and the dipole moment of isospin by

$$D_y{}^\alpha = \int dydz\, y\rho^\alpha(yz)\,.$$

The total isospin is of course

$$I^\alpha = \int dydz\, \rho^\alpha(y,z)\,.$$

In conclusion, we point out that Eq. (25) and its counterpart for $\mathcal{J}$'s pose very deep internal consistency requirements on the structure of the projections, "I bubbles," and expansion functions $e_z(Z)$ entering into the operator T. For T must have an eigenvalue equal to 1 for each current and this eigenvalue must equal 1 for all values of momentum transfer. It is not, at present, clear to us, what in the dynamical theory of Refs. 2 and 3 guarantees such eigenvalues without which the currents cannot exist. In particular it may be that a consistent theory of currents cannot be formulated without including into the theory from the start, those systems to which the currents couple (photons, leptons). At any rate, the solution will certainly await further understanding of the analytic properties of the inner products and the relation of these properties to Lorentz invariance and locality.

The theory presented here and in (3) does not explicitly deal with the problems of spin. Wherever details depend on spin we have assumed scalar particles. The complications due to spin are presently being studied by G. Frye.

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Anomalous and Normal Singularities and the Infinite-Momentum Limit*

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The behavior of the intrinsic size of a bound state as a function of its mass is derived by using an analogy between nonrelativistic two-dimensional quantum mechanics and the infinite-momentum limit of relativistic quantum theory.

ATTEMPTS have been made to understand relativistic form factors for bound states in terms of the use of wave functions.[1,2] In relativistic physics it is found that the familiar nonrelativistic behavior in the limit of loose binding is caused by the existence of anomalous thresholds in the dispersion relations for form factors which manifest themselves in ranges of charge distributions that are larger than the Compton wavelength of the constituent charged particle. The range for loose binding depends on the binding energy and masses according to the formula[1]

$$r^{-2} = 4M^2 - M^4/m^2\,, \tag{1}$$

where M is the bound-state mass, m the mass of the constituents, and r the range of the distribution of charge. We choose equal-mass constituents for simplicity only.

In the limit $M \to 2m$, (Eq. 1) agrees with the range given by the nonrelativistic Schrödinger equation, which is

$$r^{-2} = mE\,, \tag{2}$$

where E is the binding energy of the state.

As M decreases, Eq. (1) continues to hold until $M=\sqrt{2}m$, at which point r equals $(2m)^{-1}$. As M decreases further, Eq. (1) predicts that the range will begin to increase. However, at just this point the anomalous singularity undergoes its well-known disappearance into an unphysical sheet,[1,2] making Eq. (2) invalid. For $M<\sqrt{2}m$ the range is controlled by the normal singularity, which gives

$$r^{-2} = 4m^2\,. \tag{3}$$

It is a widely held belief that this behavior at $M \leq \sqrt{2}m$ represents very relativistic effects which have no simple interpretation in terms of wave functions. For this reason, it is generally believed that nonrelativistic intuitions are useless for the understanding of deeply bound systems.

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[1] R. Karplus, C. Sommerfield, and E. Wichmann, Phys. Rev. **111**, 1187 (1968); R. E. Cutkosky, J. Math. Phys. **429**, 1 (1960).

[2] F. Gross, Phys. Rev. **134B**, 406 (1964); R. Blankenbecler and L. F. Cook, Jr., *ibid.* **119**, 1745 (1960).

It is the purpose of this paper to demonstrate that exactly the opposite is true. Both the anomalous and normal behavior of the range can easily be understood in terms of a very simple wave-function picture.

In order to do this, use is made of a theorem which asserts that relativistic quantum theory in the limit of infinite momentum[3–5] is a special case of Galilean-invariant nonrelativistic quantum mechanics in two spatial dimensions. The theorem was suggested by one of us in Ref. 4.

The quantities of the infinite-momentum description of relativistic theory are as follows:

Each particle in the system has a mass m_i and a z component of momentum given by $\alpha_i L$, where L is a very large positive number. Each α_i is positive definite, since we are using a reference frame moving so rapidly in the $-z$ direction that all momenta are boosted up to infinite positive values.[4] Each particle also carries a transverse momentum (k_{ix},k_{iy}).

The identification between the quantities (α,k_{ix},k_{iy}) and the quantities of a Galilean-invariant quantum mechanics is: (1) $\mathbf{k}_i$= 2-dimensional momentum of the ith particle; (2) α_i= twice the mass of the ith particle of the two-dimensional theory; and (3) m_i^2/α_i= internal energy of the ith patricle.

The nonrelativistic two-dimensional energy would then be

$$H=\sum_i \frac{|\mathbf{k}_i|^2+m^2|}{\alpha_i} \tag{4}$$

for a system of free particles. It was shown in Refs. 4 and 5 that the Hamiltonian of Eq. (4) correctly describes the motion of the infinite-momentum relativistic system if one counteracts the time dilation for systems moving near the speed of light by defining a time scale T given by $T=t/L$.

We shall apply the above identification to the problem of the range of the charge distribution for a relativistic state consisting of a charged and an uncharged particle, each having mass m.

The bound-state mass is M and satisfies

$$M<2m. \tag{5}$$

[3] S. Weinberg, Phys. Rev. **165**, 1313 (1966).

[4] L. Susskind, Phys. Rev. **165**, 1535 (1968). In this reference the identification of 2-dimensional nonrelativistic physics and infinite-momentum physics is worked out in detail.

[5] L. Susskind, this issue, Phys. Rev. **165**, 1535 (1968).

For a nonrelativistic pair of particles of mass μ_1 and μ_2 ,bound in a state with binding energy B, the range of the charge distribution in the bound state satisfies

$$r^{-1}=(-8\mu_0 B)^{1/2}\left(\frac{(\mu_1+\mu_2)}{\mu_1}\right). \tag{6}$$

Here μ_0 is the reduced mass $\mu_1\mu_2/(\mu_1+\mu_2)$. The first factor is the inverse range of the distribution of the relative coordinate, and the second factor is the reciprocal of the relative distance from the charged particle (labeled 2) to the center of mass.

Applying the identification $2\mu_1=\alpha_1$, $2\mu_2=\alpha_2$, and using the invariance of the infinite-momentum limit under scale transformations[4] of the α coordinate to set $\alpha_1+\alpha_2=1$, Eq. (6) gives

$$r^{-1}(\alpha)=[4B\alpha(1-\alpha)]^{1/2}. \tag{7}$$

The binding energy B in the two-dimensional theory is the value of the difference between the energy of a bound state at rest and the energy of two free constituent particles at rest. In the relativistic theory, the identification of a particle at rest is a particle with $k_x=k_y=0$. Hence the binding energy can be gotten by using Eq. (4) for both the bound state and the two-particle system to give

$$\begin{aligned} -B&=M^2-m^2/\alpha-m^2/(1-\alpha)\\ &=M^2-m^2/\alpha(1-\alpha). \end{aligned} \tag{8}$$

Using Eq. (8) in Eq. (7) gives

$$r^{-1}(\alpha)=2[-\alpha(1-\alpha)M^2+m^2]^{1/2}. \tag{9}$$

Now, in general, the constituent particles in the bound state will be in a superposition of states with all values of α for which both α and $(1-\alpha)$ are positive. Equation (9) gives the range only for that part of the state correlated to a given value of α for the charged particle. Hence the actual range will correspond to the minimum value of $r^{-1}(\alpha)$ on the interval $0\leq\alpha\leq 1$.

A minimum of $r^{-1}(\alpha)$ occurs at $\alpha=2m^2/M^2$, which lies in the interval if $M>\sqrt{2}m$. Then the range given by Eq. (9) and $\alpha=2m^2/M^2$ agrees exactly with Eq. (1). If $M<\sqrt{2}m$, the minimum of $r^{-1}(\alpha)$ occurs at the endpoint $\alpha=1$. Equation (9) then gives a range in exact agreement with the normal-threshold answer of Eq. (3). Hence it is possible to understand both the anomalous and the normal behavior of the form factors of a bound state very simply.