Approximation Method for Three-Body Collisions*

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A new approximation for the three-body collision problem is described, which allows practical calculation of elastic, inelastic, rearrangement, and breakup amplitudes. The approximation is derived from a novel form of the Lovelace-Faddeev equations in which all two-body bound states appear explicitly. It appears in practice as a simple modification of the impulse-pickup approximation, and is expected to considerably extend the useful energy range of that approximation. It is shown that unitarity is satisfied within a certain approximation, and that the submatrix of the S matrix that excludes the breakup channel satisfies a unitary constraint. The symmetry requirements for identical particles can be treated rigorously, and lead to a practical simplification of the method. Numerical calculations of bound-state scattering in an exactly soluble model show encouraging results.

1. INTRODUCTION

In recent years the formal understanding of the three-body collision problem has been greatly advanced through the work of Faddeev and others.[1–3] On the other hand, the complexity of the problem still prevents exact solutions, except for a restricted class of two-body interactions, those of separable type. Lovelace[4] has shown that the approximation of separable interactions is useful if the two-body scattering is dominated by bound-state or resonance poles, as is the case with low-energy nucleon-nucleon scattering. Nevertheless, it is very clear that there are many situations in which it is necessary to consider more general interactions, for which approximate methods of solution are needed.

We here develop[5] a practical method for the approximate solution of the Lovelace-Faddeev equations, with arbitrary two-body interactions. The method appears in practice as a modification of the impulse-pickup approximation (described in Sec. 3), but it is expected to considerably extend the useful energy range of that approximation. The method allows practical calculation of all physical amplitudes (except free-free amplitudes) if the two-body amplitudes are known, and it satisfies the important constraint of unitarity in a certain approximate sense (Sec. 4).

Many theoretical approaches and approximation methods run into difficulties when more than one of the three pair interactions supports a bound state. For example, the Born series fails to converge.[6] The present method, on the other hand, takes advantage of the existence of bound states, and uses the knowledge of bound-state energies and wave functions as valuable input information.

The Lovelace–Faddeev equations contain the two-body bound-state information in a concealed form. In Sec. 2 we therefore recast the exact equations into a form in which the bound states between each pair of particles appear explicitly, so that it becomes possible to approximate without losing the bound-state information. Two versions of the approximation are described in Sec. 3. In this approximation the physical amplitudes satisfy coupled integral equations, in which the inhomogeneous terms and kernels are the on-shell impulse-pickup amplitudes.

In the final section we consider bound-state scattering for a system of identical particles, and show that the symmetry requirements lead to practical simplification of the method. The method is then illustrated with numerical calculations on an exactly soluble model.

2. EXACT EQUATIONS

The particles are denoted by 1, 2, 3, their masses by m_1, m_2, m_3, and the pair interactions by V_1, V_2, V_3, where $V_1=V_{23}$ is the interaction between particles 2 and 3. Following Lovelace,[2] we also define

$$V_0=0. \tag{2.1}$$

The channel in which particle α is free and the other pair bound by V_α is called channel α, and the channel in which the three particles are free is channel 0. Unless noted otherwise, Greek letters run from 0–3.

We require the operators

$$G(s)=(s-H)^{-1}, \tag{2.2}$$

$$G_\alpha(s)=(s-H_\alpha)^{-1}, \tag{2.3}$$

$$T_\alpha(s)=V_\alpha+V_\alpha G_\alpha(s)V_\alpha, \tag{2.4}$$

where H_0 is the three-body kinetic energy in the center-of-mass system,

$$H_\alpha=H_0+V_\alpha, \tag{2.5}$$

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[1] L. D. Faddeev, Zh. Eksperim. i Teor. Fiz. **39**, 1459 (1960) [English transl.: Soviet Phys.—JETP **12**, 1014 (1961)].

[2] C. Lovelace, in *Strong Interactions and High Energy Physics*, edited by R. G. Moorehouse (Oliver and Boyd, London, 1964).

[3] S. Weinberg, Phys. Rev. **133**, B232 (1964).

[4] C. Lovelace, Phys. Rev. **135**, B1225 (1964).

[5] A preliminary report of this work has appeared in Phys. Letters **25B**, 84 (1967). The method described there is that referred to as approximation II in Sec. 3 of this paper.

[6] R. Aaron, R. D. Amado, and B. W. Lee, Phys. Rev. **121**, 319 (1961).

and H is the total Hamiltonian,

$$H=H_0+V_1+V_2+V_3. \tag{2.6}$$

For future reference we note the relations[2]

$$\begin{aligned} G_0(s)T_\alpha(s)&=G_\alpha(s)V_\alpha, \\ T_\alpha(s)G_0(s)&=V_\alpha G_\alpha(s). \end{aligned} \tag{2.7}$$

Equations (2.3) and (2.4) define two-body propagators and transition operators in the three-body space. The corresponding operators in the two-body space are

$$g_\alpha(s)=(s-h_\alpha)^{-1}, \tag{2.8}$$

$$t_\alpha(s)=V_\alpha+V_\alpha g_\alpha(s)V_\alpha, \quad (\alpha=1,2,3) \tag{2.9}$$

where h_α is the Hamiltonian of the pair of particles other than α in their center-of-mass system. The operators T_α and t_α are simply related in the free-particle representation. Let $\hbar\mathbf{p}_\alpha$ be the momentum of particle α with respect to the center of mass of the other pair, and let $\hbar\mathbf{q}_\alpha$ be the relative momentum within that pair. Then

$$\begin{aligned} &\langle\mathbf{p}_\alpha'\mathbf{q}_\alpha'|\,T_\alpha(s)\,|\mathbf{p}_\alpha\mathbf{q}_\alpha\rangle \\ &\quad=\delta(\mathbf{p}_\alpha'-\mathbf{p}_\alpha)\langle\mathbf{q}_\alpha'|\,t_\alpha(s-(\hbar^2/2M_\alpha)p_\alpha^2)\,|\mathbf{q}_\alpha\rangle, \end{aligned} \tag{2.10}$$

where M_α is the reduced mass in channel α, for example,

$$M_1=m_1(m_2+m_3)/(m_1+m_2+m_3), \tag{2.11}$$

and $\langle\mathbf{q}_\alpha'|\,t_\alpha(s)\,|\mathbf{q}_\alpha\rangle$ is the off-shell two-body amplitude.

The possible three-body processes may all be expressed in terms of transition operators $U_{\beta\alpha}{}^{\pm}$ (in the notation of Ref. 2), where

$$U_{\beta\alpha}{}^{+}(s)=(1-\delta_{\beta\alpha})V_\alpha+W_{\beta\alpha}(s), \tag{2.12}$$

$$U_{\beta\alpha}{}^{-}(s)=(1-\delta_{\beta\alpha})V_\beta+W_{\beta\alpha}(s), \tag{2.13}$$

$$W_{\beta\alpha}(s)=\sum_{\beta\neq\gamma\neq\alpha}V_\gamma+\sum_{\beta\neq\gamma}\sum_{\delta\neq\alpha}V_\gamma G(s)V_\delta. \tag{2.14}$$

Let $\Phi_{\alpha\nu}(E)$ be a suitably normalized unperturbed state in channel α, with quantum numbers ν and energy E, i.e.,

$$H_\alpha\Phi_{\alpha\nu}(E)=E\Phi_{\alpha\nu}(E). \tag{2.15}$$

Then the physical amplitude for a transition from channel α to channel β is the on-shell matrix element

$$\langle\beta\nu'|\,T\,|\alpha\nu\rangle=(\Phi_{\beta\nu'}(E),\,U_{\beta\alpha}{}^{\pm}(E+i\epsilon)\Phi_{\alpha\nu}(E)). \tag{2.16}$$

The operators $U_{\beta\alpha}{}^{+}$ and $U_{\beta\alpha}{}^{-}$ have the same on-shell matrix elements, as follows from Eqs. (2.12) and (2.13), and the identity

$$(\Phi_{\beta\nu'}(E),V_\beta\Phi_{\alpha\nu}(E))=(\Phi_{\beta\nu'}(E),V_\alpha\Phi_{\alpha\nu}(E)), \tag{2.17}$$

which itself follows from Eqs. (2.5) and (2.15). However, the off-shell matrix elements of $U_{\beta\alpha}{}^{+}$ and $U_{\beta\alpha}{}^{-}$ are different.

Lovelace[2] has derived coupled equations for the operators $U_{\beta\alpha}{}^{\pm}$, similar to the Faddeev equations,[1] and has demonstrated their superiority over the three-body Lippmann–Schwinger equations.[7] The Lovelace–Faddeev equations are

$$U_{\beta\alpha}{}^{+}(s)=\sum_{\beta\neq\gamma}V_\gamma+\sum_{\gamma\neq\alpha}U_{\beta\gamma}{}^{+}(s)G_0(s)T_\gamma(s), \tag{2.18}$$

$$U_{\beta\alpha}{}^{-}(s)=\sum_{\gamma\neq\alpha}V_\gamma+\sum_{\beta\neq\gamma}T_\gamma(s)G_0(s)U_{\gamma\alpha}{}^{-}(s). \tag{2.19}$$

We find it convenient to express the equations in terms of $W_{\beta\alpha}$, through Eqs. (2.12) and (2.13). With the aid of Eqs. (2.4) and (2.7) the equations become

$$W_{\beta\alpha}=\sum_{\beta\neq\gamma\neq\alpha}T_\gamma+\sum_{\gamma\neq\alpha}W_{\beta\gamma}G_0T_\gamma \tag{2.20}$$

$$=\sum_{\beta\neq\gamma\neq\alpha}T_\gamma+\sum_{\beta\neq\gamma}T_\gamma G_0W_{\gamma\alpha}, \tag{2.21}$$

where for clarity the variable s is suppressed. (Iteration of either equation yields directly the general multiple-scattering expansion.)

We now recast the equations into a form more suitable for approximation. Let T_γ be expressed as the sum of two parts,

$$T_\gamma=T_\gamma{}^{(1)}+T_\gamma{}^{(2)}, \tag{2.22}$$

with $T_\gamma{}^{(1)}$ and $T_\gamma{}^{(2)}$ to be defined later. We define operators $W_{\beta\alpha}{}^{(1)}$ and $U_{\beta\alpha}{}^{(1)}$ in terms of $T_\gamma{}^{(1)}$ through equations analogous to Eqs. (2.20) and (2.12):

$$W_{\beta\alpha}{}^{(1)}=\sum_{\beta\neq\gamma\neq\alpha}T_\gamma{}^{(1)}+\sum_{\gamma\neq\alpha}W_{\beta\gamma}{}^{(1)}G_0T_\gamma{}^{(1)}, \tag{2.23}$$

$$U_{\beta\alpha}{}^{(1)}=(1-\delta_{\beta\alpha})V_\alpha+W_{\beta\alpha}{}^{(1)}. \tag{2.24}$$

Then it is proved in the Appendix that Eqs. (2.20) and (2.21) lead to the identity

$$\begin{aligned} W_{\beta\alpha}-W_{\beta\alpha}{}^{(1)}=\sum_\gamma\,&[(1-\delta_{\beta\gamma})+W_{\beta\gamma}{}^{(1)}G_0]T_\gamma{}^{(2)} \\ &\times[(1-\delta_{\gamma\alpha})+G_0W_{\gamma\alpha}], \end{aligned} \tag{2.25}$$

which is valid for any choice of $T_\gamma{}^{(1)}$ and $T_\gamma{}^{(2)}$ satisfying Eq. (2.22). A variety of formulations may be generated by decomposing T_γ in different ways.

We now restrict attention to the physical energies $s=E+i\epsilon$, and choose

$$T_\gamma{}^{(2)}(E+i\epsilon)=V_\gamma G_\gamma'(E)V_\gamma, \tag{2.26}$$

where

$$G_\gamma'(E)=-i\pi\delta(E-H_\gamma)P_\gamma, \quad (\gamma=1,2,3) \tag{2.27}$$

and P_γ is the projection operator onto the channel γ eigenstates (i.e., the eigenstates of H_γ that include a

[7] B. Lippmann and J. Schwinger, Phys. Rev. **79**, 469 (1950).

bound state). For notational convenience we also define

$$G_0'(E)=0. \tag{2.28}$$

The first factor of Eq. (2.27) is the energy-conserving part of the propagator $G_\gamma(E+i\epsilon)$ [Eq. (2.3)]. We define $T_\gamma^{(2)}$ to include just the energy-conserving part rather than the full propagator, in order to obtain an integral equation that involves only the on-shell three-body amplitudes.

With this choice of $T_\gamma^{(2)}$, Eq. (2.25) can be simplified by using the relations

$$\begin{aligned} G_\gamma'(E)V_\gamma G_0(E+i\epsilon)&=G_\gamma'(E), \\ G_0(E+i\epsilon)V_\gamma G_\gamma'(E)&=G_\gamma'(E), \end{aligned} \tag{2.29}$$

which themselves follow from the identity

$$G_0(E+i\epsilon)V_\gamma\Phi_{\gamma\nu}(E)=\Phi_{\gamma\nu}(E)+O(\epsilon). \tag{2.30}$$

With the use of Eqs. (2.12), (2.13), and (2.24), Eq. (2.25) then becomes

$$U_{\beta\alpha}^{+}(E+i\epsilon)=U_{\beta\alpha}^{(1)}(E+i\epsilon) +\sum_\gamma U_{\beta\gamma}^{(1)}(E+i\epsilon)G_\gamma'(E)U_{\gamma\alpha}^{-}(E+i\epsilon). \tag{2.31}$$

Now because of the δ function in G_γ', the on-shell matrix elements of this equation involve only the on-shell matrix elements of $U_{\beta\alpha}^{+}$ and $U_{\beta\alpha}^{-}$, which are both equal to the physical amplitude [Eq. (2.16)]. It follows that the equation

$$U_{\beta\alpha}(E+i\epsilon)=U_{\beta\alpha}^{(1)}(E+i\epsilon) +\sum_\gamma U_{\beta\gamma}^{(1)}(E+i\epsilon)G_\gamma'(E)U_{\gamma\alpha}(E+i\epsilon) \tag{2.32}$$

defines a new set of transition operators $U_{\beta\alpha}$, whose on-shell matrix elements are the physical amplitudes. This is the exact equation that we shall approximate in the next section. The on-shell matrix elements of Eq. (2.32) yield coupled integral equations relating the physical amplitudes to the on-shell matrix elements of $U_{\beta\alpha}^{(1)}$.

It remains to investigate the operator $T_\gamma^{(1)}$, which is involved in the definition of $U_{\beta\alpha}^{(1)}$. From Eq. (2.26) we have

$$T_\gamma^{(1)}(E+i\epsilon)=T_\gamma(E+i\epsilon)-V_\gamma G_\gamma'(E)V_\gamma. \tag{2.33}$$

To clarify the meaning of $T_\gamma^{(1)}$ we first discuss T_γ, using the spectral representation of the two-body operator t_γ [Eq. (2.9)];

$$t_\gamma(s)=V_\gamma+\textstyle\sum_n V_\gamma|\phi_{\gamma n}\rangle[s+e_{\gamma n}]^{-1}\langle\phi_{\gamma n}|V_\gamma +\int d\kappa\, V_\gamma|\phi_{\gamma\kappa}\rangle[s-e_{\gamma\kappa}]^{-1}\langle\phi_{\gamma\kappa}|V_\gamma, \tag{2.34}$$

where the $\phi_{\gamma n}$ are the bound eigenstates of h_γ, with binding energies $e_{\gamma n}$, and the $\phi_{\gamma\kappa}$ are the continuum eigenstates. The last term exhibits the cut for real positive two-body energies s, and the second term the bound-state poles at $s=-e_{\gamma n}$. Now the matrix elements of $T_\gamma(E+i\epsilon)$ in the free-particle representation are seen from Eq. (2.10) to involve $t_\gamma(s)$ with the two-body energy

$$\begin{aligned} s&=E-(\hbar^2/2M_\gamma)p_\gamma^2+i\epsilon \\ &=e+i\epsilon. \end{aligned} \tag{2.35}$$

Thus integration with respect to p_γ causes s to follow a contour parallel to and above the real axis, extending from $E+i\epsilon$ to $-\infty+i\epsilon$. For e positive, the small positive imaginary term $i\epsilon$ in Eq. (2.35) has the effect, familiar in two-body scattering theory, of specifying the branch of the function $t_\gamma(s)$. But in the three-body application the imaginary term has an additional function: It prescribes the contour in s past the bound-state poles.

From Eqs. (2.33), (2.10), (2.34), and (2.27) we can write, in analogy with Eq. (2.10),

$$\langle\mathbf{p}_\gamma'\mathbf{q}_\gamma'|T_\gamma^{(1)}(E+i\epsilon)|\mathbf{p}_\gamma\mathbf{q}_\gamma\rangle =\delta(\mathbf{p}_\gamma'-\mathbf{p}_\gamma)\langle\mathbf{q}_\gamma'|t_\gamma^{(1)}(e+i\epsilon)|\mathbf{q}_\gamma\rangle, \tag{2.36}$$

where e is given by Eq. (2.35), and

$$t_\gamma^{(1)}(e+i\epsilon)=V_\gamma+P\textstyle\sum_n V_\gamma|\phi_{\gamma n}\rangle[e+e_{\gamma n}]^{-1}\langle\phi_{\gamma n}|V_\gamma +\int d\kappa\, V_\gamma|\phi_{\gamma\kappa}\rangle[e+i\epsilon-e_{\gamma\kappa}]^{-1}\langle\phi_{\gamma\kappa}|V_\gamma, \tag{2.37}$$

with P denoting the principal value. That is, $t_\gamma^{(1)}(e+i\epsilon)$ differs from $t_\gamma(e+i\epsilon)$ in only one respect—the pole terms are replaced by principal values. Therefore there is no special difficulty in calculating the matrix elements of $t_\gamma^{(1)}$: In fact, in the limit $\epsilon\to 0$,

$$\langle\mathbf{q}_\gamma'|t_\gamma^{(1)}(e+i\epsilon)|\mathbf{q}_\gamma\rangle =\langle\mathbf{q}_\gamma'|t_\gamma(e+i\epsilon)|\mathbf{q}_\gamma\rangle \quad \text{for } e\neq -e_{\gamma n}. \tag{2.38}$$

The practical effect of replacing t_γ by $t_\gamma^{(1)}$ is seen only in subsequent integration over $s=e+i\epsilon$. With t_γ we have seen that the contour in s passes above the bound-state poles, whereas with $t_\gamma^{(1)}$ the principal values must be taken at the bound-state poles.

3. APPROXIMATIONS

The goal of the approximation is to approximate $U_{\beta\alpha}^{(1)}$ [Eq. (2.24)], and then to solve exactly for the physical amplitudes using Eq. (2.32). From the multiple-scattering point of view the exact solution of Eq. (2.32) provides a simple way of including a wide class of multiple-scattering corrections. Furthermore, these corrections are important in connection with unitarity, as we shall see in the following section.

Specifically, we retain only the leading terms in the iterated form of Eq. (2.24), i.e., we replace $U_{\beta\alpha}^{(1)}$ by

$$I_{\beta\alpha}=(1-\delta_{\beta\alpha})V_\alpha+\sum_{\beta\neq\gamma\neq\alpha}T_\gamma^{(1)}. \tag{3.1}$$

Equation (2.32) for $U_{\beta\alpha}$ then becomes (with energy dependence suppressed)

$$U_{\beta\alpha}=I_{\beta\alpha}+\sum_\delta I_{\beta\delta}G_\delta' U_{\delta\alpha}. \tag{3.2}$$

The operators $I_{\beta\alpha}$ constitute what we refer to as the impulse-pickup approximation. The reason for the name is perhaps made clearer by writing typical operators explicitly. For the case of particle 1 incident on a bound state, the operators for elastic, rearrangement, and breakup processes, respectively, are

$$\begin{aligned} I_{11}&=T_{13}^{(1)}+T_{12}^{(1)},\\ I_{21}&=V_{23}+T_{12}^{(1)},\\ I_{01}&=T_{13}+T_{12}, \end{aligned} \tag{3.3}$$

where for clarity we have used the alternative notation $T_2=T_{13}$, etc. The direct operator I_{11} is seen to be closely related to the impulse approximation.[8,9] The rearrangement operator I_{21} is the sum of two terms, the first of which is the Born approximation for the pickup process,[10] in which 1 collides with 3 to form a bound state, leaving 2 free, and the second is an impulse-approximation term describing the ejection of 2 by 1 through direct collision. The breakup operator I_{01} is again a version of the impulse approximation. In writing Eq. (3.3) we have used the fact that the on-shell matrix elements of I_{01} involve only the two-body amplitudes with positive energies, so that it is unnecessary to distinguish between T_γ and $T_\gamma^{(1)}$. In addition, the term V_{23} has been omitted from I_{01}, since Eq. (2.17) with $\beta=0$ shows that the potential term does not contribute to the on-shell breakup amplitude.

On taking $\alpha=1$ in Eq. (3.2), corresponding to particle 1 incident on a bound state of 2 and 3, we see that U_{11}, U_{21}, and U_{31} satisfy a set of three coupled equations, and U_{01} is expressed in terms of U_{11}, U_{21}, and U_{31}. If V_{13} and V_{12} do not support bound states, so that rearrangement collisions cannot occur, then the first set of equations reduces to the single equation

$$U_{11}=(T_{13}+T_{12})+(T_{13}+T_{12})G_1'U_{11}, \tag{3.4}$$

which is a practical form of the method proposed by Rosenberg.[11] In the general case, however, it is easy to see by iterating Eqs. (3.2) and (3.4) and comparing the terms in the expansion of U_{11}, that the present method retains a very much wider class of multiple-scattering terms. If only the terms in which T_{12} and T_{13} alternate are retained in the iterated form of Eq. (3.4), the result is equivalent to the method proposed by Queen.[12] The relation between the Rosenberg and Queen methods has been discussed more fully elsewhere.[13]

The matrix elements of Eq. (3.2) between initial and final states in channels α and β yield equations expressing the on-shell amplitudes for all processes in terms of the on-shell impulse-pickup amplitudes. If the latter are known it is in general possible to solve the equations exactly. (An example is worked out in detail in Sec. 5.) However, it is not usually possible to calculate the impulse-pickup amplitudes without further approximation, since they involve the off-shell two-body amplitudes $\langle\mathbf{q}_\gamma'|t_\gamma(s)|\mathbf{q}_\gamma\rangle$, which even in partial-wave form are functions of three variables. We therefore consider a further approximation of a more practical nature. To distinguish the approximations, that described above will be referred to as approximation I, and that to be described below as approximation II.

Approximation II is obtained by replacing $T_\gamma^{(1)}(E+i\epsilon)$ in the impulse-pickup bound-to-bound amplitudes by τ_γ, where [cf. Eq. (2.10)]

$$\begin{aligned}\langle\mathbf{p}_\gamma'\mathbf{q}_\gamma'|\tau_\gamma|\mathbf{p}_\gamma\mathbf{q}_\gamma\rangle&=\delta(\mathbf{p}_\gamma'-\mathbf{p}_\gamma)\\ &\times\langle\mathbf{q}_\gamma'|\tfrac{1}{2}[t_\gamma(e+i\epsilon)+t_\gamma(e'+i\epsilon)]|\mathbf{q}_\gamma\rangle,\end{aligned} \tag{3.5}$$

and e and e' are the two-body relative energies corresponding to $\mathbf{q}_\gamma$ and $\mathbf{q}_\gamma'$. Thus

$$\begin{aligned} e&=(\hbar^2/2\mu_\gamma)q_\gamma^2,\\ e'&=(\hbar^2/2\mu_\gamma)q_\gamma'^2, \end{aligned} \tag{3.6}$$

where μ_γ is the reduced mass of the colliding particles; for example,

$$\mu_1=m_2m_3/(m_2+m_3). \tag{3.7}$$

Here, the two-body energy in Eq. (2.10) is replaced by the on-shell energy for the initial or final two-body scattering state. [The symmetric expression in Eq. (3.5) is adopted to maintain time-reversal invariance.[9]] The energy shift is suggested merely to simplify the calculation, and is a standard procedure with the impulse approximation; indeed, the usual procedure is to go further, and replace the two-body amplitudes by the fully on-shell value, with $q_\gamma'=q_\gamma$. Though the energy shift is without justification, we shall show in the following section that unitarity, in the sense of a constraint on the set of bound-to-bound amplitudes, remains satisfied.

[8] G. F. Chew, Phys. Rev. **80**, 196 (1950); G. F. Chew and G. C. Wick, *ibid.* **85**, 636 (1952).
[9] G. F. Chew and M. L. Goldberger, Phys. Rev. **87**, 778 (1952).
[10] G. F. Chew and M. L. Goldberger, Phys. Rev. **77**, 470 (1950).
[11] L. Rosenberg, Phys. Rev. **135**, B715 (1964).
[12] N. M. Queen, Nucl. Phys. **55**, 177 (1964); *ibid.* **80**, 593 (1966).
[13] I. H. Sloan, Phys. Rev. **162**, 855 (1967). It should be noted that the *ad hoc* symmetrization of the impulse approximation in that paper widens the class of multiple-scattering terms included in the approximations, but the class remains a subset of that for the present method.

4. UNITARITY

We consider first the unitarity relations for T_γ and $T_\gamma^{(1)}$. From Eqs. (2.4) and (2.7) we can write

$$T_\gamma = V_\gamma + V_\gamma G_0 T_\gamma, \tag{4.1}$$

and hence by the usual argument obtain the unitarity relation

$$T_\gamma - T_\gamma^\dagger = T_\gamma^\dagger [G_0(E+i\epsilon) - G_0(E-i\epsilon)] T_\gamma, \tag{4.2}$$

where $T_\gamma = T_\gamma(E+i\epsilon)$. Some care is required in taking the limit $\epsilon \to 0$ if V_γ supports bound states, because of the resulting poles in the two-body amplitude [see Eq. (2.34)]. With the aid of Eqs. (2.10) and (2.34) it can be shown that the correct limit is[14]

$$T_\gamma - T_\gamma^\dagger = -2\pi i T_\gamma^\dagger \delta(E-H_0) T_\gamma - 2\pi i V_\gamma \delta(E-H_\gamma) P_\gamma V_\gamma, \tag{4.3}$$

where the second term arises from the bound-state poles. It then follows from Eqs. (2.33) and (2.27) that $T_\gamma^{(1)}$ satisfies the simpler relation

$$T_\gamma^{(1)} - T_\gamma^{(1)\dagger} = -2\pi i T_\gamma^\dagger \delta(E-H_0) T_\gamma. \tag{4.4}$$

This result is not at all surprising, since $T_\gamma^{(1)}$ differs from T_γ precisely in that the troublesome part of T_γ (i.e., the δ-function contribution to the bound-state pole term) has been subtracted off.

We turn now to the unitarity relation for the approximate three-body transition operators $U_{\beta\alpha}$, considering first approximation I. By algebraic manipulation of Eq. (3.2), the bound-to-bound operators ($\beta \neq 0 \neq \alpha$) satisfy

$$U_{\beta\alpha} - U_{\alpha\beta}^\dagger = -2\pi i \sum_{\delta\neq 0} U_{\delta\beta}^\dagger \delta(E-H_\delta) P_\delta U_{\delta\alpha} + \sum_{\gamma\neq 0} \sum_{\delta\neq 0} (\delta_{\beta\gamma} + U_{\gamma\beta}^\dagger G_\gamma'^\dagger) \times (I_{\gamma\delta} - I_{\delta\gamma}^\dagger)(\delta_{\delta\alpha} + G_\delta' U_{\delta\alpha}). \tag{4.5}$$

Taking advantage of the fact that the matrix elements of $U_{\beta\alpha}$ and $I_{\beta\alpha}$ are required only on shell, so that Eq. (2.17) can be used, we obtain from Eq. (3.1)

$$I_{\gamma\delta} - I_{\delta\gamma}^\dagger = \sum_{\gamma\neq\lambda\neq\delta} (T_\lambda^{(1)} - T_\lambda^{(1)\dagger}) \tag{4.6}$$

$$= -2\pi i \sum_{\gamma\neq\lambda\neq\delta} T_\lambda^\dagger \delta(E-H_0) T_\lambda, \tag{4.7}$$

with Eq. (4.4) used in the last step.

The resulting unitarity relation may be simplified by writing the breakup transition operator $U_{0\alpha}$, using Eqs. (3.2) and (3.3), in the form

$$U_{0\alpha} = \sum_{\lambda\neq 0} U_{0\alpha}^{(\lambda)}, \tag{4.8}$$

with

$$U_{0\alpha}^{(\lambda)} = T_\lambda \sum_{\delta\neq\lambda} (\delta_{\delta\alpha} + G_\delta' U_{\delta\alpha}). \tag{4.9}$$

The unitarity relation can then be expressed in the simple form

$$U_{\beta\alpha} - U_{\alpha\beta}^\dagger = -2\pi i \sum_{\delta\neq 0} U_{\delta\beta}^\dagger \delta(E-H_\delta) P_\delta U_{\delta\alpha} - 2\pi i \sum_{\lambda\neq 0} U_{0\beta}^{(\lambda)\dagger} \delta(E-H_0) U_{0\alpha}^{(\lambda)}. \tag{4.10}$$

It differs from the correct on-shell unitarity relation

$$U_{\beta\alpha} - U_{\alpha\beta}^\dagger = -2\pi i \sum_{\delta\neq 0} U_{\delta\beta}^\dagger \delta(E-H_\delta) P_\delta U_{\delta\alpha} - 2\pi i U_{0\beta}^\dagger \delta(E-H_0) U_{0\alpha} \tag{4.11}$$

only in the last term on the right-hand side.

Equation (4.10) would be the correct unitarity relation if there were three distinct three-free-particle channels, distinguished by the three values of λ. It is therefore clear that the set of amplitudes for bound-to-bound transitions ($\beta \neq 0 \neq \alpha$) satisfy the constraint imposed by unitarity—the corresponding submatrix of the S matrix is a submatrix of a unitary matrix. We can draw a stronger conclusion by noting that $U_{0\alpha}^{(\lambda)}$ is the contribution to $U_{0\alpha}$, in which the pair of particles other than λ interact in the final state. Therefore, unitarity is essentially exact at energies at which the final-state interactions of different pairs of particles do not significantly overlap. It is also clear from Eq. (4.10) that unitarity is exact below the breakup threshold.

With the further approximation $T_\gamma^{(1)} \approx \tau_\gamma$ (approximation II), Eq. (4.4) is no longer valid. From Eq. (3.5) we deduce instead

$$\langle \mathbf{p}_\gamma' \mathbf{q}_\gamma' | \tau_\gamma - \tau_\gamma^\dagger | \mathbf{p}_\gamma \mathbf{q}_\gamma \rangle = -2\pi i \delta(\mathbf{p}_\gamma' - \mathbf{p}_\gamma) \tfrac{1}{2} \times [\langle \mathbf{q}_\gamma' | t_\gamma^\dagger(e+i\epsilon) \delta(e-h_\gamma+V_\gamma) t_\gamma(e+i\epsilon) | \mathbf{q}_\gamma \rangle + \langle \mathbf{q}_\gamma' | t_\gamma^\dagger(e'+i\epsilon) \delta(e'-h_\gamma+V_\gamma) t_\gamma(e'+i\epsilon) | \mathbf{q}_\gamma \rangle], \tag{4.12}$$

where e and e' are given by Eq. (3.6). On substituting into Eq. (4.6) and then into Eq. (4.5), it is easy to see that the main result above still holds: the set of bound-to-bound amplitudes satisfies the unitary constraint. However, the absorption processes from the bound-state channels, contained in the last term of (4.5), no longer conserve energy. In particular, with approximation II unitarity is no longer exact below the breakup threshold.

5. IDENTICAL PARTICLES AND NUMERICAL EXAMPLE

We consider specifically bound-state scattering for a system of identical particles. The symmetry require-

[14] A rigorous proof of Eq. (4.3) for generalized Yukawa two-body interactions has been given in Ref. 2.

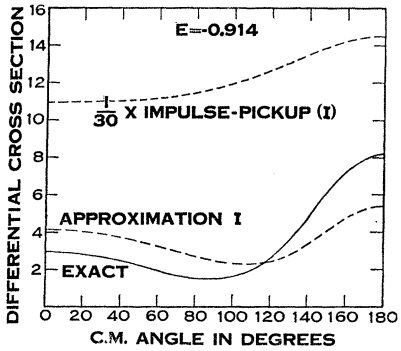

FIG. 1. Angular distributions at $E=-0.914$ (corresponding to a laboratory energy of 1.30 MeV if the two-body binding energy is the deuteron binding energy). Approximation I satisfies unitarity exactly below the breakup threshold, and is seen to yield great improvement over the corresponding impulse-pickup result. (Note the reduced scale of the impulse-pickup curve.) The exact result is from Ref. 16.

ments for either bosons or fermions lead to a simplification, without any need for further approximation.

We denote the on-shell matrix elements of $U_{\beta\alpha}$ and $I_{\beta\alpha}$ (with $\beta\neq 0\neq\alpha$) by $\langle\beta\mathbf{k}_\beta' n_\beta'|T|\alpha\mathbf{k}_\alpha n_\alpha\rangle$ and $\langle\beta\mathbf{k}_\beta' n_\beta'|\times I|\alpha\mathbf{k}_\alpha n_\alpha\rangle$, respectively, where $\mathbf{k}_\alpha$ is the channel momentum in channel α, and n_α denotes the remaining quantum numbers. Then Eq. (3.2) gives

$$\langle\beta\mathbf{k}_\beta' n_\beta'|T|\alpha\mathbf{k}_\alpha n_\alpha\rangle = \langle\beta\mathbf{k}_\beta' n_\beta'|I|\alpha\mathbf{k}_\alpha n_\alpha\rangle - i\pi\sum_{\delta=1}^{3}\sum_{n_\delta''}\int d\mathbf{k}_\delta''\langle\beta\mathbf{k}_\beta' n_\beta'|\times I|\delta\mathbf{k}_\delta'' n_\delta''\rangle\delta(E-E_\delta'')\langle\delta\mathbf{k}_\delta'' n_\delta''|T|\alpha\mathbf{k}_\alpha n_\alpha\rangle. \quad (5.1)$$

Following Lovelace,[4] we adopt the ordering convention that channel 1 denotes particle 1 plus the ordered pair (2,3), channel 2 denotes 2+(3,1), and channel 3 denotes 3+(1,2), so that it is unnecessary to distinguish between Bose and Fermi statistics. In either case there are just two distinct amplitudes for a system of identical particles, the direct and rearrangement:

$$\langle\mathbf{k}'n'|T^D|\mathbf{k}n\rangle=\langle\alpha\mathbf{k}'n'|T|\alpha\mathbf{k}n\rangle, \quad (\alpha=1,2,3) \quad (5.2)$$

$$\langle\mathbf{k}'n'|T^R|\mathbf{k}n\rangle=\langle\beta\mathbf{k}'n'|T|\alpha\mathbf{k}n\rangle, \quad (\beta\neq\alpha) \quad (5.3)$$

and the same result holds for the impulse-pickup amplitudes.

The coupling between channels in Eq. (5.1) can then be removed in exactly the same way as in Ref. 4. In particular, the symmetrized or antisymmetrized amplitude

$$\langle\mathbf{k}'n'|T|\mathbf{k}n\rangle=\langle\mathbf{k}'n'|T^D|\mathbf{k}n\rangle+2\langle\mathbf{k}'n'|T^R|\mathbf{k}n\rangle \quad (5.4)$$

satisfies

$$\langle\mathbf{k}'n'|T|\mathbf{k}n\rangle = \langle\mathbf{k}'n'|I|\mathbf{k}n\rangle - i\pi\sum_{n''}\int d\mathbf{k}''\langle\mathbf{k}'n'|I|\mathbf{k}''n''\rangle\times\delta(E-E'')\langle\mathbf{k}''n''|T|\mathbf{k}n\rangle, \quad (5.5)$$

where $\langle\mathbf{k}'n'|I|\mathbf{k}n\rangle$ is the symmetrized or antisymmetrized impulse-pickup amplitude,

$$\langle\mathbf{k}'n'|I|\mathbf{k}n\rangle=\langle\mathbf{k}'n'|I^D|\mathbf{k}n\rangle+2\langle\mathbf{k}'n'|I^R|\mathbf{k}n\rangle. \quad (5.6)$$

We perform detailed calculations for a system of spinless bosons, with separable two-body s-wave interactions of the Yamaguchi[15] form

$$\langle\mathbf{q}'|V|\mathbf{q}\rangle=-(\hbar^2\lambda/m)g(q')g(q), \quad (5.7)$$

$$g(q)=1/(q^2+\beta^2). \quad (5.8)$$

There is then a single two-body bound state at energy $-e_0$,

$$e_0=(\hbar^2/m)\alpha^2, \quad (5.9)$$

where

$$\lambda=\pi^{-2}\beta(\beta+\alpha)^2, \quad (5.10)$$

so that the label n in Eq. (5.5) is redundant. The bound-state wave function is

$$\phi_0(\mathbf{q})=Ng(q)/(q^2+\alpha^2), \quad (5.11)$$

and the two-body amplitude is

$$\langle\mathbf{q}'|t(e+i\epsilon)|\mathbf{q}\rangle = -\frac{\hbar^2\lambda}{m}g(q')[1-(\beta+\alpha)^2/(\beta-i\kappa)^2]^{-1}g(q), \quad (5.12)$$

where

$$e=(\hbar^2/m)\kappa^2, \quad \arg\kappa=0 \text{ or } \pi/2. \quad (5.13)$$

The bound-state pole at $\kappa=i\alpha$, or $e=-e_0$, is seen explicitly.

Previous approximate calculations on this system, using symmetrized versions of the Rosenberg and Queen methods, have been described in Ref. 13. Again we compare the approximate calculations with the exact results of Aaron, Amado, and Yam,[16] using the units of that paper ($\hbar=2m=1$, $e_0=1.5$) and $\beta=5$.

The main numerical problem is to calculate the symmetrized impulse-pickup amplitude [Eq. (5.6)] as a function of scattering angle. From Eq. (3.1) it may be written as

$$\langle\mathbf{k}'|I|\mathbf{k}\rangle=2[\langle 1\mathbf{k}'|T_{12}^{(1)}|1\mathbf{k}\rangle+\langle 2\mathbf{k}'|T_{12}^{(1)}|1\mathbf{k}\rangle+\langle 2\mathbf{k}'|V_{23}|1\mathbf{k}\rangle]. \quad (5.14)$$

The first term is expressible as a three-dimensional integral,

$$\langle 1\mathbf{k}'|T_{12}^{(1)}|1\mathbf{k}\rangle=\int d\mathbf{p}_3\,\phi_0^*(\tfrac{1}{2}\mathbf{k}'+\mathbf{p}_3)\langle\mathbf{k}'+\tfrac{1}{2}\mathbf{p}_3|\times t^{(1)}(e+i\epsilon)|\mathbf{k}+\tfrac{1}{2}\mathbf{p}_3\rangle\phi_0(\tfrac{1}{2}\mathbf{k}+\mathbf{p}_3), \quad (5.15)$$

[15] Y. Yamaguchi, Phys. Rev. **95**, 1628 (1954).
[16] R. Aaron, R. D. Amado, and Y. Y. Yam, Phys. Rev. **136**, B650 (1964).

where

$$e=E-(3\hbar^2/4m)p_3^2, \quad \text{(approximation I)}$$
$$=(\hbar^2/m)(\mathbf{k}+\tfrac{1}{2}\mathbf{p}_3)^2, \quad \text{(approximation II)}. \tag{5.16}$$

With approximation I some care is necessary in the integration with respect to p_3 because of the branch point at $e=0$, and the pole at $e=-e_0$. We recall from Sec. 2 that the superscript in $t^{(1)}$ means that the principal value must be taken at the bound-state pole. No difficulties arise with approximation II because the two-body energy is always positive. The second term of Eq. (5.14) is equal to the first for s-wave interactions, and the third can be expressed in terms of the bound-state wave function ϕ_0.[17]

Once the impulse-pickup amplitudes are known, the integral equation (5.5) may be readily solved with the aid of the partial-wave expansion

$$\langle \mathbf{k}'|I|\mathbf{k}\rangle=-(3\hbar^2/8\pi^2m)\frac{1}{k}\sum_{l=0}^{\infty}(2l+1)I_lP_l(\hat{k}'\cdot\hat{k}), \tag{5.17}$$

and a similar expansion for $\langle \mathbf{k}'|T|\mathbf{k}\rangle$, to yield

$$T_l=I_l/(1-iI_l). \tag{5.18}$$

The techniques used for the partial-wave expansion are described in Ref. 13.

The resulting angular distributions at two energies are shown in Figs. 1 and 2. (If e_0 is the deuteron binding energy 2.226 MeV, then the corresponding laboratory energies of the incident particle are 1.30 and 14.1 MeV, respectively.) The predictions of the impulse-pickup approximation are also shown for comparison. At the lower energy, which is below the break-up threshold, only the approximation I result is shown, because only approximation I gives exact unitarity below threshold[18] (see Sec. 4). We recall from Sec. 4, however, that both approximations satisfy the constraint imposed by unitarity: In partial-wave form,[13]

$$\mathrm{Im}T_l > |T_l|^2.$$

In contrast, the s-wave parts of the impulse-pickup amplitudes are found to strongly violate the constraint.

The results show that the approximation methods described in this paper yield at least qualitative agreement with the exact results, even at such low energies that the impulse-pickup results are wildly astray. In multiple-scattering language it is therefore clear that the methods provide a simple and effective way of summing an important class of multiple-scattering terms. Exact results on this model are unfortunately not available at higher energies, but it is reasonable to expect improved agreement at higher energies as the importance of the neglected multiple-scattering terms diminishes.

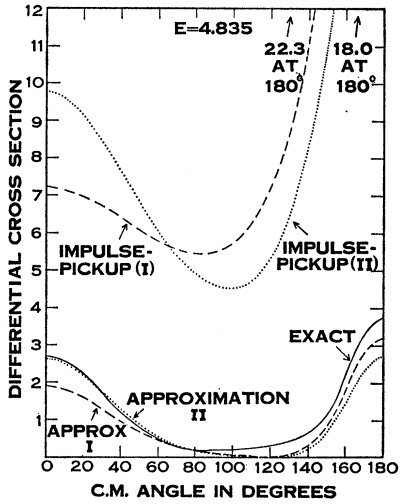

FIG. 2. Angular distributions at $E=4.835$ (corresponding to a laboratory energy of 14.1 MeV if the two-body energy is the deuteron binding energy). Approximations I and II both satisfy the unitary constraint, whereas the corresponding impulse-pickup results strongly violate unitarity.

Note added in proof. Since the submission of this paper we have learned of the work of Alt, Grassberger, and Sandhas [Nucl. Phys. **B2**, 167 (1967)], in which formal equations closely related to those of Sec. 2 are derived by a somewhat different argument.

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APPENDIX

To prove Eq. (2.25) we write Eqs. (2.20), (2.21), and (2.23) in the form

$$W_{\beta\alpha}=\sum_{\gamma\neq\alpha}Q_{\beta\gamma}T_\gamma \tag{A1}$$

$$=\sum_{\beta\neq\gamma}T_\gamma P_{\gamma\alpha}, \tag{A2}$$

$$W_{\beta\alpha}{}^{(1)}=\sum_{\gamma\neq\alpha}Q_{\beta\gamma}{}^{(1)}T_\gamma{}^{(1)}, \tag{A3}$$

[17] L. Rosenberg, Phys. Rev. **131**, 874 (1963), Eq. (4.22).

[18] In partial-wave form the approximation-I phase shifts are easily seen to be real below the breakup threshold, since the impulse-pickup amplitudes are then real. Note that this would not be the case if T_γ appeared instead of $T_\gamma{}^{(1)}$ in Eq. (3.1), since the bound-state pole in the two-body amplitude [Eq. (5.12)] would then give rise to an imaginary term in the impulse-pickup amplitude.

with

$$P_{\gamma\alpha}=(1-\delta_{\gamma\alpha})+G_0W_{\gamma\alpha}, \quad \text{(A4)}$$

$$Q_{\beta\gamma}=(1-\delta_{\beta\gamma})+W_{\beta\gamma}G_0, \quad \text{(A5)}$$

$$Q_{\beta\gamma}{}^{(1)}=(1-\delta_{\beta\gamma})+W_{\beta\gamma}{}^{(1)}G_0. \quad \text{(A6)}$$

On using $T_\gamma=T_\gamma{}^{(1)}+T_\gamma{}^{(2)}$ [Eq. (2.22)], Eqs. (A1) and (A3) give

$$W_{\beta\alpha}-W_{\beta\alpha}{}^{(1)}=\sum_{\gamma\neq\alpha}Q_{\beta\gamma}{}^{(1)}T_\gamma{}^{(2)}+\sum_{\gamma\neq\alpha}(Q_{\beta\gamma}-Q_{\beta\gamma}{}^{(1)})T_\gamma. \quad \text{(A7)}$$

With the use of Eqs. (A5) and (A6), and writing

$$A_{\beta\alpha}=W_{\beta\alpha}-W_{\beta\alpha}{}^{(1)}, \quad \text{(A8)}$$

we obtain

$$A_{\beta\alpha}=\sum_{\gamma\neq\alpha}Q_{\beta\gamma}{}^{(1)}T_\gamma{}^{(2)}+\sum_{\gamma\neq\alpha}A_{\beta\gamma}G_0T_\gamma. \quad \text{(A9)}$$

Now, with the successive use of Eqs. (A4), (A2), (A9), and (A2) we obtain

$$\begin{aligned}\sum_{\gamma\neq\alpha}A_{\beta\gamma}G_0T_\gamma&=\sum_\gamma A_{\beta\gamma}G_0T_\gamma(P_{\gamma\alpha}-G_0W_{\gamma\alpha})\\&=\sum_\gamma A_{\beta\gamma}G_0T_\gamma(P_{\gamma\alpha}-G_0\sum_{\gamma\neq\delta}T_\delta P_{\delta\alpha})\\&=\sum_\delta(A_{\beta\delta}-\sum_{\gamma\neq\delta}A_{\beta\gamma}G_0T_\gamma)G_0T_\delta P_{\delta\alpha}\\&=\sum_{\gamma\neq\delta}\sum Q_{\beta\gamma}{}^{(1)}T_\gamma{}^{(2)}G_0T_\delta P_{\delta\alpha}\\&=\sum_\gamma Q_{\beta\gamma}{}^{(1)}T_\gamma{}^{(2)}G_0W_{\gamma\alpha}. \quad \text{(A10)}\end{aligned}$$

The result of substituting (A10) into (A9),

$$A_{\beta\alpha}=\sum_\gamma Q_{\beta\gamma}{}^{(1)}T_\gamma{}^{(2)}P_{\gamma\alpha},$$

is equivalent to Eq. (2.25).

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Low-Energy Theorems, Dispersion Relations, and Superconvergence Sum Rules for Compton Scattering*

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A derivation of low-energy theorems for Compton scattering from spin-0 and spin-$\frac{1}{2}$ targets is given within the framework of dispersion theory. We work exclusively with physical helicity amplitudes and utilize the zeros of these amplitudes forced by angular momentum conservation to write unsubtracted dispersion relations. The conventional requirement of gauge invariance is replaced in our work by Lorentz invariance together with the knowledge that the photon is a massless spin-1 particle. From the dispersion relations we extract a number of sum rules of the superconvergence type, one example of which reduces the Drell–Hearn result in the forward direction.

I. INTRODUCTION

THE amplitude for the scattering of low-energy photons by spin-$\frac{1}{2}$ systems has been given by Low[1] and by Gell-Mann and Goldberger.[2] Using the full machinery of quantum field theory and, in particular, the gauge invariance of photon emission and absorption matrix elements, the following theorem was proved: The Compton amplitude, regarded as a function of the photon energy, at fixed scattering angle (and given target-photon polarizations), can be exactly specified in terms of the static properties of the target (i.e., charge, mass, and magnetic moment) provided only terms of zero and first order in photon energy are retained. The feature which distinguishes this result from a number of low-energy theorems recently derived from current algebra[3] is that it yields the amplitude in the physical

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[1] F. E. Low, Phys. Rev. **96**, 1428 (1954).

[2] M. Gell-Mann and M. L. Goldberger, Phys. Rev. **96**, 1433 (1954).

[3] A thorough exposition of the methods involved in obtaining such theorems and a critical analysis of the results will be found in the forthcoming book by S. L. Adler and R. F. Dashen, *Current Algebra* (W. A. Benjamin, Inc., New York, 1967).