Hadronic Currents*

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Questions concerning the hadronic currents are studied in the context of the infinite-momentum limit of the theory of self-induced strong interactions. Representations for the matrix elements of currents are derived from the composite nature of the particles. The kinematic structure in the transverse plane is shown to be Galilean-invariant, and conclusions for the theory of current algebra are drawn.

I. INTRODUCTION

INTERACTION currents and densities of the hadrons are studied from the point of view of the infinite-momentum limit of the theory of self-induced strong interactions as developed by Frye and the author.[1–3]

Two Lorentz observers W and W' with coordinates x, y, z, t and x', y', z', t' observe the system of hadrons under consideration. The observer W is an ordinary Lorentz observer who ascribes to each particle a finite momentum vector p_x, p_y, p_z and a finite energy, $(p_x^2+p_y^2+p_z^2+m^2)^{1/2}$. The frame W' is obtained from W by velocity transformation in the x direction.[4] It is assumed that the relative velocity of W and W' is so near to 1 (we take $c=1$) that the spacelike hypersurface $t'=0$ practically touches the light cone from the point of view of W.

The frame W' is not unique, since any Lorentz transformation in the x direction with $v<1$ will transform W' into another observer, W'', similar to W' itself. The assumption that an infinite-momentum limit exists is the assumption that the description in W' is related to that in W'' by a particularly simple set of rules and that covariance under such transformations is manifest. In Ref. 3, the limit was studied in terms of a set of variables α and k which could be used by W' to describe a system. When relevant, a discrete index a or b will be used to denote particle type and spin state.

The transformations between two observers W' and W'' were shown in Ref. 3 to be equivalent to a change of scale of α so that from the point of view of W', an infinite-momentum limit means that the quantities of the theory are expressed in terms of ratios of the α. More generally, the existence of an infinite-momentum limit requires manifest invariance with respect to the subgroup F, of the inhomogeneous Lorentz group which leaves invariant the hypersurface $x+t=0$.

In this paper the infinite-momentum limit of the matrix elements representing hadronic currents is examined.

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[1] L. Susskind, Phys. Rev. **154**, 1411 (1967).
[2] L. Susskind and G. Frye, Phys. Rev. **164**, 2003 (1967).
[3] L. Susskind, Phys. Rev. this issue, **165**, 1535 (1968).
[4] In discussing the subgroup F we use the notation of Ref. 3 except for the generator which was called Q in Ref. 3. Here we call it P. Also the x direction is used as the longitudinal direction in place of the z direction used in Ref. 3.

The other aspect of hadronic currents which we discuss is the structure implied by the composite nature of the hadrons. Briefly, we may expect the matrix elements of currents between particle states to be linearly related to matrix elements taken between the constituents of the composites, the hadrons themselves. The kernel of the linear relation depends on the amplitudes which describe the particles as composites.

We first consider the case of total charges such as electronic, baryonic, or SU_3 charges. The theory of charges is then generalized into a theory of currents and densities.

II. CHARGES

Charges are the space integrals of the fourth components of current vectors. As such, their matrix elements connect only states with equal spatial momentum. They generate, through exponentiation, Lie groups which act independently on subsystems.

Suppose

$$e^{i\alpha Q}|i\rangle=U_{il}|l\rangle\,, \tag{1}$$

where the notation used is that of the abstract formulation of Ref. 3. We shall assume that when acting on a composite state $|ij\rangle$, $e^{i\alpha Q}$ acts separately on the two subsystems i and j:

$$e^{i\alpha Q}|ij\rangle=U_{il}U_{jm}|lm\rangle \tag{2}$$

or

$$Q|ij\rangle=Q_{il}|lj\rangle+Q_{jl}|il\rangle\,. \tag{3}$$

Equation (3) defines a *strictly additive* quantity. The word "strictly" is added to distinguish between quantities like Q and quantities like energy which are only additive for well-separated systems. It is easy to verify that the commutator of two strictly additive operators is strictly additive.

The charge as seen by W' is an integral over all x', y', z' for t' fixed. From the point of view of W, the charge seen by W' is an integral over the hypersurface left invariant by the subgroup F.[4] It is not difficult to show that for any element of F, $[F,Q]=0$. We shall demonstrate the method which makes use of the observer W. The time component of the W' current vector is given in terms of the W current vector by

$$j_t'=(1-v^2)^{-1/2}(j_t+j_x)\,. \tag{4}$$

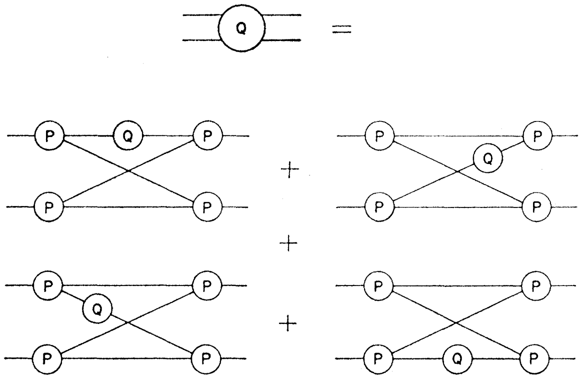

FIG. 1. Two-cluster representation for the matrix elements of strictly additive operators.

Integrating j_t' over $x'y'z'$ can be carried out in W by integration over the hypersurface $x+t=0$. The measure of x' distance for given distance x on this surface is $\Delta x'=(1-v^2)^{1/2}\Delta x$ so that

$$Q=\int dxdydzdt\,\delta(x+t)(j_t+j_x). \qquad (5)$$

The effect of F on Q can be expressed through the commutators $[F,Q]$:

$$[\alpha,Q]=\left[p_x+p_t,\int dxdydzdt\,\delta(x+t)(j_x+j_t)\right]$$
$$=i\int dxdydzdt\,\delta(x+t)\left(\frac{\partial}{\partial x}-\frac{\partial}{\partial t}\right)(j_x+j_t)$$
$$=0,$$
$$[\Lambda,Q]=\left[L_x,\int dxdydzdt\,\delta(x+t)(j_x+j_t)\right] \qquad (6)$$
$$=0,$$
$$[k_y,Q]=\int dxdydzdt\,\delta(x+t)\frac{\partial}{\partial y}(j_x+j_t)=0,$$
$$[k_z,Q]=0,$$
$$[P_y,Q]=[L_y,Q]+[R_z,Q]=0.$$

We shall introduce a method of labeling eigenstates of the operators α and k which were introduced in Ref.

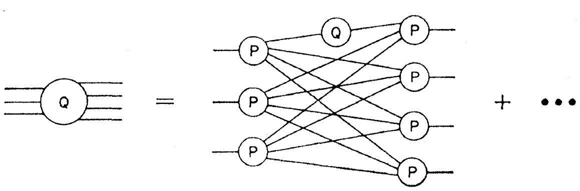

FIG. 2. Many-cluster representation for the matrix elements of a strictly additive operator. For an m-n cluster matrix element there are mn terms of the type shown.

3. The operators α and k have as their eigenvalues the sum of the longitudinal ratios for each particle and the sum of the transverse momenta, respectively.

We also assumed invariance of the theory under the replacement of each α_i by $|\Lambda|\alpha_i$ and the replacement of each k by $k_i+\alpha_i P$ with Λ and P arbitrary. Consider any state which is an eigenvector of α and k with eigenvalues α and k. Applying a longitudinal scale transformation with $\Lambda=\alpha^{-1}$ and following this by a transverse scale transformation with $P=-k$ produces a state $|\psi\rangle$ with $\alpha=1$ and $k=0$ which will be used to label the original state in the form $|\alpha k\psi\rangle$. This is analogous to labeling a state with a set of variables which describe the configuration as seen by the center-of-mass observer together with the total momentum of the configuration. Note that $|1,0,\psi\rangle=|\psi\rangle$.

A special case of this labeling is to label single-particle states by their longitudinal ratio α, transverse momenta k, and particle type a as in $|\alpha,k,a\rangle$. The matrix elements of Q have the form

$$\langle\phi\beta l|Q|\psi\alpha k\rangle=Q_{\phi\psi}\delta(1-\beta/\alpha)\delta(k-l). \qquad (7)$$

In (3) it was shown that strictly additive operators satisfy an internal particle representation shown in Fig. 1.

It is an easy matter to extend Fig. 1 to matrix

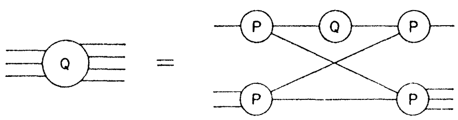

FIG. 3. Another representation for a strictly additive operator. For an m-n particle matrix element these are mn terms of the type shown.

elements involving arbitrarily many particles on the left or right. The generalization is shown in Fig. 2.

Figure 2 may be used as the starting point for an iteration method of computing matrix elements. An n-m particle matrix element of Q consists of nm terms, each of which contains an internal matrix element of Q inserted into the line connecting two external particles. Figure 2 shows one such term. The single term shown in Fig. 2 can be reexpressed with the help of the *PIP* rules of Sec. 7 of Ref. 2 to give Fig. 3.

Summing all nm terms gives

$$\langle w_1\cdots w_n|Q|z_1\cdots z_m\rangle$$
$$=\sum_{ij}\langle w_1\cdots w_{i-1},w_{i+1}\cdots w_n|A_{Z_1}{}^{\dagger}A_{W_1}$$
$$\times|z_1\cdots z_{j-1},z_{j+1}\cdots z_m\rangle I(Z_1;W_1')I(Z_1';W_1)$$
$$\times\langle w_i|Z_1'Z_2\rangle\langle W_1'W_2|z_j\rangle$$
$$\times I(Z_2;W)I(Z;W_2)\langle W|Q|Z\rangle, \qquad (8)$$

where the cluster notation of Ref. 3 is used. Equation (8) is nontrivial if n or m or both are greater than 1.

The I coefficients[2] are chosen so as to include as much of the sum as possible in the single-particle part. To do this, we break the unit operator into the sum of the projection operator for the space spanned by the one-particle states and I', the projection operator for the orthogonal subspace:

$$I=\int\frac{d\alpha}{\alpha}\frac{d\beta}{\beta}dkdl|\alpha k\rangle\langle\beta l|\delta(1-\alpha/\beta)\delta(k-l)+\sum_{Z,W=2}^{\infty}I'(Z;W)|Z\rangle\langle W|+|0\rangle\langle 0|. \quad (9)$$

The notation $\sum_{Z,W=2}^{\infty}$ means a sum over multiparticle states starting with two-body states. We replace $I(Z_2;W)$ and $I(W;Z_2)$ in Eq. (8) by the two terms in Eq. (9) giving $\langle w_1\cdots w_n|Q|z_1\cdots z_m\rangle$ as a sum of four terms. The first term contains only single-particle matrix elements of Q and equals

$$\sum_{ij}\langle w_1\cdots w_{i-1},w_{i+1}\cdots w_n|A_{Z_1}{}^{\dagger}A_{W_1}|z_1\cdots z_{j-1},z_{j+1}\cdots z_m\rangle \times I(Z_1;W_1')I(Z_1';W_1)\langle w_i'|Z_1',w\rangle\langle W_1',z|z_j\rangle \times\langle w|Q|z\rangle. \quad (10)$$

In the other terms, matrix elements of Q involving more than one particle either on the bra side or ket side or both are contained linearly.

For the matrix elements involving single particles in both bra and ket, Eq. (8) is trivial. For all other matrix elements, Eq. (8) is a nontrivial representation of the internal structure of Q. Therefore it is possible to regard Eq. (10) as a starting point for an iteration to obtain the matrix elements of Q. We denote the single-particle contribution by Q_1 and represent the linear operations which are applied to $\langle w|Q|z\rangle$ in Eq. (10) as $B_{wz}{}^1(w_1\cdots w_n;z_1\cdots z_m)$ so that

$$Q_1(w_1\cdots w_n;z_1\cdots z_m)=B_{wz}{}^1(w_1\cdots w_n;z_1\cdots z_m)\langle w|Q|z\rangle$$

or more simply

$$Q_1=B_{wz}{}^1\langle w|Q|z\rangle. \quad (11)$$

The three terms involving multiparticle matrix elements of Q are linear transforms of $\langle w_1\cdots|Q|z_1\cdots\rangle$ which we express as

$$\sum_{W,Z=2}^{\infty}B_{WZ}{}^M(w_1\cdots w_n;z_1\cdots z_m)\langle W|Q|Z\rangle.$$

The superscript, M, means many-particle contribution.

In terms of these definitions, Eq. (8) becomes

$$\langle W|Q|Z\rangle=Q_1(W;Z)+B_{W';Z'}{}^M\langle W'|Q|Z'\rangle$$

or

$$\langle W|Q|Z\rangle=[1-B^M]^{-1}Q_1=\{[1-B^M]^{-1}B^1\}\langle w|Q|z\rangle. \quad (12)$$

Equation (12) can be used to give a linear homogeneous equation for the single-particle matrix element by observing that the single-particle states are expandable.[2,3]

$$\langle w'|Q|z'\rangle=e_{z'}(Z)e_{w'}(W)^*\langle W|Q|Z\rangle =e_{z'}(Z)e_{w'}(W)^*\{[1-B^M]^{-1}B^1\}\langle w|Q|z\rangle. \quad (13)$$

Collecting the entire set of linear operations on $\langle w|Q|z\rangle$ implied in Eq. (13) into an operator $T_{w'z';\,wz}$ gives

$$\langle w'b'|Q|z'a'\rangle=T_{w'b'z'a';\,wbza}\langle wb|Q|za\rangle. \quad (14)$$

In Eq. (14), the full dependence on the discrete particle variables has been reinstated through the indices a and b.

An inspection of the terms entering into an iteration construction of T shows that $T_{w'b'z'a';\,wbza}$ is of the form

$$T_{b'a';\,ba}(w'z';wz)\delta(z-z';w-w'), \quad (15)$$

with $T_{b'a';\,ba}$ being an F-invariant function and $\delta(z-z';w-w')$ being the F-invariant δ function defined in Ref. 3.

Since the matrix elements of a charge are F-invariant and momentum-conserving, we have

$$\langle wb|Q|za\rangle=Q_{ba}\delta(w;z). \quad (16)$$

Inserting Eqs. (16) and (15) into (14) gives

$$Q_{b'a'}=\int T_{b'a';\,ba}(z'z';zz)dzQ_{ba}. \quad (17)$$

The quantity $\int T_{b'a';\,ba}(z'z';zz)dzQ_{ba}$ appears to have a dependence on z', but it is actually independent of z' since the only F-invariant function of z' is constant.

Each solution of Eq. (17) can be used to begin an iteration toward an operator whose single-particle matrix elements equal Q_{ba}. The entire set of such operators is closed under commutations.

The number of such operators is the number of eigenvectors with unit eigenvalue of the matrix

$$\int T_{b'a';\,ba}(z'z';zz)dz, \quad \text{where} \quad dz=d\alpha dk/\alpha.$$

The dimensionality of this matrix is the square of the total number of independent particle types. However, not every solution is actually a charge, where by charge we mean integrated time component of a four-vector. For example, α and k, the total longitudinal ratio and transverse momenta, are solutions.

III. LOCAL SCALAR DENSITIES

The most important advantage of the infinite-momentum limit is the elimination of vacuum structure from the theory.[3] The infinite-momentum external particles in the X or $\mathfrak{X}$ representation cannot connect to the vacuum structure if the vacuum structure graphs fall off sufficiently rapidly with energy. We assumed this to be the case for charges also. However, this

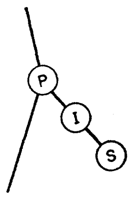

FIG. 4. A vacuum contribution to the matrix elements of a local scalar. Such terms disappear when S is longitudinally integrated and transformed to infinite momentum.

assumption cannot be maintained for the local quantities such as $s(x)$, a scalar density. The vacuum can connect to states of arbitrarily large momentum through $s(x)$, since $s(x)$ is a local scalar. For example,

$$\langle p|s(x)|0\rangle = e^{ipx}.$$

Because of this, the matrix element $\langle \beta l|s(0)|\alpha k\rangle$ will contain a vacuum term shown in Fig. (4), in which the intermediate I bubble carries infinite momentum. Similar diagrams could not contribute to $\langle \beta l|\alpha k\rangle$ or $\langle \beta l|H|\alpha k\rangle$ because both $\langle \psi|H|0\rangle$ and $\langle \psi|0\rangle$ vanish unless the momentum of ψ is zero and $\langle w|z, z-w\rangle$ goes to zero as $\alpha-\beta$ goes to zero. However, for local quantities, $\langle \psi|s|0\rangle$ is independent of the momentum of ψ. On the other hand, integration of s over space will cause those matrix elements $\langle \psi|s|0\rangle$ to vanish if the momentum of $\psi\neq 0$. An example of this is the lack of vacuum structure in the integrated charges Q.

One of the reasons we like to eliminate vacuum effects is that this will lead to strictly additive operators which will close under commutation. However, we should like to maintain information about the local structure of the operators which make up the commutator algebra. The important aspect of the spatial integration is eliminating the matrix element $\langle \psi|s|0\rangle$ for large longitudinal momenta of ψ since this will prevent the vacuum graphs from connecting into the structure of matrix elements between states of very large longitudinal momentum. This can be accomplished by integrating $s(x'y'z')$ over the longitudinal direction x' in the frame W'. We tentatively define the "transverse scalar current" by such an integration.

$$\mathcal{S}(y',z') = \int_{-\infty}^{\infty} s(x',y',z')dx'. \tag{18}$$

From the point of view of the observer W, the integral in Eq. (18) is

$$\mathcal{S}(y,z) = (1-v^2)^{1/2}\int_{-\infty}^{\infty} s(x,y,z,t)\delta(x+t)dxdt. \tag{19}$$

The factor $(1-v^2)^{1/2}$ which goes to zero as $v\rightarrow 1$ is a Lorentz contraction factor.

This factor causes the operator $\mathcal{S}$ to be zero. It is possible to understand this as follows. Any physical system in space time which contributes to a presence of s will be contracted into infinitely thin distributions in the x direction. Thus an integral over x' will give zero. In order to avoid this we simply scale up the definition of $\mathcal{S}$ by the factor $(1-v^2)^{-1/2}$. We redefine $\mathcal{S}$ by

$$\mathcal{S}(y',z') = (1-v^2)^{-1/2}\int_{-\infty}^{\infty} s(x',y',z')dx' \tag{18'}$$

and

$$\mathcal{S}(y,z) = \int_{-\infty}^{\infty} s(x,y,z,t)\delta(x+t)dxdt. \tag{19'}$$

The transformation properties of $\mathcal{S}$ under the group F are contained in the commutation relations

$$[\alpha,\mathcal{S}]=0, \tag{20a}$$

$$[k_z,\mathcal{S}]=-(\partial\mathcal{S}/\partial z), \tag{20b}$$

$$[k_y,\mathcal{S}]=-i(\partial\mathcal{S}/\partial y), \tag{20c}$$

$$[P,\mathcal{S}]=0, \tag{20d}$$

$$[\Lambda,\mathcal{S}]=i\mathcal{S}. \tag{20e}$$

The matrix element of $\mathcal{S}$ between configuration ϕ and ψ with ψ carrying momentum αk and ϕ carrying βl is

$$\langle \beta l\phi|\mathcal{S}(y,z)|\alpha k\psi\rangle = \delta(\alpha-\beta)e^{i(k_y-l_y)y+i(k_z-l_z)z} \times\langle \beta l\phi|s(0)|\alpha k\psi\rangle. \tag{21}$$

Assuming that the longitudinal integration removed the vacuum terms from the structure of s, the transverse $\mathcal{S}$ is strictly additive and therefore satisfies Eqs. (3), (8), (12), and (14) and Figs. (1)–(3).

Using Eqs. (15) and (21), Eq. (14) takes on the following form for the operator $\mathcal{S}$.

$$\langle \beta'l'b'|s(0)|\alpha'k'a'\rangle\delta(\alpha'-\beta') = \delta(\alpha'-\beta')T_{a'b';\,ab}(\alpha'\alpha k'l';\alpha\alpha kl) \times\langle \alpha lb|s(0)|\alpha ka\rangle\delta(k'-l'-k+l). \tag{22}$$

By concentrating only on the matrix elements of s which are diagonal in α, a certain amount of information concerning the local structure of s is lost. The information which is contained in $\mathcal{S}$ is exactly the information contained in the usual matrix elements of s for spacelike momentum transfer. It is easy to see that matrix elements of $\mathcal{S}$ do not contain information about timelike momentum transfer. The momentum-transfer squared between αk and βl is easily evaluated using the identifications[3]

$$\alpha = p_x+p_t,\quad k_z=p_z,\quad k=p_y$$
$$\beta = q_x+q_t,\quad l_z=q_z,\quad l_y=q_y$$
$$(k^2+m^2)/\alpha = p_t-p_x,\quad (l^2+n^2)/\beta = q_t-q_x.$$

(Invariant mass, m, is carried by p and n by q.) Then simply evaluate

$$t=(p_t-q_t)^2-(p_x-q_x)^2-(p_y-q_y)^2-(p_z-q_z)^2,$$

which gives

$$t=(\alpha-\beta)\left[\frac{k^2+m^2}{\alpha}-\frac{l^2+n^2}{\beta}\right]-(k-l)^2. \tag{23}$$

For matrix elements of $\mathcal{S}$, $\alpha=\beta$ and $t=-(k-l)^2$ and is spacelike.

Also it is not difficult to show that for any two states ψ and φ carrying 4-momenta p and q with spacelike momentum transfer between p and q, a rotation can be made which will bring p and q into a new configuration with $p_x+p_t=q_x+q_t$. Lorentz transformation in the x direction will bring ψ and φ to infinite-momentum states with equal longitudinal ratios. Since $s(0)$ is a scalar, its matrix elements do not change under this set of operations. Hence it follows that $\mathcal{S}(y,z)$ contains just that information contained in $s(x,y,z)$ for spacelike momentum transfer.

For scalar particles, the matrix elements of $s(0)$ depend only on momentum transfer and particle type:

$$\langle\beta,l,b|s(0)|\alpha,k,a\rangle=F_{b,a}((k-l)^2), \tag{24}$$

which when used in Eq. (22) gives

$$F_{b'a'}((k'-l')^2)=\int T_{b'a';\,ba}(\alpha'\alpha'k'l';\alpha\alpha kl)\times\delta(k'-l'-k+l)(d\alpha/\alpha)dkdlF_{ba}((k-l)^2). \tag{25}$$

The right-hand side is independent of α' by virtue of the F-invariance of T.

Hence the homogeneous linear equations for matrix elements of $\mathcal{S}$ which derive from the strict additivity of $\mathcal{S}$ are uncoupled for different values of invariant momentum transfer and leave solutions arbitrary up to multiplication by any $f(t)$.

This is analogous to theories of currents which make use of "sidewise" dispersion relations to obtain coupled linear equations for matrix elements of currents.[5] In order to determine the function $f(t)$, a more powerful technique, making full use of the local properties of the currents, is necessary.

IV. VECTOR CURRENTS

As a second example of the use of the infinite-momentum frame in the study of local currents, we consider a 4-vector current such as the electromagnetic, baryonic, isospin, strangeness, or SU_3 currents. The current j has components, j_x, j_y, j_z, and j_t in W and j_x', j_y', j_z', and j_t' in W'. The Lorentz transformation on j gives

$$\begin{aligned} j_y'(0)&=j_y(0),\\ j_z'(0)&=j_z(0),\\ j_x'(0)&=(1-v^2)^{-1/2}(j_x+vj_t),\\ j_t'(0)&=(1-v^2)^{-1/2}(j_t+vj_x). \end{aligned} \tag{26}$$

In the limit $v\to 1$ the last two equations of Eqs. (26) become

$$\begin{aligned} j_x'(0)&=(1-v^2)^{-1/2}(j_x+j_t),\\ j_t'(0)&=(1-v^2)^{-1/2}(j_x+j_t). \end{aligned} \tag{26'}$$

The transverse currents are defined by longitudinal integration:

$$\begin{aligned} \mathcal{J}_y(y',z')&=(1-v^2)^{-1/2}\int j_y'(x',y',z')dx',\\ \mathcal{J}_z(y',z')&=(1-v^2)^{-1/2}\int j_z'(x',y',z')dx',\\ \rho(y',z')&=\frac{1}{2}\left[\int j_t'(x',y',z')dx'+\int j_x'(x',y',z')dx'\right]. \end{aligned} \tag{27}$$

In terms of the frame W, the definitions in Eqs. (27) are

$$\begin{aligned} \mathcal{J}_z(y,z)&=\int j_z(x,y,z,t)\,\delta(x+t)dxdt,\\ \mathcal{J}_y(y,z)&=\int j_y(x,y,z,t)\,\delta(x+t)dxdt,\\ \rho(y,z)&=\frac{1}{2}\int [j_x+j_t]\,\delta(x+t)dxdt. \end{aligned} \tag{28}$$

The matrix elements have been scaled in order to make their value finite at infinite momentum. As with the scalar density, it is expected that the longitudinal integration will eliminate the vacuum structure from the matrix elements of $\mathcal{J}$ and ρ. We assume as with $\mathcal{S}$ that the transverse currents $\mathcal{J}_y$ and $\mathcal{J}_z$ and the transverse density ρ are strictly additive quantities and that they satisfy an equation similar to Eq. (25).

The transformation properties of the $\mathcal{J}$'s under the group F can be derived most easily by making the identification with the relevant transformations in W. They are

$$\begin{aligned} [\alpha,\mathcal{J}_i]&=0,\quad i=y,z\\ [\alpha,\rho]&=0,\\ [k_j,\rho]&=-i\partial^j\rho,\quad j=y,z\\ [k_j,\mathcal{J}_i]&=-i\partial^j\mathcal{J}_i,\quad j,i=y,z\\ [\Lambda,\rho]&=0,\\ [\Lambda,\mathcal{J}_i]&=i\mathcal{J}_i,\quad i=y,z\\ [P,\rho]&=0,\\ [P_j,\mathcal{J}_i]&=-i\delta_{ij}\rho,\quad j,i=y,z. \end{aligned} \tag{29}$$

The reader who has followed the arguments beginning with those in Ref. 3 and concluding with the above commutation relations may have discovered an essential simplification in going to infinite momentum. The simplification is exactly this: The structure of the theory with respect to the transverse y-z plane is

[5] R. Dashen and S. Frautschi, Phys. Rev. **143**, 1171 (1966); in particular compare Eq. (5) in Dashen and Frautschi with Eq. (25) in the present article.

completely nonrelativistic and Galilean-invariant. First of all, consider the effective single-particle energy introduced in (3):

$$H=(k_y^2+k_z^2)/\alpha+m^2/\alpha.$$

Interpreting α as twice the effective transverse mass (which depends on the state of longitudinal motion) and m^2/α as an internal energy, we see that the transverse motion is given by the nonrelativistic formula $H=p^2/2m+B$. As is well known, Galilean invariance requires total mass conservation which here means conservation of effective transverse mass $\frac{1}{2}\alpha$. The motion of single particles is therefore nonrelativistic motion in two dimensions with each particle possessing an additional degree of freedom—its mass.

Formally the group F together with H contains the two-dimensional Galilean group as a subgroup.[3] The identification is

$$H\rightarrow \text{time displacement},$$
$$k\rightarrow \text{space displacement},$$
$$y,\,z \text{ rotation}\rightarrow y,\,z \text{ rotation},$$
$$P_y\rightarrow \text{velocity transformation in } y \text{ direction},$$
$$P_z\rightarrow \text{velocity transformation in } z \text{ direction},$$
$$\alpha\rightarrow 2\times\text{mass}.$$

The commutation relations given in (3) can be directly checked to see that this subgroup is the Galilean group. Simply stated, P generates a change in k proportional to the effective transverse mass α. This is just what a nonrelativistic velocity transformation does: momentum $\rightarrow$ momentum+mass$\times$velocity.

The lack of vacuum structure at infinite momentum is demanded by a nonrelativistic interpretation. The currents lose their vacuum structure by longitudinal integration which removes the longitudinal dependence of the currents and produces local transverse currents. In fact, there are only three components of the current vectors at infinite momentum, $\mathcal{J}_y$, $\mathcal{J}_z$, and ρ, which serve as the two-component current and the density in the two-dimensional analog. Suppose, for example, that $j(x,y,z,t)$ is a current satisfying a differential conservation law $\partial^i j_i=0$ or

$$g_{ij}[j_i,p_j]=0. \qquad (30)$$

From this it follows that

$$[\mathcal{J}_z,k_z]+[\mathcal{J}_y,k_y]-[\rho,H]=0. \qquad (31)$$

Equation (31) is the local conservation law in two dimensions.

We note that the invariant upon which the matrix elements of the transverse currents $\mathcal{J}$, ρ, and $\mathcal{S}$ depend is $t=-(k-l)^2$, the square of the two-dimensional spatial momentum transfer which bears close analogy to the matrix elements of currents in nonrelativistic quantum mechanics and, unlike relativistic form factors, does not depend on energy differences. As in nonrelativistic physics, the matrix elements of $\mathcal{J}$, ρ, and $\mathcal{S}$ are defined only for $t\leq 0$.

A final remark concerning the nonrelativistic behavior of $\mathcal{J}$ and ρ is that nonrelativistically we think of a current as a density ρ times a local velocity vector u_i. Under a velocity transformation u_i becomes $u_i+\epsilon_i$ and the current changes by $\rho\epsilon_i$. Hence the commutator of a current with the generator of velocity transformation must be $-i\rho$. Furthermore, under velocity transformation, densities do not change nonrelativistically and so the commutator of velocity transformation with ρ should be zero. These physical requirements are reflected in the last two commutation relations of Eqs. (29).

The nonrelativistic behavior of the transverse plane should not be looked upon as a mere curiosity. It is possible that it can be helpful in guessing properties and relations by analogy that could only be understood from conventional methods by laborious calculation. A possible application is to the theory of current algebra.[6]

Since the transverse currents contain no vacuum structure, commutators between such currents should not have the vacuum problems of "Schwinger's disease." We suggest therefore that extensions of current algebra which are attempting to go beyond integrated commutation relations be carried out in terms of the transverse currents $\mathcal{J}(y,z)$ and $\rho(y,z)$. While the determination of commutation relations of such currents may require a model, these relations must be consistent with nonrelativistic commutation relations. It is certainly easier to work with a nonrelativistic two-dimensional model than a relativistic three-dimensional model. For example, commutation relations for the isotopic-spin vector currents may be based on a simple model of nonrelativistic fermion or boson quarks of isospin $\frac{1}{2}$. The results are

$$\begin{aligned}[][\rho^\alpha(y,z),\rho^\beta(y',z')]&=i\epsilon_{\alpha\beta\gamma}\rho^\gamma(y,z)\delta(y-y')\delta(z-z'),\\ \epsilon_{\alpha\beta\gamma}[\rho^\alpha(y,z),\mathcal{J}_i^\beta(y',z')]&=i\mathcal{J}_i^\gamma(y,z)\delta(y-y')\delta(z-z').\end{aligned} \qquad (32)$$

The usual moments of currents can be calculated from the $\mathcal{J}$ and commutation relations can be worked out. For example, the magnetic moment in the x direction is given by

$$\mu_x^\alpha=\int dydz[z\mathcal{J}_y^\alpha-y\mathcal{J}_z^\alpha].$$

The radius squared of the isospin distribution is given by

$$\langle r^2\rangle^\alpha=\tfrac{3}{2}\int dydz\,\rho^\alpha(y,z)(y^2+z^2),$$

[6] M. Gell-Mann, Physics 1, 63 (1964).

and the dipole moment of isospin by

$$D_y{}^\alpha = \int dydz\, y\rho^\alpha(yz).$$

The total isospin is of course

$$I^\alpha = \int dydz\, \rho^\alpha(y,z).$$

In conclusion, we point out that Eq. (25) and its counterpart for $\mathcal{J}$'s pose very deep internal consistency requirements on the structure of the projections, "I bubbles," and expansion functions $e_z(Z)$ entering into the operator T. For T must have an eigenvalue equal to 1 for each current and this eigenvalue must equal 1 for all values of momentum transfer. It is not, at present, clear to us, what in the dynamical theory of Refs. 2 and 3 guarantees such eigenvalues without which the currents cannot exist. In particular it may be that a consistent theory of currents cannot be formulated without including into the theory from the start, those systems to which the currents couple (photons, leptons). At any rate, the solution will certainly await further understanding of the analytic properties of the inner products and the relation of these properties to Lorentz invariance and locality.

The theory presented here and in (3) does not explicitly deal with the problems of spin. Wherever details depend on spin we have assumed scalar particles. The complications due to spin are presently being studied by G. Frye.

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Anomalous and Normal Singularities and the Infinite-Momentum Limit*

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The behavior of the intrinsic size of a bound state as a function of its mass is derived by using an analogy between nonrelativistic two-dimensional quantum mechanics and the infinite-momentum limit of relativistic quantum theory.

ATTEMPTS have been made to understand relativistic form factors for bound states in terms of the use of wave functions.[1,2] In relativistic physics it is found that the familiar nonrelativistic behavior in the limit of loose binding is caused by the existence of anomalous thresholds in the dispersion relations for form factors which manifest themselves in ranges of charge distributions that are larger than the Compton wavelength of the constituent charged particle. The range for loose binding depends on the binding energy and masses according to the formula[1]

$$r^{-2} = 4M^2 - M^4/m^2, \tag{1}$$

where M is the bound-state mass, m the mass of the constituents, and r the range of the distribution of charge. We choose equal-mass constituents for simplicity only.

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[1] R. Karplus, C. Sommerfield, and E. Wichmann, Phys. Rev. **111**, 1187 (1968); R. E. Cutkosky, J. Math. Phys. **429**, 1 (1960).

[2] F. Gross, Phys. Rev. **134B**, 406 (1964); R. Blankenbecler and L. F. Cook, Jr., *ibid.* **119**, 1745 (1960).

In the limit $M \to 2m$, (Eq. 1) agrees with the range given by the nonrelativistic Schrödinger equation, which is

$$r^{-2} = mE, \tag{2}$$

where E is the binding energy of the state.

As M decreases, Eq. (1) continues to hold until $M = \sqrt{2}m$, at which point r equals $(2m)^{-1}$. As M decreases further, Eq. (1) predicts that the range will begin to increase. However, at just this point the anomalous singularity undergoes its well-known disappearance into an unphysical sheet,[1,2] making Eq. (2) invalid. For $M < \sqrt{2}m$ the range is controlled by the normal singularity, which gives

$$r^{-2} = 4m^2. \tag{3}$$

It is a widely held belief that this behavior at $M \leq \sqrt{2}m$ represents very relativistic effects which have no simple interpretation in terms of wave functions. For this reason, it is generally believed that nonrelativistic intuitions are useless for the understanding of deeply bound systems.