathcal{P}_{J-2}'''\\
&\quad-\{gp'\}_{\beta\alpha_1\alpha_2}\mathcal{P}_J''-[(2J+1)p_\beta'(K)\{p'p\}_{\alpha_1\alpha_2}\\
&\quad+p'^2(K)\{gp\}_{\beta\alpha_1\alpha_2}]\mathcal{P}_{J-1}''-(2J+1)p'^2(K)\\
&\quad\times p_\beta'(K)p_{\alpha_1}(K)p_{\alpha_2}(K)\mathcal{P}_{J-2}''\\
&\qquad+(2J+1)p_\beta'(K)g_{\alpha_1\alpha_2}(K)\mathcal{P}_{J-1}'\,,
\end{aligned}
\tag{23}
$$

and so on, with $\Delta_{;\alpha_1\alpha_2}(J)\to\Delta_{\beta_1\beta_2;}(J)$ and $\Delta_{\beta;\alpha_1\alpha_2}(J)\to\Delta_{\beta_1\beta_2;\alpha}(J)$ when $\alpha\leftrightarrow\beta$ and $p\leftrightarrow p'$, where $\{\quad\}$ indicates symmetric combinations such as

$$
\begin{aligned}
\{p'p'p\}_{\beta\alpha_1\alpha_2}&=p_\beta'(K)p_{\alpha_1}'(K)p_{\alpha_2}(K)\\
&\quad+p_\beta'(K)p_{\alpha_1}(K)p_{\alpha_2}'(K)\\
&\qquad+p_\beta(K)p_{\alpha_1}'(K)p_{\alpha_2}'(K)\,,\\
\{p'p'p';p^2\}_{\beta\alpha_1\alpha_2}&=p^2(K)p_\beta'(K)p_{\alpha_1}'(K)p_{\alpha_2}'(K)\\
&\quad+p'^2(K)[p_\beta(K)p_{\alpha_1}'(K)p_{\alpha_2}(K)\\
&\quad+p_\beta(K)p_{\alpha_1}(K)p_{\alpha_2}'(K)\\
&\qquad+p_\beta'(K)p_{\alpha_1}(K)p_{\alpha_2}(K)]\,.
\end{aligned}
$$

For the interaction $s_1+s_2\to J\to s_1'+s_2'$, one needs to consider the above boson propagator formulas with at most s_1+s_2 free α labels and $s_1'+s_2'$ free β labels.

Next we investigate spin-$J+\frac{1}{2}$ fermion propagators. In the rest frame one can repeat the arguments leading to Eq. (9) for $s=J+\frac{1}{2}$ and obtain

$$\mathcal{P}^{J+\frac{1}{2}}(p',p;M)=T_{J+\frac{1}{2}}{}^*(\mathbf{p}'){:}T_{J+\frac{1}{2}}(\mathbf{p})\,. \tag{24}$$

Using the spinor projection operator[2]

$$T_{J+\frac{1}{2}}(\mathbf{p})=(J+1+\boldsymbol{\sigma}\cdot\mathbf{S})/(2J+1)T_J(\mathbf{p})\,,$$

where $\mathbf{S}=-i\mathbf{p}\times\boldsymbol{\nabla}_p$, we can write

$$\mathcal{P}^{J+1/2}(p',p;M)=\frac{c_J}{2J+1}[(J+1)\mathcal{P}_J-i\boldsymbol{\sigma}\cdot\mathbf{p}'\times\mathbf{p}\mathcal{P}_J'] \tag{25}$$

$$=\frac{c_{J+1}}{J+1}[\mathcal{P}_{J+1}'-\boldsymbol{\sigma}\cdot\mathbf{p}'\boldsymbol{\sigma}\cdot\mathbf{p}\mathcal{P}_J']\,. \tag{26}$$

Equation (26) is in a form which can easily be boosted into the covariant Dirac formalism. Given the normalization $\bar{u}u=2M$, the rest frame spin matrices 1 and $\sigma_i\sigma_j$ can be replaced by

$$1\to\boldsymbol{K}+M\,, \tag{27}$$

$$\boldsymbol{\sigma}\cdot\mathbf{p}'\boldsymbol{\sigma}\cdot\mathbf{p}\to\boldsymbol{p}'(K)(\boldsymbol{K}-M)\boldsymbol{p}(K) \tag{28}$$

(with $\boldsymbol{K}\equiv\gamma^\mu K_\mu$), where the form $\boldsymbol{p}(K)=\boldsymbol{p}-(p\cdot K/M^2)\boldsymbol{K}$ can be written as

$$\boldsymbol{p}(K)=\boldsymbol{p}+p\cdot K/M \tag{29}$$

because it will always be next to the operator $\boldsymbol{K}-M$, and $\boldsymbol{K}(\boldsymbol{K}-M)=-M(\boldsymbol{K}-M)$ on the mass shell. It should be noted here that Eq. (26) is in such a form that the usual γ-algebra manipulations are reduced to a minimum. That is to say, if the external lines p and p' are on-shell fermions with masses m and m', then

$$\boldsymbol{p}'(K)(\boldsymbol{K}-M)\boldsymbol{p}(K)=\left(m'+\frac{p'\cdot K}{M}\right)\left(m+\frac{p\cdot K}{M}\right)(\boldsymbol{K}-M)\,. \tag{30}$$

This means, for example, that the usual spin-$\frac{3}{2}$ propagator numerator

$$\mathcal{P}_{\beta;\alpha}{}^{3/2}(K)=-\left[g_{\beta\alpha}-\tfrac{1}{3}\gamma_\beta\gamma_\alpha-\frac{1}{3M}(K_\beta\gamma_\alpha-K_\alpha\gamma_\beta)-\frac{2}{3M^2}K_\alpha K_\beta\right](\boldsymbol{K}+M)$$

is best written as[13]

$$\mathcal{P}_{\beta;\alpha}{}^{3/2}=-[g_{\beta\alpha}(K)(\boldsymbol{K}+M)+\tfrac{1}{3}\gamma_\beta(K)(\boldsymbol{K}-M)\gamma_\alpha(K)]\,. \tag{31}$$

Using the differential technique again with the added condition $\boldsymbol{p}(K)\to\boldsymbol{p}(K)+\epsilon\gamma_\alpha(K)$, where

$$\gamma_\alpha(K)=\gamma_\alpha+K_\alpha/M\,, \tag{32}$$

[13] Eq. (43) is also valid off shell with $\gamma_\alpha(K)=\gamma_\alpha-K_\alpha K^{-2}\mathbf{K}$.

we can free initial and final momenta from the spin-$J+\frac{1}{2}$ fermion propagator. The resulting formulas are:

$$
\begin{aligned}
\mathcal{P}^{J+\frac{1}{2}}(p',p;K)&=\frac{c_{J+1}}{J+1}\{\Delta'(J+1)(\mathbf{K}+M)-\Delta'(J)\boldsymbol{p}'(K)(\mathbf{K}-M)\boldsymbol{p}(K)\}\,,\\
\mathcal{P}_{;\alpha}{}^{J+\frac{1}{2}}(p',p;K)&=\frac{c_{J+1}}{(J+1)J}\{\Delta_{;\alpha}'(J+1)(\mathbf{K}+M)-\Delta_{;\alpha}'(J)\boldsymbol{p}'(K)(\mathbf{K}-M)\boldsymbol{p}(K)-\Delta'(J)\boldsymbol{p}'(K)(\mathbf{K}-M)\gamma_\alpha(K)\}\,,\\
\mathcal{P}_{\beta;}{}^{J+\frac{1}{2}}(p',p;K)&=\frac{c_{J+1}}{(J+1)J}\{\Delta_{\beta;}'(J+1)(\mathbf{K}+M)-\Delta_{\beta;}'(J)\boldsymbol{p}(K)(\mathbf{K}-M)\boldsymbol{p}(K)-\Delta'(J)\gamma_\beta(K)(\mathbf{K}-M)\boldsymbol{p}(K)\}\,,\\
\mathcal{P}_{\beta;\alpha}{}^{J+\frac{1}{2}}(p',p;K)&=\frac{c_{J+1}}{(J+1)J^2}\{\Delta_{\beta;\alpha}'(J+1)(\mathbf{K}+M)-\Delta_{\beta;\alpha}'(J)\boldsymbol{p}'(K)(\mathbf{K}-M)\boldsymbol{p}(K)-\Delta_{\beta;}'(J)\boldsymbol{p}'(K)(\mathbf{K}-M)\gamma_\alpha(K)\\
&\qquad-\Delta_{;\alpha}'(J)\gamma_\beta(K)(\mathbf{K}-M)\boldsymbol{p}(K)-\Delta'(J)\gamma_\beta(K)(\mathbf{K}-M)\gamma_\alpha(K)\}\,,\\
\mathcal{P}_{;\alpha_1\alpha_2}{}^{J+\frac{1}{2}}(p',p;K)&=\frac{c_{J+1}}{(J+1)J(J-1)}\{\Delta_{;\alpha_1\alpha_2}'(J+1)(\mathbf{K}+M)-\Delta_{;\alpha_1\alpha_2}'(J)\boldsymbol{p}'(K)(\mathbf{K}-M)\boldsymbol{p}(K)\\
&\qquad-\Delta_{;\alpha_1}'(J)\boldsymbol{p}'(K)(\mathbf{K}-M)\gamma_{\alpha_2}(K)-\Delta_{;\alpha_2}'(J)\boldsymbol{p}'(K)(\mathbf{K}-M)\gamma_{\alpha_1}(K)\}\,,\\
\mathcal{P}_{\beta;\alpha_1\alpha_2}{}^{J+\frac{1}{2}}(p',p;K)&=\frac{c_{J+1}}{(J+1)J^2(J-1)}\{\Delta_{\beta;\alpha_1\alpha_2}'(J+1)(\mathbf{K}+M)-\Delta_{\beta;\alpha_1\alpha_2}'(J)\boldsymbol{p}'(K)(\mathbf{K}-M)\boldsymbol{p}(K)\\
&\qquad-[\Delta_{\beta;\alpha_1}'(J)\boldsymbol{p}'(K)+\Delta_{;\alpha_1}'(J)\gamma_\beta(K)](\mathbf{K}-M)\gamma_{\alpha_2}(K)-[\Delta_{\beta;\alpha_2}'(J)\boldsymbol{p}'(K)+\Delta_{;\alpha_2}'(J)\gamma_\beta(K)]\\
&\qquad\times(\mathbf{K}-M)\gamma_{\alpha_1}(K)-\Delta_{;\alpha_1\alpha_2}'(J)\gamma_\beta(K)(\mathbf{K}-M)\boldsymbol{p}(K)\}\,,
\end{aligned}
\tag{33}
$$

where $\Delta'(J)$, $\Delta'_{;\beta}(J)$, etc. are just the $\Delta_{\ldots}(J)$ of Eqs. (23), with the solid harmonic derivatives taken to one higher order, as $\Delta'(J)=\mathcal{P}_J'$, or $\Delta_{;\alpha}'(J)=-[p_\alpha'(K)\mathcal{P}_J''+p'^2(K)p_\alpha(K)\mathcal{P}_{J-1}'']$.

Given the interaction $s_1+s_2\rightarrow(J+\frac{1}{2})\rightarrow s_1'+s_2'$, one need consider the above fermion propagator formulas with at most J_1+J_2 free α labels and $J_1'+J_2'$ free β labels.

It will prove convenient to evaluate these contracted propagators in the forward direction with $p'=p$. One can either use the value of the nth derivative of the Legendre polynomial

$$
P_J{}^{(n)}(1)=\frac{(J+n)!}{(J-n)!}\frac{1}{2^n n!}\,,
\tag{34}
$$

or use new differential techniques when $p=p'$. Defining $\xi=-p^2(K)\rightarrow|\boldsymbol{p}|^2$ in the propagator rest frame, the resulting forward direction formulas for bosons are:

$$
\begin{aligned}
\mathcal{P}^J(p,p;K)&=c_J\xi^J\,,\\
\mathcal{P}_{;\alpha}{}^J(p,p;K)&=-c_J\xi^{J-1}p_\alpha(K)\,,\\
\mathcal{P}_{\beta;}{}^J(p,p;K)&=-c_J\xi^{J-1}p_\beta(K)\,,\\
\mathcal{P}_{\beta;\alpha}{}^J(p,p;K)&=\frac{c_J}{2J}\xi^{J-2}\{(J-1)p_\beta(K)p_\alpha(K)+(J+1)g_{\beta\alpha}(K)p^2(K)\}\,,\\
\mathcal{P}_{;\alpha_1\alpha_2}{}^J(p,p;K)&=\frac{3c_J}{2}\xi^{J-2}\{p_{\alpha_1}(K)p_{\alpha_2}(K)-\tfrac{1}{3}p^2(K)g_{\alpha_1\alpha_2}(K)\}\,,\\
\mathcal{P}_{;\alpha_1\cdots\alpha_n}{}^J(p,p;K)&=(-)^n\frac{c_J}{c_n}\xi^{J-n}T_{\alpha_1\cdots\alpha_n}{}^{(n)}(p(K))\,,\\
\mathcal{P}_{\beta;\alpha_1\cdots\alpha_n}{}^J(p,p;K)&=(-)^{n+1}\frac{c_J}{Jc_n(n+1)}\xi^{J-n-1}\{(2j+1)(n+1)p_\beta(K)T_{\alpha_1\cdots\alpha_n}{}^{(n)}(p(K))\\
&\qquad-(j+1)(2n+1)T_{\beta\alpha_1\cdots\alpha_n}{}^{(n+1)}(p(K))\}\,,
\end{aligned}
\tag{35}
$$

and for fermions:

$$\mathcal{P}^{J+\frac{1}{2}}(p,p;K)=c_{J+1}\xi^{J}(K+M),$$

$$\mathcal{P}_{;\alpha}{}^{J+\frac{1}{2}}(p,p;K)=-\tfrac{1}{2}c_{J+1}\xi^{J-1}\{3p_\alpha(K)(K+M)+p(K)(K-M)\gamma_\alpha(K)\},$$

$$\mathcal{P}_{\beta;}{}^{J+\frac{1}{2}}(p,p;K)=-\tfrac{1}{2}c_{J+1}\xi^{J-1}\{3p_\beta(K)(K+M)+\gamma_\beta(K)(K-M)p(K)\},$$

$$\mathcal{P}_{\beta;\alpha}{}^{J+\frac{1}{2}}(p,p;K)=\frac{c_{J+1}}{2J}\xi^{J-2}\{[3(J-1)p_\beta(K)p_\alpha(K)+(J+2)p^2(K)g_{\beta\alpha}(K)](K+M)$$
$$+(J-1)[p_\beta(K)p(K)(K-M)\gamma_\alpha(K)+p_\alpha(K)\gamma_\beta(K)(K-M)p(K)]+p^2(K)\gamma_\beta(K)(K-M)\gamma_\alpha(K)\},$$

$$\mathcal{P}_{;\alpha_1\alpha_2}{}^{J+\frac{1}{2}}(p,p;K)=\tfrac{1}{2}c_{J+1}\xi^{J-2}\{[5p_{\alpha_1}(K)p_{\alpha_2}(K)-g_{\alpha_1\alpha_2}(K)p^2(K)](K+M)+p(K)(K-M)$$
$$\times[p_{\alpha_1}(K)\gamma_{\alpha_2}(K)+p_{\alpha_2}(K)\gamma_{\alpha_1}(K)]\},$$

$$\mathcal{P}_{;\alpha_1\cdots\alpha_n}{}^{J+\frac{1}{2}}(p,p;K)=\frac{(-)^n c_{J+1}}{c_n(n+1)}\xi^{J-n}\{(2n+1)T_{\alpha_1\cdots\alpha_n}{}^{(n)}(p(K))(K+M)$$
$$+np(K)(K-M)T_{\alpha_1\cdots\alpha_n}{}^{(n)}(p(K)\cdots p(K)\gamma(K))\}. \quad (36)$$

IV. COVARIANT VERTEX FUNCTIONS—ON-SHELL

Before applying the propagator formulas of Sec. III to any specific problem, one must know how a high-spin J or $J+\frac{1}{2}$ particle couples to other particles. We define the "normality" n of a particle of spin J or $J+\frac{1}{2}$ as $n=(-)^J\times$(intrinsic parity),[14] so a "normal" ($n=1$) particle has parity $(-)^J$ ($J^P=0^+$, 1^-, $2^+\cdots$ and $(J+\frac{1}{2})^P=\frac{1}{2}^+$, $\frac{3}{2}^-$, $\frac{5}{2}^+\cdots$), and an "abnormal" ($n=-1$) particle has parity $-(-)^J$. Then we define the normality of a three-point Yukawa vertex as the product of the normalities of each particle, $n_v=n_1n_2n_3$. Parity conservation then divides vertex functions into the normal class with $n_v=1$ and the abnormal class with $n_v=-1$.

If the coupling is of the form $0+s\to s'$ in spin space, the number of independent on-shell couplings as counted from the rest frame of one of the particles is clearly $2s_m+1$, where $s_m=\min(s,s')$. If s and s' are fermions then $\frac{1}{2}(2s_m+1)$ couplings are normal and $\frac{1}{2}(2s_m+1)$ couplings are abnormal,[15] whereas if s and s' are bosons, then $\frac{1}{2}[(2s_m+1)+1]$ couplings are normal and $\frac{1}{2}[(2s_m+1)-1]$ are abnormal.

We can generalize this counting procedure to the general coupling $s_1+s_2\to s_3$. Take s_1 and s_2 to be the lowest of the three spins and combine them into $\mathbf{s}=\mathbf{s}_1+\mathbf{s}_2$ so that the problem is reduced to couplings $0+s\to s_3$ for $|s_1-s_2|\leq s\leq s_1+s_2$. Hence the total number of reduced couplings is $\sum_{s_m}(2s_m+1)$, where $s_m=\min(s,s_3)$. It is then easy to show that if $s_1+s_2\leq s_3$, there are

$$N^{\pm}=\tfrac{1}{2}(2s_1+1)(2s_2+1) \quad (37)$$

independent normal (N^+) or abnormal (N^-) couplings for FFB interactions ($F\equiv$fermion; $B\equiv$boson), and

$$N^{\pm}=\tfrac{1}{2}[(2s_1+1)(2s_2+1)\pm1] \quad (38)$$

independent normal (N^+) or abnormal (N^-) couplings for BBB interactions.[16] However, if $s_1+s_2>s_3$ (but s_1, $s_2\leq s_3$) there are $\mathcal{J}(\mathcal{J}+1)$ "nonsense states," and therefore

$$N^{\pm}-\tfrac{1}{2}\mathcal{J}(\mathcal{J}+1) \quad (39)$$

normal or abnormal couplings for FFB or BBB interactions with $\mathcal{J}=s_1+s_2-s_3$. These counting rules will apply to covariant three-point functions providing all three particles are on shell with $p_i^2=m_i^2$. (Photon couplings are discussed in Sec. V.)

The above facts indicate that as long as we let just one spin at a vertex become arbitrarily large, the number and structure of the couplings depend only upon the other two (low) spins. In momentum space we consider $s_1(p)+s_2(q)\to s_3(K)$ with $K=p+q$ and $\Lambda=\frac{1}{2}(p-q)$ and write effective Lagrangian interactions as,

$$\mathcal{L}\sim\bar{\psi}^{\alpha_1\cdots\alpha}\mathcal{C}_{\alpha\alpha_1\cdots\alpha_{J3};\mu_1\cdots\mu_{J3};\nu_1\nu_{J2}}\times\psi^{\mu_1\cdots\mu_{J1}}(p)\psi^{\nu_1\cdots\nu_{J2}}(q)+\text{H.c.} \quad (40)$$

where $\mathcal{C}$ is what we shall call the covariant coupling function or "$\mathcal{C}$ function" which will depend upon various combinations of momenta, metric tensors, and γ matrices (kinematic covariants) and upon the independent on-shell vertex functions (coupling constants). The subsidiary conditions on the high-spin wave functions (Sec. II) imply that all the momentum covariants can be written as Λ with $\Lambda_\mu=-\frac{1}{2}q_\mu=-\frac{1}{2}K_\mu$, $\Lambda_\nu=\frac{1}{2}p_\nu=\frac{1}{2}K_\nu$, and $\Lambda_\alpha=p_\alpha=-q_\alpha$.

For BBB interactions the abnormal couplings ($\mathcal{C}^-$) also depend upon the covariants $\epsilon_{\mu\nu}(K\Lambda)\equiv\epsilon_{\mu\nu\sigma\tau}K^\sigma\Lambda^\tau$ and $\epsilon_{\alpha\mu\nu}(K)\equiv\epsilon_{\alpha\mu\nu\sigma}K^\sigma$. We list such interactions with

[14] Normality is related to "γ parity" as defined by P. Carruthers, Phys. Rev. **152**, 1345 (1966). See also Ref. 22.

[15] L. Durand, P. DeCelles, and R. B. Marr, Phys. Rev. **126**, 1882 (1962).

[16] This has been derived in a more formal manner by J. S. Lamont, J. Math. Phys. **1**, 237 (1960).

$\mathcal{C}(s_1(p)_\mu+s_2(q)_\nu\to s_3(K)_\alpha)$ written as $\mathcal{C}(s_1,s_2,s_3)$:

$$
\begin{aligned}
&\mathcal{C}^+(0,0,J)=g\Lambda_{\alpha_1}\cdots\Lambda_{\alpha_J}\\
&\mathcal{C}^-(0,0,J)=0\\
&\mathcal{C}^+(1,0,J)=\Lambda_{\alpha_2}\cdots\Lambda_{\alpha_J}\{g_1g_{\alpha_1\mu}+g_2\Lambda_{\alpha_1}\Lambda_\mu\}\\
&\mathcal{C}^-(1,0,J)=\Lambda_{\alpha_2}\cdots\Lambda_{\alpha_J}\{g\epsilon_{\alpha_1\mu}(K\Lambda)\}\\
&\mathcal{C}^+(2,0,J)=\Lambda_{\alpha_3}\cdots\Lambda_{\alpha_J}\{g_1g_{\alpha_1\mu_1}g_{\alpha_2\mu_2}+g_2g_{\alpha_1\mu_1}\Lambda_{\alpha_2}\Lambda_{\mu_2}+g_3\Lambda_{\alpha_1}\Lambda_{\alpha_2}\Lambda_{\mu_1}\Lambda_{\mu_2}\}\\
&\mathcal{C}^-(2,0,J)=\Lambda_{\alpha_3}\cdots\Lambda_{\alpha_J}\epsilon_{\alpha_1\mu_1}(K\Lambda)\{g_1g_{\alpha_2\mu_2}+g_2\Lambda_{\alpha_2}\Lambda_{\mu_2}\}\\
&\mathcal{C}^+(1,1,J)=\Lambda_{\alpha_3}\cdots\Lambda_{\alpha_J}\{g_1g_{\alpha_1\mu}g_{\alpha_2\nu}+g_2g_{\alpha_1\mu}\Lambda_{\alpha_2}\Lambda_\nu+g_3g_{\alpha_1\nu}\Lambda_{\alpha_2}\Lambda_\mu+g_4g_{\mu\nu}\Lambda_{\alpha_1}\Lambda_{\alpha_2}+g_5\Lambda_{\alpha_1}\Lambda_{\alpha_2}\Lambda_\mu\Lambda_\nu\}\\
&\mathcal{C}^-(1,1,J)=\Lambda_{\alpha_3}\cdots\Lambda_{\alpha_J}\{g_1\epsilon_{\alpha_1\mu}(K\Lambda)g_{\alpha_2\nu}+g_2\epsilon_{\mu\nu\alpha_1}(K)\Lambda_{\alpha_2}+g_3\epsilon_{\mu\nu\alpha_1}(\Lambda)\Lambda_{\alpha_2}+g_4\epsilon_{\mu\nu}(K\Lambda)\Lambda_{\alpha_1}\Lambda_{\alpha_2}\}\\
&\mathcal{C}^+(2,1,J)=\Lambda_{\alpha_4}\cdots\Lambda_{\alpha_J}\{g_1g_{\alpha_1\mu_1}g_{\alpha_2\mu_2}g_{\alpha_3\nu}+g_2g_{\alpha_1\mu_1}g_{\alpha_2\mu_2}\Lambda_{\alpha_3}\Lambda_\nu+g_3g_{\alpha_1\mu_1}g_{\alpha_2\nu}\Lambda_{\alpha_3}\Lambda_{\mu_2}+g_4g_{\alpha_1\mu_1}g_{\mu_2\nu}\Lambda_{\alpha_2}\Lambda_{\alpha_3}\\
&\qquad+g_5g_{\alpha_1\mu_1}\Lambda_{\alpha_2}\Lambda_{\alpha_3}\Lambda_{\mu_2}\Lambda_\nu+g_6g_{\alpha_1\nu}\Lambda_{\alpha_2}\Lambda_{\alpha_3}\Lambda_{\mu_1}\Lambda_{\mu_2}+g_7g_{\mu_1\nu}\Lambda_{\alpha_1}\Lambda_{\alpha_2}\Lambda_{\alpha_3}\Lambda_{\mu_2}+g_8\Lambda_{\alpha_1}\Lambda_{\alpha_2}\Lambda_{\alpha_3}\Lambda_{\mu_1}\Lambda_{\mu_2}\Lambda_\nu\}\\
&\mathcal{C}^-(2,1,J)=\Lambda_{\alpha_4}\cdots\Lambda_{\alpha_J}\{g_1\epsilon_{\alpha_1\nu}(K\Lambda)g_{\alpha_2\mu_1}g_{\alpha_3\mu_2}+g_2\epsilon_{\alpha_1\mu_1\nu}(K)g_{\alpha_2\mu_2}\Lambda_{\alpha_3}+g_3\epsilon_{\alpha_1\mu_1\nu}(\Lambda)g_{\alpha_2\mu_2}\Lambda_{\alpha_3}+g_4\epsilon_{\mu_1\nu}(K\Lambda)g_{\alpha_1\mu_2}\Lambda_{\alpha_2}\Lambda_{\alpha_3}\\
&\qquad+g_5\epsilon_{\alpha_1\mu_1\nu}(K)\Lambda_{\alpha_2}\Lambda_{\alpha_3}\Lambda_{\mu_2}+g_6\epsilon_{\alpha_1\mu_1\nu}(\Lambda)\Lambda_{\alpha_2}\Lambda_{\alpha_3}\Lambda_{\mu_2}+g_7\epsilon_{\mu_1\nu}(K\Lambda)\Lambda_{\alpha_1}\Lambda_{\alpha_2}\Lambda_{\alpha_3}\Lambda_{\mu_2}\}\,.
\end{aligned}
\tag{41}
$$

As the wave functions in Eq. (40) are symmetric in their indices, the $\mathcal{C}$ functions need not be symmetric in the μ, ν, or α labels.

Fermion $\mathcal{C}$ functions are quite similar in structure to boson $\mathcal{C}$ functions but contain an extra kinematic covariant, γ_ρ, where ρ must be a boson label. We need consider only normal coupling functions $\mathcal{C}^+$, as $\mathcal{C}^+\to\mathcal{C}^-$ when $1\to\gamma_5$ in the Dirac spin space. Note that covariant like $\gamma_\mu\gamma_\nu$, $\sigma_{\mu\nu}$, $\epsilon_{\mu\nu\alpha\beta}$, or $\gamma_5\epsilon_{\mu\nu\alpha\beta}$ are never necessary in on-shell coupling functions. For *FFB* interactions with high-spin bosons,[16a]

$$
\begin{aligned}
&\mathcal{C}^+(\tfrac12,\tfrac12,J)=\Lambda_{\alpha_2}\cdots\Lambda_{\alpha_J}\{g_1\gamma_{\alpha_1}+g_2\Lambda_{\alpha_1}\}\\
&\mathcal{C}^+(\tfrac12,\tfrac32,J)=\Lambda_{\alpha_3}\cdots\Lambda_{\alpha_J}\{g_1g_{\alpha_1\mu}\gamma_{\alpha_2}+g_2g_{\alpha_1\mu}\Lambda_{\alpha_2}+g_3\gamma_{\alpha_1}\Lambda_{\alpha_2}\Lambda_\mu+g_4\Lambda_{\alpha_1}\Lambda_{\alpha_2}\Lambda_\mu\}\\
&\mathcal{C}^+(\tfrac12,\tfrac52,J)=\Lambda_{\alpha_4}\cdots\Lambda_J\{g_1g_{\alpha_1\mu_2}g_{\alpha_2\mu_2}\gamma_{\alpha_3}+g_2g_{\alpha_1\mu_1}g_{\alpha_2\mu_2}\Lambda_{\alpha_3}+g_3g_{\alpha_1\mu_1}\gamma_{\alpha_2}\Lambda_{\alpha_3}\Lambda_{\mu_2}+g_4g_{\alpha_1\mu_1}\Lambda_{\alpha_2}\Lambda_{\alpha_3}\Lambda_{\mu_2}\\
&\qquad+g_5\gamma_{\alpha_1}\Lambda_{\alpha_2}\Lambda_{\alpha_3}\Lambda_{\mu_1}\Lambda_{\mu_2}+g_6\Lambda_{\alpha_1}\Lambda_{\alpha_2}\Lambda_{\alpha_3}\Lambda_{\mu_1}\Lambda_{\mu_2}\}\\
&\mathcal{C}^+(\tfrac32,\tfrac32,J)=\Lambda_{\alpha_4}\cdots\Lambda_{\alpha_J}\{g_{\alpha_1\mu}g_{\alpha_2\nu}\gamma_{\alpha_3}+g_2g_{\alpha_1\mu}g_{\alpha_2\nu}\Lambda_{\alpha_3}+g_3g_{\alpha_1\mu}\gamma_{\alpha_2}\Lambda_{\alpha_3}\Lambda_\nu+g_4g_{\mu\nu}\gamma_{\alpha_1}\Lambda_{\alpha_2}\Lambda_{\alpha_3}\\
&\qquad+g_5g_{\alpha_1\mu}\Lambda_{\alpha_2}\Lambda_{\alpha_3}\Lambda_\nu+g_6g_{\mu\nu}\Lambda_{\alpha_1}\Lambda_{\alpha_2}\Lambda_{\alpha_3}+g_7\gamma_{\alpha_1}\Lambda_{\alpha_2}\Lambda_{\alpha_3}\Lambda_\mu\Lambda_\nu+g_8\Lambda_{\alpha_1}\Lambda_{\alpha_2}\Lambda_{\alpha_3}\Lambda_\mu\Lambda_\nu\}\,,
\end{aligned}
\tag{42}
$$

and for *FBF* interactions with high-spin fermions,

$$
\begin{aligned}
&\mathcal{C}^+(\tfrac12,0,J+\tfrac12)=g\Lambda_{\alpha_1}\cdots\Lambda_{\alpha_J}\\
&\mathcal{C}^+(\tfrac32,0,J+\tfrac12)=\Lambda_{\alpha_2}\cdots\Lambda_{\alpha_J}\{g_1g_{\alpha_1\mu}+g_2\Lambda_{\alpha_1}\Lambda_\mu\}\\
&\mathcal{C}^+(\tfrac52,0,J+\tfrac12)=\Lambda_{\alpha_3}\cdots\Lambda_{\alpha_J}\{g_1g_{\alpha_1\mu_1}g_{\alpha_2\mu_2}+g_2g_{\alpha_1\mu_1}\Lambda_{\alpha_2}\Lambda_{\mu_2}+g_3\Lambda_{\alpha_1}\Lambda_{\alpha_2}\Lambda_{\mu_1}\Lambda_{\mu_2}\}\\
&\mathcal{C}^+(\tfrac12,1,J+\tfrac12)=\Lambda_{\alpha_2}\cdots\Lambda_{\alpha_J}\{g_1g_{\alpha_1\nu}+g_2\gamma_\nu\Lambda_{\alpha_1}+g_3\Lambda_{\alpha_1}\Lambda_\nu\}\\
&\mathcal{C}^+(\tfrac32,1,J+\tfrac12)=\Lambda_{\alpha_3}\cdots\Lambda_{\alpha_J}\{g_1g_{\alpha_1\mu}g_{\alpha_2\nu}+g_2g_{\alpha_1\mu}\gamma_\nu\Lambda_{\alpha_2}+g_3g_{\alpha_1\mu}\Lambda_{\alpha_2}\Lambda_\nu+g_4g_{\mu\nu}\Lambda_{\alpha_1}\Lambda_{\alpha_2}+g_5\gamma_\nu\Lambda_{\alpha_1}\Lambda_{\alpha_2}\Lambda_\mu+g_6\Lambda_{\alpha_1}\Lambda_{\alpha_2}\Lambda_\mu\Lambda_\nu\}\,.
\end{aligned}
\tag{43}
$$

More couplings can be written in $\mathcal{C}^-(1,1,J)$, $\mathcal{C}^-(2,1,J)$, $\mathcal{C}(\frac32,\frac32,J)$, and $\mathcal{C}(\frac32,1,J+\frac12)$ but are related to the independent sets we have chosen. This point will be considered in more detail in Sec. V. Note that, if J or $J+\frac12$ takes on values less than s_1+s_2, then some of the couplings in Eqs .(41), (42), and (43) vanish (beginning with g_1) in accordance with the counting rules Eq. (39) or Eq. (38).

Further restrictions on the couplings occur when we impose the discrete symmetries on $\mathcal{C}$. Bose or Fermi statistics relate like couplings; $\mathcal{C}^-(1,1,J)$ and $\mathcal{C}(\frac32,\frac32,J)$ should then be written in a form which manifests the symmetry.[16b] Time reversal invariance ensures the reality of the coupling constants. Charge conjugation on self-conjugate fermion states (in the crossed channel) eliminates half the abnormal fermion couplings because $\gamma_5(\gamma_\beta,P_\beta)\to\gamma_5(\gamma_\beta,-P_\beta)$, whereas $(\gamma_\beta,P_\beta)\to-(\gamma_\beta,P_\beta)$.

The coupling constants that we have chosen in Eqs. (41–43) are not dimensionless; however, one can show that

$$\dim g_1^{\pm}=m^{\mathcal{J}\pm1}\tag{44}$$

for *BBB* couplings, and

$$\dim g_1^{\pm}=m^{\mathcal{J}}\tag{45}$$

[16a] Strictly speaking, if both fermions are incoming, $\mathcal{C}(F,F,B)$ must include the charge conjugation matrix C.

[16b] According to Eq. (49), the g_1 term in $\mathcal{C}^-(1,1,J)$ should then be symmetrized in the labels μ and ν. In this case, Bose statistics implies g_1 and g_3 vanish for even J whereas g_2 and g_4 vanish for odd J.

for FFB couplings, where $\mathcal{J}\equiv s_1+s_2-s_3\leq 0$ [$g_1=0$ for $\mathcal{J}>0$ by Eq. (39)] and

$$\dim g_{\max}{}^{\pm}=m^{1-(s_1+s_2+s_3)} \quad (46)$$

for both BBB and FFB couplings, where $g_{\max}$ couplings contain the maximum number of momenta, $J_3+J_1+J_2$, and g_1 couplings contain the minimum, $s_3-(s_1+s_2)$ [except for abnormal BBB couplings with $s_3-(s_1+s_2)+2$ minimal momenta].

V. OFF-SHELL VERTEX FUNCTIONS AND PHOTONS

One possible prescription to take the propagator momentum off shell, with $K^2\neq M^2$, is to keep the propagator numerators of Sec. III on shell and alter the couplings of Sec. IV. That is to say, since $K^\alpha\mathcal{P}_\alpha(K)$ is no longer zero, we relax the subsidiary conditions on $\mathcal{C}_\alpha$ and add to it terms proportional to K_α. If the off-shell particle is a fermion, we must also add couplings proportional to $\gamma\cdot p$. The coupling function $\mathcal{C}$ then becomes a vertex function $\mathcal{V}$, where the coupling constants become complex form factors depending on $K^2=s$.

In the usual perturbative approach to vertex functions, the set of kinematic covariants reduced to simplest form yields form factors free of kinematic singularities. For low enough values of the other two spins s_1 and s_2 (the off-shell particle being $s_3=J$ or $J+\frac{1}{2}$), this process is straightforward, the number of possible kinematic covariants that one can write down just agreeing with the necessary count. However, for higher spins beginning with $\mathcal{C}(1,1,1)$ for abnormal BBB couplings and $\mathcal{C}(\frac{3}{2},1,\frac{3}{2})$ for FBF couplings, these exist more kinematic covariants than the count requires. Such covariants are related by the abnormal BBB (on-shell) "equivalence theorems"

$$\epsilon_{\alpha\mu}(K\Lambda)\Lambda_\nu-\epsilon_{\alpha\nu}(K\Lambda)\Lambda_\mu=\epsilon_{\alpha\mu\nu}(\Lambda)K\cdot\Lambda-\epsilon_{\alpha\mu\nu}(K)\Lambda^2-\epsilon_{\mu\nu}(K\Lambda)\Lambda_\alpha\,, \quad (47)$$

$$\epsilon_{\alpha\mu}(K\Lambda)\Lambda_\nu+\epsilon_{\alpha\nu}(K\Lambda)\Lambda_\mu=\tfrac{1}{2}\epsilon_{\alpha\mu\nu}(\Lambda)K^2-\tfrac{1}{2}\epsilon_{\alpha\mu\nu}(K)K\cdot\Lambda\,, \quad (48)$$

$$\epsilon_{\alpha_1\mu}(K\Lambda)g_{\alpha_2\nu}-\epsilon_{\alpha_1\nu}(K\Lambda)g_{\alpha_2\mu}=-\epsilon_{\alpha_1\mu\nu}(K)\Lambda_{\alpha_2}\,, \quad (49)$$

which follow from the identity $\epsilon_{\mu\nu\sigma\tau}g_{\alpha\beta}=\epsilon_{\alpha\nu\sigma\tau}g_{\mu\beta}+\epsilon_{\mu\alpha\sigma\tau}g_{\nu\beta}+\epsilon_{\mu\nu\alpha\tau}g_{\sigma\beta}+\epsilon_{\mu\nu\sigma\alpha}g_{\tau\beta}$, and the FBF equivalence theorem

$$2mg_{\alpha\nu}\Lambda_\mu=[g_{\alpha\mu}(K\cdot p+mM)+2\Lambda_\alpha\Lambda_\mu]\gamma_\nu+Mg_{\mu\nu}\Lambda_\alpha-2(m+M)g_{\alpha\mu}\Lambda_\nu\,, \quad (50)$$

which follows from the double epsilon form $\epsilon_{\alpha\xi}(K\gamma)\times\epsilon_{\xi\mu\nu}(p)$ taken between spinors $\bar{u}_\alpha(K)$ and $u_\mu(p)$.[16c] The normal BBB equivalence theorem first occurs in $\mathcal{C}(2,2,2)$ and can be obtained by consideration of four episilons each contracted with one momentum.

In such cases, when $s_3(K)$ goes off-shell with $K^2\to s$ and $M\to\sqrt{s}$, one must take care to eliminate, by means of these equivalence theorems, only those covariants which do not introduce any kinematic singularities in s. Clearly these are the terms on the left-hand side of Eqs. (47)-(50). Hence our choice of coupling functions in Sec. IV always leads to vertex functions free of kinematic singularities.

Consider now an off-shell photon in the cross channel (see the Appendix) with square mass $t=\Delta^2$ and $p(m)+\gamma_\beta(\Delta)\to p'(m')$ where $\Delta=p'-p$ and $P=\frac{1}{2}(p+p')$. If $m'\neq m$, the added condition of current conservation at the vertex necessitates off-shell terms in Δ_β. That is, we may regard the off-shell photon as an on-shell 1^- particle of mass $\sqrt{t}$ with

$$\mathcal{V}_\beta(t)=(g_{\beta\beta'}-\Delta_\beta\Delta_{\beta'}/t)\mathcal{C}_{\beta'}{}^{(1^-)}(t)\,. \quad (51)$$

Of course current conservation just ensures the gauge independence of the photon propagator. If the Landau gauge is used then $\mathcal{C}_\beta{}^{(1^-)}(t)$ alone is sufficient to describe photon vertex functions.

Alternatively we could have used manifestly gauge invariant "helicity covariants" to describe photon couplings,[17,18] such as $R_\beta=\Delta^2P_\beta-P\cdot\Delta\Delta_\beta$, $r_\beta=2\gamma_5\epsilon_\beta(P\Delta\gamma)$ and $S_{\beta\mu}(P\Delta)=\epsilon_{\beta\xi}(P\Delta)\epsilon_{\xi\mu}(P\Delta)$ or $\gamma_5\epsilon_{\beta\mu}(P\Delta)$. They have the advantage of leaving cross sections as a sum of squares but contain kinematic singularities in t and are not independent at threshold.[19,20]

VI. APPLICATIONS

A. Feynman Pole Amplitudes

Consider the s-channel process $s_1(p)+s_2(q)\to s_1'(p')+s_2'(q')$. Its covariant $\mathfrak{M}$ function is defined from the T matrix as

$$\langle p'\lambda_1'q'\lambda_2'|T|p\lambda_1q\lambda_2\rangle=\bar{\psi}_{\mu_1\cdots}{}^{(\lambda_1')}(p')\psi_{\nu_1'\cdots}{}^{(\lambda_2')}(q')\times\mathfrak{M}_{\mu_1'\cdots\nu_1'\cdots;\mu_1\cdots\nu_1\cdots}\psi_{\mu_1\cdots}{}^{(\lambda_1)}(p)\psi_{\nu_1\cdots}{}^{(\lambda_2)}(q)\,, \quad (52)$$

with s-channel pole contributions given by

$$\mathfrak{M}_{fi}\sim\frac{1}{s-M^2}\bar{\mathcal{C}}_{n'f}\mathcal{P}_{n'n}\mathcal{C}_{ni}\,, \quad (53)$$

where i, f, n, and n' indicate all possible covariant spin labels. Using the formulas of Secs. III and IV, $\mathfrak{M}_{fi}$ can be expressed as sums of Legendre polynomials and their derivatives.

[16c] For equal fermion masses, Eq. (50) in the crossed channel [between wave functions $\bar{u}_{\mu'}(p')u_\mu(p)\epsilon_\beta(\Delta)$]

$$m(g_{\mu'\beta}P_\mu+g_{\mu\beta}P_{\mu'})=[-P^2g_{\mu'\mu}+2P_{\mu'}P_\mu]\gamma_\beta+mg_{\mu'\mu}P_\beta.$$

Note that such equivalence theorems [including (47)-(49)] always relate covariants which transform the same way under charge conjugation.

[17] Helicity covariants could also be used to describe the massive particle couplings of Sec. IV, with $\Lambda_\mu\to R_\mu$, $g_{\mu\nu}\to S_{\mu\nu}$, and $\gamma_\mu\to r_\mu$.

[18] Helicity vertex functions are considered in great detail by I. Ketley and R. King (to be published).

[19] Recall that $G_E=G_M$ at $t=4m^2$ for nucleon form factors.

[20] For the case of electromagnetic production of high-spin nucleon isobars, see J. Bjorken and J. Walecka, Ann. Phys. (N. Y.) 38, 35 (1966).

B. Unpolarized Cross Sections

The unpolarized spin sum is of the form

$$\sum_{\text{all }\lambda} |T|^2 \sim \operatorname{Tr} \mathfrak{M}_{fi} \mathcal{P}_{ii'} \overline{\mathfrak{M}}_{f'i'} \mathcal{P}_{f'f}, \tag{54}$$

where $\mathcal{P}_{ii'}$ or $\mathcal{P}_{f'f}$ is the product of the initial or final propagator numerators (on-shell). If one of the external particles is of high spin, and if $\mathfrak{M}_{fi}$ is given by either s- or t-channel poles in the form of Eq. (53), then the spin sum reduces to "forward direction" propagator terms [Eqs. (35) or (36)]. Thus, in a simple manner, one can obtain generalized Rosenbluth formulas.[20,21]

C. Decay Rates

For the process $s(K) \to s_1(p)+s_2(q)$, the total decay width in the rest frame of the decay particle is

$$\Gamma = \frac{p}{2M^2}\frac{1}{4\pi}\sum |T|^2,$$

where

$$\sum |T|^2 = \frac{1}{2s+1} \operatorname{Tr} \mathcal{C}_{fi} \mathcal{P}_{ii'}{}^{s} \overline{\mathcal{C}}_{f'i'} \mathcal{P}_{f'f}. \tag{55}$$

The high-spin propagator terms will always be of the "forward" type and with the aid of the contraction property[5]

$$g_{\mu\nu} \mathcal{P}_{\mu\cdots;\nu\cdots}{}^{s} = -\frac{2s+1}{2s-1} \mathcal{P}_{\cdots;\cdots}{}^{s-1} \tag{56}$$

for $s=J$ or $J+\frac{1}{2}$, one can readily calculate decay rates of high-spin bosons[22] or fermions.[23]

D. Invariant Amplitude Separation

Consider the $\mathfrak{M}$ function development

$$\mathfrak{M}_{\mu\cdots,\nu\cdots} = \sum A_i(s,t) \mathcal{K}_{\mu\cdots,\nu\cdots}{}^{i}, \tag{57}$$

where the A_i are invariant amplitudes free of kinematic singularities[24,1] and $\mathcal{K}_{\mu\cdots,\nu\cdots}{}^{i}$ are kinematic covariants depending on the momenta P, Q, and Δ. Expressing s-channel poles with $K=P+Q$ or t-channel poles with $\Delta \to P$ or Q (depending upon the subsidiary conditions) enables one to isolate the kinematic covariants of Eq. (57) and hence to "pick off" the pole contributions to the discontinuities of the invariant amplitudes. For the analysis of superconvergence relations[25,26] at $t=0$, one must first isolate the invariant amplitudes for $t\neq 0$ ($p'\neq p$) according to Eq. (22) or (33), after which one may set $t=0$ and use Eq. (34).

[21] Polarization or higher moments of the density matrix can be obtained in our relativistic formalism with the aid of the covariant version of Table III of Ref. 2.

[22] R. King, thesis, University of London, Imperial College, 1964 (unpublished).

[23] J. Rushbrooke, Phys. Rev. **143**, 1345 (1966).

[24] A. Hearn, Nuovo Cimento **21**, 333 (1961).

[25] V. de Alfaro, S. Fubini, G. Rossetti, and G. Furlan, Phys. Letters **21**, 576 (1966).

[26] For examples of this technique, see H. Jones and M. Scadron, Imperial College, report 1967 and R. Rivers, Imperial College report 1967.

E. Isobar Expansion and Reggeism

Assume that the amplitude can be approximated by a sum of resonances of the form of Eq. (52). There is then a one-to-one correspondence with the exact helicity partial-wave expansion only when the sum of the initial (or final) spins is less than one. If the isobar expansion is made in the crossed t channel, a possible "covariant Regge prescription" is to let $J\to\alpha(t)$ in the resulting Legendre polynomials of the propagators in Sec. III.[27]

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APPENDIX

From the point of view of dispersion theory, our formulas for propagators and vertex functions have been given in terms of the s-channel $p+q\to p'+q'$ with $s=(p+q)^2$, $t=(p'-p)^2$, and $u=(p-q')^2$. Then the s-channel resonance poles of Fig. 1 become the t-channel force poles of Fig. 2 when $p\leftrightarrow -q'$. The s-channel

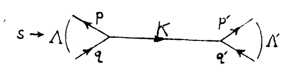

FIG. 1. Diagram for s-channel poles.

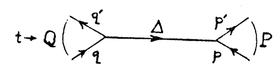

FIG. 2. Diagram for t-channel poles.

pole variables $K=p+q=p'+q'$, $\Lambda=\frac{1}{2}(p-q)$, and $\Lambda'=\frac{1}{2}(p'-q')$ become $\Delta=q-q'=p'-p$, $Q=\frac{1}{2}(q+q')$, and $P=\frac{1}{2}(p+p')$ with $K\to\Delta$, $\Lambda\to -Q$, and $\Lambda'\to P$. We have chosen p and p' in the propagator formulas to be nucleons when they exist (then $p\to m$, $p'\to m'$); nevertheless p and p' could be replaced by Λ and Λ' due to the subsidiary conditions. Hence, for $s\to t$, $p(K)\to -Q(\Delta)$, $p'(K)\to P(\Delta)$, and $p'\cdot p\to -\mathbf{P}\cdot\mathbf{Q}$ in the solid harmonics, which have the on-shell invariant forms

$$\mathbf{p}'\cdot\mathbf{p} = \mathbf{\Lambda}'\cdot\mathbf{\Lambda} \to -\Lambda'\cdot\Lambda + \frac{\Lambda'\cdot K \Lambda\cdot K}{K^2} = \frac{1}{4}\left[t-u+\frac{(m'^2-\mu'^2)(m^2-\mu^2)}{s}\right]$$

in the s channel ($p^2=m^2$, $p'^2=m'^2$, $q^2=\mu^2$, $q'^2=\mu'^2$), and

$$-\mathbf{P}\cdot\mathbf{Q} \to P\cdot Q - \frac{P\cdot\Delta Q\cdot\Delta}{\Delta^2} = \frac{1}{4}\left[s-u+\frac{(m'^2-m^2)(\mu'^2-\mu^2)}{t}\right]$$

in the t channel. If instead, one defines the t channel by $q\leftrightarrow -p'$ then $\mathbf{\Lambda}\to P$ and $\mathbf{\Lambda}'\to -Q$.

[27] L. Durand, Phys. Rev. **154**, 1537 (1967); Phys. Rev. Letters **18**, 58 (1967); J. G. Taylor, Oxford Report (unpublished); H. Jones and M. Scadron, Nucl. Phys. (to be published).