Model of Self-Induced Strong Interactions*

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It is postulated that the theory of self-induced strong interactions possesses an infinite-momentum limit. This limit is defined and its consequences are examined. It is found that some important simplifications occur if the limit exists. The limiting form of the theory is shown to have Galilean invariance with respect to motions transverse to the direction in which the momentum is infinite.

I. INTRODUCTION

IN two recent articles,[1,2] the author,[1] with Frye,[2] proposed a theory of "self-induced strong interactions." The form of the theory is greatly complicated by the presence of vacuum structure effects. Furthermore, because the theory is not manifestly covariant, in each approximation the input form of the energy-momentum relation

$$E(p)=(p^2+m^2)^{1/2}$$

will not be reproduced in the output.

Stimulated by Weinberg's[3] observation that vacuum effects disappear in perturbation theory, when the limit of infinite momentum is taken, I have examined the possibility of taking the same limit in the theory of "self-induced strong interactions." Besides eliminating vacuum effects the limiting process is found to have the additional advantage that the form of the energy spectrum is manifestly reproduced at each stage of approximation.

In Sec. II we introduce the infinite-momentum method in terms of a system of free scalar bosons. A remarkable analogy between two-dimensional Galilean invariance and the invariances of the infinite-momentum limit is demonstrated.

In Sec. III the infinite-momentum limit is defined for the theory of "self-induced strong interactions," and it is shown that in the limit the transformation properties of the theory are the same as for free particles for those transformations which were discussed.

The limiting forms of the dynamical equations of Ref. (2) are developed in Sec. IV.

In Sec. V, a formulation of the theory is given in which the search for solutions is reduced to a search for a set of infinite-dimensional matrices satisfying certain algebraic properties.

II. FREE PARTICLES AT INFINITE MOMENTUM

There are two types of reference frames from which we shall describe a system of particles. "Ordinary reference frames" (ORF) are frames in which each particle has a finite-momentum vector. Limiting reference frames (LRF) are frames which move along the z axis in the negative z direction with velocity close to the speed of light (which we take to be 1) relative to the ORF. Of course, for fixed $v<1$ such a frame is also an ORF. However, we are interested in the limit $v\to 1$. A LRF then is one whose velocity is sufficiently near 1 that the limit of all the quantities under consideration have almost been attained.

Quantities measured in the LRF will be primed and those in the ORF will be unprimed. The connections between primed and unprimed quantities are given below.

$$\begin{aligned} x'&=x,\\ y'&=y,\\ z'&=(z+vt)/(1-v^2)^{1/2},\\ t'&=(t+vt)/(1-v^2)^{1/2},\end{aligned}\tag{1}$$

and

$$\begin{aligned} p_x'&=p_x,\\ p_y'&=p_y,\\ p_z'&=(p_z+vp_t)/(1-v^2)^{1/2},\\ p_t'&=(p_t+vp_z)/(1-v^2)^{1/2}.\end{aligned}\tag{1$'$}$$

We consider a scalar particle moving with momentum p as seen by an ORF. From the point of view of the LRF, the z' component of momentum given in Eq. (1′) becomes infinite. Therefore, the description of the motion in the z' direction shall be in terms of the "longitudinal ratio" α, defined by $\alpha=p_z'(1-v^2)^{1/2}$ which tends to a finite limit as $v\to 1$.

Scaling the momentum by the factor $(1-v^2)^{1/2}$ is equivalent to stretching the z' axis by a factor $(1-v^2)^{-1/2}$ which is necessary in order to see any structure in the z' direction for systems which have Lorentz contracted by the factor $(1-v^2)^{1/2}$. In terms of the ORF, $\alpha=p_z+vp_t$, which becomes

$$\alpha=p_z+p_t$$

as $v\to 1$.

When considering the state of a system of free particles from the point of view of ORF, we shall use the notation

$$|\mathbf{p}_1\cdots\mathbf{p}_n\rangle.$$

The description of the same state from the LRF is

$$|(\mathbf{k}\alpha)_1\cdots(\mathbf{k}\alpha)_n\rangle,$$

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[1] L. Susskind, Phys. Rev. **154**, 1411 (1967).
[2] L. Susskind and G. Fyre, Phys. Rev. **164**, 2003 (1967).
[3] S. Weinberg, Phys. Rev. **150**, 1313 (1966).

where $\mathbf{k}$ is a 2-vector in the x,y plane equal to (p_x,p_y), and α_i is the longitudinal ratio for the ith particle, $\alpha_i=(p_z)_i+(p_t)_i$. We note that $\alpha>0$. Whenever possible the combination $(\alpha,\mathbf{k})_i$ will be replaced by a single symbol z_i. Hence a typical state in the LRF is

$$|z_1\cdots z_n\rangle.$$

We are interested in the transformation properties of the $|z_1\cdots z_n\rangle$ under the inhomogeneous Lorentz group.

Let us first consider (finite) boosts in the z direction with velocity u. For such boosts each free-particle momentum transforms as follows:

$$\begin{aligned} p_x &\to p_x,\\ p_y &\to p_y,\\ p_z &\to (p_z+up_t)/(1-u^2)^{1/2},\\ p_t &\to (p_t+up_z)/(1-u^2)^{1/2}, \end{aligned}$$

so that

$$p_z+p_t\to\frac{1+u}{(1-u^2)^{1/2}}(p_z+p_t).$$

Or, from the point of view of LRF:

$$\begin{aligned} \mathbf{k}&\to\mathbf{k},\\ \alpha&\to\lambda(u)\alpha, \qquad (2)\\ \lambda(u)&=(1+u)(1-u)^{-1/2}. \end{aligned}$$

Boost transformations are therefore scale transformations on the α with positive scale factor $\lambda(u)$. Define Λ to be the generator of such boosts so that the unitary boost operator $U(u)=e^{i[\Lambda\ln\lambda(u)]}$ gives

$$U(u)|(\alpha,\mathbf{k})_1\cdots(\alpha,\mathbf{k})_n\rangle=|(\lambda\alpha,\mathbf{k})_1\cdots(\lambda\alpha,\mathbf{k})_n\rangle.$$

Translations in the x,y plane are as usual generated by the x,y components of momentum which are conserved

$$T(\mathbf{a})|z_1\cdots z_n\rangle=\exp(i\mathbf{a}\cdot\textstyle\sum\mathbf{k}_i)|z_1\cdots z_n\rangle,$$

where $T(\mathbf{a})$ is the unitary translator along the 2-vector $\mathbf{a}$. Also, since $\sum\alpha_i$ is proportional to p_z' it is conserved.

Rotations about the z' axis in the LRF are represented in the obvious manner, $\mathbf{k}$ forming a 2-vector and α being invariant.

Two interesting symmetries are derived as follows.

The direction in which the momentum is infinite in a state $|\alpha k\rangle$ may be changed infinitesimally. Consider an infinitesimal rotation of the system through angle ϵ about the y axis. This induces a transformation among the p' components.

$$\begin{aligned} p_y'&\to p_y',\\ p_x'&\to p_x'+\epsilon p_z',\\ p_z'&\to p_z'-\epsilon p_x'. \end{aligned}$$

Expressed in terms of α and $\mathbf{k}$ the transformation is given by

$$\begin{aligned} k_y&\to k_y,\\ k_x&\to k_x+\alpha Q_x, \qquad (3)\\ \alpha&\to\alpha, \end{aligned}$$

where in the limit $v\to1$, ϵ has been allowed to go to zero so that $\epsilon/(1-v^2)^{1/2}\to Q_x$.

Since these transformations are rotations in the LRF, the theory must be manifestly symmetric under them. Similar considerations apply to the rotation about x which mixes p'_z and p'_y, so that Eq. (3) can be generalized to

$$\mathbf{k}\to\mathbf{k}+\alpha Q, \qquad (4)$$

where k and Q are 2-vectors.

In terms of the ORF the transformation (3) is given by

$$\begin{aligned} p_x&\to p_x+Q_x(p_z+p_t),\\ p_z+p_t&\to p_z+p_t,\\ p_y&\to p_y. \end{aligned}$$

This transformation is a homogeneous Lorentz transformation generated by R_y+L_x, where R_y is the rotation generator about y and L_x is the x-direction boost generator. We shall refer to this combination as Q_x.

The motion of the particles in the LRF is generated by p_t' which equals a sum over particles:

$$\begin{aligned} p_t'&=\sum_i[(\mathbf{p}_i')^2+m_i^2]^{1/2}\\ &=\sum_i[\alpha_i^2/(1-v^2)+\mathbf{k}_i^2+m_i^2]^{1/2}\\ &=\frac{1}{(1-v^2)^{1/2}}\sum_i\alpha_i+\sum_i\frac{\mathbf{k}_i^2+m_i}{2\alpha_i}(1-v^2)^{1/2}\\ &\qquad+\text{higher order in }(1-v^2)^{1/2}. \qquad (5) \end{aligned}$$

Consider the possibility of a coherent superposition of two states with different total α. The leading terms in the energies of the states are $\alpha_1/(1-v^2)^{1/2}$ and $\alpha_2/(1-v^2)^{1/2}$. Hence, as $v\to1$ the relative phase of the two terms in the superposition will change infinitely rapidly so that this relative phase is meaningless. Therefore, in the limit $v\to1$ one should introduce a *superselection rule* for total α. In doing so there is no loss of generality.

For interacting particles, which we have not yet introduced, the amplitude for a real transition need only be computed for eigenstates of α. Using the Lorentz invariance of the scattering amplitude we automatically obtain the transition amplitude in an ORF. If the state in question happens to be a coherent superposition of different 4-momentum in the ORF, we can superpose the ORF amplitudes in the customary manner.

Since the leading term $\sum\alpha_i/(1-v^2)^{1/2}$ only multiplies the state vector by an uninteresting phase for eigenstates of $\sum\alpha$, we subtract it out of the expression for the energy in the LRF. Transitions among states of the relative coordinates and transverse x,y motions of the system are generated by the rest of the energy which goes to zero as $(1-v^2)^{1/2}$ corresponding to the fact that all such motions are slowed down by a corresponding factor due to time dilation. Hence it is natural to introduce a new time scale T for the LRF, defined by

$$T=\tfrac{1}{2}t'(1-v^2)^{1/2}.$$

In order to compensate for this, the generator of motion must be multiplied by $2/(1-v^2)^{1/2}$. Hence, we define an effective Hamiltonian by

$$H=\sum_i \frac{k_i^2+m_i^2}{\alpha_i}. \qquad (6)$$

In terms of the ORF's momenta

$$H=\sum_i [(p_x)_i^2+(p_y)_i^2+m_i^2]/[(p_z)_i+(p_t)_i] \\ =(p_t)_i-(p_z)_i. \qquad (7)$$

Thus we have found the following combinations of the elements of the Lorentz algebra of interest:

(1) $k_x=p_x$, $k_y=p_y$; the generators of transverse translations.

(2) $\alpha=p_z+p_t$; the sum of the generators of translation in the z direction and t directions.

(3) R_z, the generator of rotations about the z' axis.

(4) Λ, the generator of boosts in the z direction.

(5) Q_x (Q_y), the sum of the generators of boosts in the x (y) direction and rotations about the y (x) axis.

(6) H, the difference of the translation generators in the time and z directions, p_t-p_z.

This set of eight generators forms a subalgebra of the 10-element Lorentz algebra. The commutation relations can be obtained from the usual Lorentz algebra by making the above identifications and are given below:

$$\begin{gathered}[\alpha,\mathbf{k}]=[\alpha,H]=[\alpha,\mathbf{Q}]=[k,\Lambda]=[\mathbf{k},H] \\ =[R_z,\alpha]=[R_z,H]=0, \\ [\alpha,\Lambda]=-i\alpha, \quad [\mathbf{k},\mathbf{Q}]=-i\alpha, \\ [\Lambda,\mathbf{Q}]=i\mathbf{Q}, \quad [\Lambda,H]=-iH, \\ [\mathbf{Q},H]=2i\mathbf{k}, \quad [R_z,k_i]=i\epsilon_{ij}k_j, \\ [R_z,Q_i]=i\epsilon_{ij}Q_j.\end{gathered} \qquad (8)$$

The seven generators excluding H also form a subalgebra which we call F. In fact, it is easy to show that F generates the subgroup of the Lorentz group which leaves invariant the hypersurface $z+t=0$ in Minkowski space. This hyperplane is tangent to the light cone and is the limit of the surface $t'=0$ for the LRF as $v\to 1$. Hence one may think of the LRF as a reference frame in which "space" is tangent to the light cone. The subgroup F plays the same role in the infinite-momentum limit as the rotations and translations do in an ORF in which they leave invariant the surface $t=0$.

The geometrical significance of the generator $H=p_t-p_z$ from the point of view of the ORF is that it generates motions along the lightlike direction given by

$$dx/dt=dy/dt=0, \quad dz/dt=1.$$

Hence the effective Hamiltonian H generates motions of the surfaces $x+t=c\to x+t=c+\delta$.

Unlike the subgroup of rotations and translations in the ORF which cannot change the magnitude of the spatial momentum of a particle, the subgroup F can act on any particle state $|\alpha,k\rangle$ to give a state of motion of the same particle with any other values of α and k.

Finally, we wish to call attention to a remarkable analogy between infinite-momentum transformation properties and Galilean invariance.

Consider a two-dimensional system of nonrelativistic particles invariant under the Galilean group consisting of the following transformations:

(1) translations generated by the two components of linear momentum called q_x and q_y;

(2) rotations in the x,y plane generated by L; and

(3) velocity boosts (Galilean, not Lorentzian) generated by G_x and G_y.

The energy is

$$E=\sum_i (\mathbf{q}_i^2/2\mu_i)+\sum_{i\neq j} v_{ij}(|\mathbf{r}_i-\mathbf{r}_j|),$$

where μ_i is the mass of the ith particle.

A velocity boost with velocity v translates the momentum of each particle by μv. Hence the unitary operators representing the boosts are

$$e^{iG_x}=\exp[iv\sum_i \mu_i x_i],$$

$$e^{iG_y}=\exp[iv\sum_i \mu_i y_i],$$

where x_i and y_i are the spatial coordinates of the ith particle and the generators are

$$G_x=\sum_i \mu_i x_i,$$
$$G_y=\sum_i \mu_i y_i.$$

From this we deduce the commutation relations for the Galilean group:

$$\begin{gathered}[q_x,G_x]=i\sum\mu_i, \\ [q_y,G_y]=i\sum\mu_i, \\ [q_x,G_y]=0, \\ [G_x,G_y]=0, \\ [G_x,L]=iG_y, \\ [G_y,L]=-iG_x, \\ [E,G_x]=iq_x, \\ [q,E]=0, \\ [L,E]=0.\end{gathered}$$

These commutation relations are identical with those of Eq. (8) if we identify q with k, G with $Q/2$, L with R_z, E with H, and μ_i with $\alpha_i/2$. Thus the infinite-momentum limit is Galilean invariant with $\alpha/2$ playing the role of effective transverse nonrelativistic mass and $Q/2$ replacing the generator of Galilean boosts.

The origin of this symmetry is not difficult to understand. The expression for the Hamiltonian in the LRF is $\sum_i (\mathbf{k}_i^2+m_i^2)/\alpha_i$. The term $\mathbf{k}_i^2/\alpha_i$ is formally identical to the $\mathbf{q}^2/2\mu$ of nonrelativistic physics, and the m_i^2/α_i can be regarded as internal binding energy. The transformations $\mathbf{k}\rightarrow\mathbf{k}+\alpha\mathbf{Q}$, generated by $\mathbf{Q}$, are formally identical to momentum $\rightarrow$ momentum + mass $\times$ velocity.

Thus any relativistic theory, if it has an infinite-momentum limit, must satisfy all the general requirements of Galilean invariance in the LRF. It is therefore in the LRF, not the center-of-mass frame, where our intuitions from nonrelativistic quantum theory may have their most useful analogs.

III. INTERACTING PARTICLES

In Refs. 1 and 2, multiparticle state vectors $|\mathbf{p}_1\cdots\mathbf{p}_n\rangle$ were introduced. The $|\mathbf{p}_1\cdots\mathbf{p}_n\rangle$ are not energy eigenvectors for interacting systems. In a convergent field theory, if $a^\dagger p$ were the polynomial in Fock-space creation operators which acts on the physical vacuum $|0\rangle$ to produce the single-particle momentum-energy eigenvector $|\mathbf{p}\rangle$, then

$$|\mathbf{p}_1\cdots\mathbf{p}_n\rangle=a^\dagger p_1\cdots a^\dagger p_n|0\rangle.$$

Unless the field theory is noninteracting, the $|\mathbf{p}_1\cdots\mathbf{p}_n\rangle$ cannot be expected to be energy eigenvectors since they do not contain the "scattered wave." They are, however, momentum eigenstates.

Under Lorentz transformations the $|\mathbf{p}_1\cdots\mathbf{p}_n\rangle$ for $n>1$ generally become very complicated superpositions of multiparticle states involving different numbers of particles as well as different values of total and submomenta. The physical reason for this is that Lorentz transformations which move the surface $t=0$ (this includes boosts and time translations) leave a region of space-time between the surfaces of simultaneity of the two observers. Interaction, absorption, creation, and scattering of particles can take place in this region causing the connection between the descriptions in the two frames to be complicated. In particular, since the presence of one system will affect the behavior of another in the region between the two spacelike surfaces, the Lorentz transformation will not act as a product transformation on the subsystems of a system.

Those Lorentz transformations which leave invariant the surface $t=0$, namely rotations and spatial translations, act separately on the coordinates of each subsystem.

We again introduce two frames ORF and LRF moving with relative velocity v along the z axis. Although the states $|\mathbf{p}_1\cdots\mathbf{p}_n\rangle$ are not energy eigenvectors, we still define $(p_t)_i\equiv[|\mathbf{p}_i|^2+m_i^2]^{1/2}$. Primed variables are related to unprimed by

$$\begin{aligned} p_x'&=p_x,\\ p_y'&\equiv p_y,\\ p_z'&\equiv[p_z+vp_t]/(1-v^2)^{1/2},\\ p_t'&\equiv[p_t+vp_z]/l(1-v^2)^{1/2}. \end{aligned}\tag{9}$$

We are interested in states $|\mathbf{p}_1'\cdots\mathbf{p}_n'\rangle$ as seen by LRF. These configurations, as observed by ORF, are not simply $|\mathbf{p}_1\cdots\mathbf{p}_n\rangle$ but are $U(v)^{-1}|\mathbf{p}_1'\cdots\mathbf{p}_n'\rangle$, where $U(v)$ is the unitary operator connecting ORF with LRF. Define $|\mathbf{p}_1\cdots\mathbf{p}_n;v\rangle$ by $U^{-1}(v)|\mathbf{p}_1'\cdots\mathbf{p}_n'\rangle=|\mathbf{p}_1\cdots\mathbf{p}_n;v\rangle$. By definition $|\mathbf{p}_1\cdots\mathbf{p}_n;0\rangle$ equals $|\mathbf{p}_1\cdots\mathbf{p}_n\rangle$ and for free particles $|\mathbf{p}_1\cdots\mathbf{p}_n\rangle=|\mathbf{p}_1\cdots\mathbf{p}_n;v\rangle$ for all v. Our basic assumption, which we introduce as a postulate, is that $\lim_{v\to1}|\mathbf{p}_1\cdots\mathbf{p}_n;v\rangle$ exists and that in the limit the structure of the vector space spanned by the limiting vectors and the relations of these vectors are unchanged. The meaning of this latter qualification will become clear as we proceed.

The limiting vectors $\lim_{v\to1}|\mathbf{p}_1\cdots\mathbf{p}_n;v\rangle$ we denote by $|(\alpha\mathbf{k})_1\cdots(\alpha\mathbf{k})_n\rangle$, where as before, $\alpha_i=(p_z)_i+(p_t)_i=\lim p_i'(1-v^2)^{1/2}$ and $k_{x,y}=p_{x,y}$.

According to the assumptions of Ref. 2 the $|\mathbf{p}_1\cdots\mathbf{p}_n;v\rangle$ are complete for any value v. As an example of the meaning of our basic postulate, we assume that the limit vectors $|(\alpha_1,\mathbf{k}_1)\cdots(\alpha_n,\mathbf{k}_n)\rangle$ are also complete.

We shall now prove that the action of the subgroup F, introduced in the previous section, on the $|(\alpha\mathbf{k})_1\cdots(\alpha\mathbf{k})_n\rangle$ is identical to the action of the free-particle transformations. This is obvious for the generators of transverse translations and rotations about z', k_x, k_y, and R_z.

The generator p_z+vp_t is $(1-v^2)^{1/2}p_z'$. Hence it acts on $|p_1\cdots p_n;v\rangle$ to give

$$(1-v^2)^{1/2}\sum_i(p_z')_i|\mathbf{p}_1\cdots\mathbf{p}_n;v\rangle.$$

In the limit $v\rightarrow1$ this gives

$$(p_z+p_t)|z_1\cdots z_n\rangle=\sum\alpha_i|z_1\cdots z_n\rangle,$$

where we have used the notation z_i for $(\alpha k)_i$.

The z-direction boost by velocity $|u|<1$ takes a state which is seen as $|\mathbf{p}_1'\cdots\mathbf{p}_n'\rangle$ by an observer moving relative to ORF with velocity $-v$ into the state $|\mathbf{p}_1'\cdots\mathbf{p}_n'\rangle$ as seen by an observer with velocity $-w=(-u-v)/(1-uv)$ relative to ORF. By definition, this state is

$$\left|p_x,p_y,\frac{p_z+up_t}{(1-u^2)^{1/2}}\cdots;w\right\rangle.$$

Now as $v\rightarrow1$ so does w, so that the boost of $|(\alpha,k)_1\cdots\rangle$ is

$$\lim_{w\to1}\left|(p_x,p_y,\frac{p_z+up_t}{(1-u^2)^{1/2}}\cdots;w\right\rangle.$$

Hence the action of the z boost (called Λ) is again a rescaling of all the α_i.

The generators Q_x, Q_y can be handled by recognizing that they are infinitesimal rotations about the y and x axes, respectively, in the LRF. Their action is therefore identical to the free-particle case.

In Refs. 1 and 2 the $|p_1\cdots p_n\rangle$ were not an orthonormal set because of the extended structure of the physical particles. The inner products were shown to possess a cluster decomposition, identical to that of the S matrix except that since the vectors $|\mathbf{p}_1\cdots\mathbf{p}_n\rangle$ for $n>1$ are not energy eigenvectors, the inner products are not diagonal in the sum of p_t. This allowed nonvanishing of inner products between single- and many-body states and vacuum and many-body states. This is forbidden, of course, for the S matrix.

The connected parts of the inner products are continuous functions apart from an over-all multiplicative δ function of initial less final total momentum which we called $C(q_1\cdots q_m;\, p_1\cdots p_n)$.

In the present development the unhealthy "vacuum pollution" represented by nonvanishing of inner products between the vacuum and multiparticle states is absent because the multiparticle states have $\alpha>0$ and are therefore orthogonal to the vacuum. Actually one must be more careful about this point because the phase space "blows up" at $\alpha\to 0$. However, it is plausible, especially in view of Weinberg's[3] result in perturbation theory, that the vacuum structure effects do disappear as $v\to 1$.

It follows immediately from the invariance of the inner product under the subgroup F of the Lorentz group that the inner products $\langle(\beta\mathbf{l})_1\cdots(\beta\mathbf{l})_n|(\alpha,\mathbf{k})_1\cdots(\alpha,\mathbf{k})_n\rangle$ are unchanged by the following substitutions:

$$\begin{Bmatrix}\alpha_i\to\lambda\alpha_i\\ \beta_j\to\lambda\beta_j\end{Bmatrix}, \tag{10}$$

$$\begin{Bmatrix}\mathbf{k}_i\to\mathbf{k}_i+\alpha_i\mathbf{Q}\\ \mathbf{l}_j\to\mathbf{l}_j+\beta_j\mathbf{Q}\end{Bmatrix}, \tag{11}$$

$$\begin{Bmatrix}k_x+ik_y\to e^{i\theta}(k_x+ik_y)\\ l_x+il_y\to e^{i\theta}(l_x+il_y\ )\end{Bmatrix}. \tag{12}$$

Conservation of total α and $\mathbf{k}$ requires the inner product to be diagonal in these quantities.

Assuming the inner products have the same cluster structure in the LRF as in the ORF, the connected parts

$$C(w_1\cdots w_n;\, z_1\cdots z_m)$$

must have the same invariances as the inner products. The notation w_i is used in bra vectors to denote $(\beta,\mathbf{l})_i$.

A function, invariant under the transformations (10), (11), and (12), which conserves α and $\mathbf{k}$ is called an F invariant. It can be expressed in terms of the invariant coordinates α_i/A, β_j/A, $\mathbf{l}_j-(\beta_j/A)\mathbf{K}$, and $\mathbf{k}_i-(\alpha_i/A)\mathbf{K}$, where $A=\sum_i\alpha_i$ and $\mathbf{K}=\sum_i\mathbf{k}_i$. The F-invariant δ function is defined to be

$$\delta(w_1\cdots w_n;\, z_1\cdots z_m)=(\textstyle\sum_i\alpha_i)\delta(\sum_i\alpha_i-\sum_j\beta_j)\times\delta^2(\sum_i\mathbf{k}_i-\sum_j\mathbf{l}_j). \tag{13}$$

The requirements of $(\alpha,\mathbf{k})$ conservation are expressed by the connected parts C being proportional to $\delta(w\cdots;\, z\cdots)$. For example

$$\langle w|z\rangle=C(w;\,z)=\delta(w;\,z)=\delta(1-\alpha/\beta)\delta^2(\mathbf{k}-\mathbf{l}). \tag{14}$$

We also know about the C's that

$$C(0;\,z\cdots)=0 \tag{15}$$

and

$$C(w\cdots;\,z\cdots)=C(z\cdots;\,w\cdots)^*. \tag{16}$$

In Refs. 1 and 2, the postulated composite nature of the particles was represented by each single-particle state being a superposition of multiparticle states:

$$|\mathbf{p}\rangle=\sum_{n=2}^{\infty}\int e_p(\mathbf{p}_1\cdots\mathbf{p}_n)|\mathbf{p}_1\cdots\mathbf{p}_n\rangle d^n\mathbf{p}. \tag{17}$$

The linear dependence of the single- and many-particle states is postulated to go over in the limit $v\to 1$ to a relation similar to Eq. (17):

$$|z\rangle=\sum_{n=2}^{\infty}\int e_z(z_1\cdots z_n)|z_1\cdots z_n\rangle d^nz. \tag{18}$$

In Eq. (18), $d^nz=dz_1\cdots dz_n$ and $dz=dk_xdk_yd\alpha/\alpha$. The differential volume element dz is F invariant.

Projecting Eq. (18) into $\langle w_1\cdots w_n|$ gives

$$\langle w_1\cdots w_m|z\rangle=\sum_{n=2}^{\infty}\int d^nz e_z(z_1\cdots z_n)\times\langle w_1\cdots w_m|z_1\cdots z_n\rangle. \tag{19}$$

From Eq. (19) and the F invariance of the inner product we conclude that $e_z(z_1\cdots z_n)$ must be an F-invariant function of z and z_i.

We shall also require a set of coefficients which resolve the identity. We call them $I(z\cdots;\,w\cdots)$.

$$\sum_{m,n=0}^{\infty}\int d^mzd^nw|z_1\cdots z_m\rangle\langle w_1\cdots w_n|\times I(z_1\cdots z_m;\,w_1\cdots w_n)=I. \tag{20}$$

Taking matrix elements of Eq. (20) gives

$$\langle w'\cdots|z'\cdots\rangle=\sum_{m,n=0}^{\infty}\int d^mzd^nw\langle w'\cdots|z_1\cdots z_m\rangle\times I(z_1\cdots z_m;\,w_1\cdots w_n)\times\langle w_1\cdots w_m|z'\cdots\rangle. \tag{21}$$

The F invariance of the inner product and differential dz require the I to be F-invariant functions of w_j and

z_i. We have formally included the vacuum projection operator into Eq. (20) by summing over $n=m=0$. The coefficients $I(0;0)$, $I(z;0)$, and $I(0,w)$ equal 1, 0, and 0, respectively.

In Ref. 2, the inner products satisfied an internal particle "X" representation. We postulate the correctness of the representation in the limit. Since the vacuum projections $\langle 0|z_1\cdots z_m\rangle$ are zero, the X representation is not complicated by vacuum terms. We use the following cluster notation:

Z or W stands for an entire cluster of particles. For example,

$$Z=(z_1\cdots z_m),\quad W=(w_1\cdots w_n).$$

The total transverse momentum of $Z(W)$ is $\mathbf{K}(\mathbf{L})$ and the total longitudinal ratio is $A(B)$. The vacuum is formally included in the cluster notation as in Ref. 2 by a "null" or "empty" cluster. Hence we have in symbolic notation that

$$\sum_{ZW}|Z\rangle I(Z;W)\langle W|=I\,,$$

where $\sum_{WZ}$ means summation and F-symmetric integration over the clusters Z and W.

The X representation is given by

$$\begin{aligned}&\langle w_1\cdots w_n|z_1\cdots z_m\rangle\\ &=\sum_{W,Z}\langle w_1|Z_{11},Z_{12}\cdots Z_{1m}\cdots\langle w_n|Z_{n1}\cdots Z_{nm}\rangle\\ &\times\langle W_{11}\cdots W_{n1}|z_1\rangle\cdots\langle W_{1m}\cdots W_{nm}|z_m\rangle\\ &\qquad\times I(Z_{11};W_{11})I(Z_{12};W_{12})\cdots I(Z_{mn};W_{mn}).\quad (22)\end{aligned}$$

We state a theorem which is proved by inspection.

Theorem: The right-hand side of Eq. (22) is F invariant if the single-particle projections, $\langle w_i|Z\cdots\rangle$ and $\langle W\cdots|z_i\rangle$, and the I's are. Hence an iteration method may be used to obtain the multiparticle overlaps, which at each stage yields F-invariant projections. The first step of the iteration is to replace the sum over intermediate I's by vacuum and single-particle contributions.

The resulting overlaps imply a set of I functions which can be obtained by the method of Ref. 2. The new I functions can be used in the second iteration to improve the projections through the X representation.

In first approximation

$$I=|0\rangle\langle 0|+|z\rangle\langle w|\,\delta(1-\alpha/\beta)\delta^2(\mathbf{k}-\mathbf{l}).\qquad (23)$$

IV. DYNAMICAL EQUATIONS

The generator of time translations in the LRF is related to the components of 4-momentum in the ORF by

$$p_t'=(p_t+vp_z)/(1-v^2)^{1/2}.\qquad (24)$$

This energy becomes infinite as $v\to 1$. However, the leading term in the energy is proportional to p_z'. If we are working with a system which is in an eigenstate of the total z' momentum we can subtract any function of p_z' and p_t' at the expense of doing nothing more drastic to the time development of the state than multiplying it by a time-dependent phase factor. We shall therefore subtract from p_t' its infinite part, p_z'. We stress that this does not prevent us from considering systems which contain subsystems with indefinite p_z' as long as the total p_z' is well defined. In terms of ORF variables, the "effective Hamiltonian" $p_t'-p_z'$ for LRF is given by

$$p_t'-p_z'=\frac{p_t+vp_z}{(1-v^2)^{1/2}}-\frac{p_z+vp_t}{(1-v^2)^{1/2}},$$

which for $v\to 1$ becomes

$$(p_t-p_z)(1-v^2)^{1/2}/2.$$

This is the "real" transition-producing part of p_t' which goes to zero as $v\to 1$ in a manner consistent with the time dilation for internal processes of a system with velocity near 1. As for free particles, a change in time scale eliminates the factor $(1-v^2)^{1/2}/2$, leaving the LRF effective Hamiltonian

$$H=p_t-p_z.\qquad (25)$$

We define the total mass squared invariant $p_t^2-p_x^2-p_y^2-p_z^2$ to be S. Using the definitions of α as p_x+p_t and H as p_x-p_t, the invariant S may be expressed in the LRF as

$$S=H\alpha-\mathbf{k}^2,$$

which gives

$$H=(\mathbf{k}^2+S)/\alpha.\qquad (26)$$

Note the similarity between Eq. (26) and Eq. (6). In fact for a system of free particles, Eq. (26) reduces to Eq. (6).

Since S is Lorentz invariant it follows that it is also F invariant.

In terms of matrix elements, Eq. (26) takes the form

$$\langle W|H|Z\rangle=\langle W|Z\rangle|\mathbf{K}|^2/A+S(W;Z)/A.\qquad (27)$$

In Eq. (27) $S(W;Z)$ are the matrix elements of the F-invariant operator S and are therefore F-invariant functions of the z_i and w_i.

In Ref. 2 it was proved that the matrix elements of the Hamiltonian satsify a cluster decomposition which follows from the cluster decomposition of the unitary time development operator. The cluster rule for energy is based on the cluster decomposition of the inner product. For each term in the cluster decomposition of $\langle W|Z\rangle$ containing N disjoint clusters, there are N terms in the decomposition of $\langle W|H|Z\rangle$, each one obtained by substituting for one of the C clusters a connected function $H_c(w\cdots;z\cdots)$, leaving the other $N-1$ factors as C functions.

We shall use just such a cluster structure for the matrix elements of the effective Hamiltonian in the LRF.

The justification for this is that both the operators p_t and p_z satsify this cluster rule. In terms of the functional methods introduced in Ref. 2, a set of functions $T(\mathbf{q}_1\cdots\mathbf{q}_m;\mathbf{p}_1\cdots\mathbf{p}_n)$ satisfies the cluster structure proposed for H if the generating functional $\mathcal{T}$ of T satisfies

$$\mathcal{T}=\mathcal{T}_c\exp[\mathcal{C}],$$

where $\mathcal{T}_c$ is a connected functional and $\mathcal{C}$ is the generating functional for the $C(W;Z)$. Hence, since p_t' and p_z' are the energy and momentum in the LRF, the functionals which generate their matrix elements satisfy

$$\mathcal{P}_t'=(\mathcal{P}_t')_c\exp\mathcal{C},$$
$$\mathcal{P}_z'=(\mathcal{P}_z')_c\exp\mathcal{C},$$

and therefore

$$\mathcal{H}=(\mathcal{P}_t'-\mathcal{P}_z')\frac{2}{(1-v^2)^{1/2}}=\mathcal{H}_c\exp\mathcal{C}.$$

The structure of the functions $H_c(W;Z)$ can be clarified by considering the fully connected part of Eq. (27). This gives

$$H_c(W;Z)=(K^2/A)C(W;Z)+S_c(W;Z)/A. \tag{28}$$

Given that H satisfies the cluster structure $\mathcal{H}=\mathcal{H}_c\exp\mathcal{C}$, Eq. (28) is necessary and sufficient for Eq. (27).

In Eq. (28), $S_c(W;Z)$ is the momentum-conserving, F-invariant, connected part of $S(W;Z)$.

For the single particle

$$\langle w|H|z\rangle=H_c(w;z)=\delta(w;z)(\mathbf{k}^2+m^2)/\alpha, \tag{29}$$

from which it follows that

$$S_c(w;z)=\delta(w;z))m^2. \tag{30}$$

A significant feature of Eqs. (29) and (28) is that the structure of Eq. (29) is reproduced under the expansion of the single-particle states:

$$\begin{aligned}\langle w|H|z\rangle&=\delta(w;z)(\mathbf{k}^2+m^2)/\alpha\\&=\sum_{ZW}e_z(Z)e_w(W)^*\langle W|H|Z\rangle.\end{aligned} \tag{31}$$

In Eq. (31), $\sum_{ZW}$ means summation and F-symmetric integration over the variables describing Z and W. In order to prove Eq. (31) consider the particular terms in the cluster expansion of H which contains N connected C clusters and one H_c. Such a term gives a contribution to the right-hand side of Eq. (31) of the form

$$\begin{aligned}\frac{1}{N!}\sum_{\text{perm}}\sum_{ZW}e_z(Z_1\cdots Z_{N+1})e_w(W_1\cdots W_{N+1})^*H_c(W_1;Z_1)\\\times C(W_2;Z_2)\cdots C(W_{N+1};Z_{N+1}).\end{aligned} \tag{32}$$

The sum over permutations means a sum over every different partition of the z_i and w_j into the clusters Z_1-Z_{N+1} and W_1-W_{N+1}.

Inserting Eq. (28) into Eq. (32) gives

$$\begin{aligned}\langle w|H|z\rangle=&\frac{1}{\alpha}\sum_N\frac{1}{(N+1)!}\sum_{\text{perm}}\sum_{ZW}e_z(Z_1\cdots Z_{N+1})e_w(W_1\cdots W_{N+1})^*\left[\frac{|\mathbf{K}_1|^2}{A_1/\alpha}+\cdots+\frac{|\mathbf{K}_{N+1}|^2}{A_{N+1}/\alpha}\right]C(W_1;Z_1)\cdots\\&\times C(W_{N+1};Z_{N+1})+\frac{1}{\alpha}\sum_N\frac{1}{N!}\sum_{\text{perm}}\sum_{ZW}e_z(Z_1\cdots Z_{N+1})e_w(W_1\cdots W_{N+1})^*\frac{\alpha}{A_1}S_c(W_1;Z_1)C(W_2;Z_2)\cdots\\&\times C(W_{N+1};Z_{N+1}).\end{aligned} \tag{33}$$

In Eq. (33), α is the longitudinal ratio for the external particle z. Since each quantity in Eq. (33) has a transverse-momentum- and longitudinal-ratio-conserving δ function, the entire expression is proportional to $\delta(w;z)$. The entire second term which multiplies $1/\alpha$ can easily be seen to be F invariant, and, therefore, has the form $g_1\delta(w;z)$, with g_1 just a number.

In the first term, $\sum_{i=1}^{N+1}|\mathbf{K}_i|^2/(A_i/\alpha)$ is written as

$$|\mathbf{k}|^2+\left[\sum_{i=1}^{N+1}\frac{|\mathbf{K}_i|^2}{A_i/\alpha}-|\mathbf{k}|^2\right],$$

where

$$\mathbf{k}=\sum_{i=1}^{N+1}\mathbf{K}_i,$$

and is, in fact, the transverse momenta associated with z. The $|\mathbf{k}|^2$ term gives a contribution to Eq. (33) of the form

$$\begin{aligned}\frac{|\mathbf{k}|^2}{\alpha}\sum_N\frac{1}{N!}\sum_{\text{perm}}\sum_{ZW}e_z(Z_1\cdots Z_N)e_w(W_1\cdots W_N)^*C(W_1;Z_1)\cdots C(W_N;Z_N)\\=\frac{|\mathbf{k}|^2}{\alpha}\sum_{WZ}e_z(Z)e_w(W)^*\langle W|Z\rangle=\tfrac{1}{2}|\mathbf{k}|^2\langle w|z\rangle=\tfrac{1}{2}|\mathbf{k}|^2\delta(w;z).\end{aligned} \tag{34}$$

The term involving $\sum |\mathbf{K}_i|^2/(A_i/\alpha)-|\mathbf{k}|^2$ is F invariant since $\sum (\alpha/A_i)|\mathbf{K}_i|^2-|\mathbf{k}|^2$ is F invariant. The invariance under Λ transformations is obvious since the only longitudinal dependence is in the A_i/α, which are F invariant. Under $\mathbf{Q}$ transformations we have

$$\alpha[\sum |\mathbf{K}_i|^2/A_i-|\mathbf{k}|^2/\alpha]\to\alpha[\sum |\mathbf{K}_i+A_i\mathbf{Q}|^2/A_i-|\mathbf{k}+\alpha\mathbf{Q}|^2/\alpha]=\alpha[\sum |\mathbf{K}_i|^2/A_i-|\mathbf{k}|^2/\alpha].$$

Hence this term is F invariant and gives a contribution to Eq. (33) of the form $(g_2/\alpha)\delta(w;z)$. The entire right-hand side of Eq. (33) is

$$[|\mathbf{k}|^2/\alpha+(g_1+g_2)/\alpha]\delta(w;z),$$

which proves the assertion.

The Schrödinger equation for the single-particle state $|z\rangle$ is

$$H|z\rangle=(|\mathbf{k}|^2+m^2)/\alpha|z\rangle. \tag{35}$$

Expanding $|z\rangle$ in multiparticle states and taking matrix elements gives

$$\sum_Z \langle W|H|Z\rangle e_z(Z)=\frac{|\mathbf{k}|^2+m^2}{\alpha}\sum_Z \langle W|Z\rangle e_z(Z). \tag{36}$$

Using the cluster decomposition of H as in Eq. (33) gives

$$\sum_N \frac{1}{N!}\sum_{\text{perm}}\sum_Z e_z(Z_1\cdots Z_{N+1})H_c(W_1;Z_1)C(W_2;Z_2)\cdots C(W_{N+1};Z_{N+1})$$
$$=\frac{|\mathbf{k}|^2+m^2}{\alpha}\sum_N\frac{1}{N!}\sum_Z\sum_{\text{perm}} e_z(Z_1\cdots Z_N)C(W_1;Z_1)\cdots C(W_N;Z_N), \tag{37}$$

which by Eq. (28) becomes after some rearrangement

$$\sum_N \frac{1}{N!}\sum_{\text{perm}}\sum_{WZ} e_z(Z_1\cdots Z_N)\left\{\left[\frac{|\mathbf{K}_1|^2}{A_1/\alpha}+\cdots+\frac{|\mathbf{K}_N|^2}{A_N/\alpha}-|\mathbf{k}|^2\right]C(W_1;Z_1)+NS_c(W_1;Z_1)\right\}C(W_2;Z_2)\cdots C(W_N;Z_N)$$
$$=m^2\sum_N\frac{1}{N!}\sum_{\text{perm}}\sum_{WZ} e_z(Z_1\cdots Z_N)C(W_1;Z_1)\cdots C(W_N;Z_N). \tag{38}$$

Both the left-hand side and right-hand side of Eq. (38) are F invariant. The equation is a linear eigenvalue, eigenvector problem for the eigenvalue m^2 and eigenvector $e_z(z_1\cdots z_m)$. *The F invariance of the equations means that the problem can be solved for a particular choice of α and $\mathbf{k}$ say $\alpha=1$ and $\mathbf{k}=0$, and the solution can be continued to all α and $\mathbf{k}$ by F transformations.*

The $\mathfrak{X}$ representation for H postulated in Refs. 1 and 2 is assumed in the present model:

$$\langle w_1\cdots w_n|H|z_1\cdots z_m\rangle=\sum_{l=1}^{n}\sum_{W,Z}\prod_{ij}\langle w_i|Z_{i1}\cdots Z_{im}\rangle\frac{\langle w_l|H|Z_{l_1}\cdots Z_{l_m}\rangle}{\langle w_l|Z_{l_1}\cdots Z_{l_m}\rangle}I(Z_{ij};W_{ij})\langle W_{i1}\cdots W_{nj}|zj\rangle$$
$$+\sum_{l=1}^{m}\sum_{WZ}\prod_{ij}\langle w_i|Z_{i1}\cdots Z_{im}\rangle I(Z_{ij};W_{ij})\frac{\langle W_{1_l}W_{2_l}\cdots W_{n_l}|H|z_l\rangle}{\langle W_{1_l},W_{2_l}\cdots W_{n_l}|z_{l\cdot}\rangle}\langle W_{ij}\cdots W_{nj}|z_j\rangle-\sum_{kl}\sum_{W'W,ZZ'}\prod_{ij}$$
$$\times\langle w_i|Z_{i1}\cdots Z_{im}\rangle I(Z_{ij};W_{ij})\langle W_{1j}\cdots W_{nj}|z_j\rangle\frac{I(Z_{kl};W')H(W';Z')I(Z';W_{kl})}{I(Z_{kl};W_{kl})}. \tag{39}$$

Substituting Eq. (28) for H produces a set of matrix elements whose connected structure is given by Eq. (28). Hence the $\mathfrak{X}$ representation is consistent with the transformation properties of H.

The matrix elements of H may be calculated by an iteration method proposed in (1) and (2). Since

$$\langle w|H|z\rangle=\langle w|Z\rangle(|\mathbf{l}|^2+m^2)/\alpha,$$

the first two terms of Eq. (39) give

$$\left[\sum_1^m\frac{|\mathbf{K}_i|^2}{\alpha_i}+\sum_1^n\frac{|\mathbf{l}_j|^2}{\beta_j}+\sum_i\frac{m^2}{\alpha_i}+\sum_j\frac{m^2}{\beta_j}\right]$$
$$\times\langle w_1\cdots w_n|z_1\cdots z_m\rangle.$$

The third term is approximated by including only the effect of the single-particle configurations in the intermediate sum over H. The resulting H may be used in the $\mathfrak{X}$ representation to give an improved approximation.

From the discussion beginning with Eq. (31) and ending above, we see that the manifest F invariance of the equations guarantees that the form of the energy spectrum used as input into the cycle of equations is reproduced in the output.

V. ABSTRACT FORMULATION

In this section we assume the space of states to be a Hilbert space with a countable orthonormal basis. The vacuum $|0\rangle$ is chosen as one of the basis elements and the remainder of the space is spanned by the vectors $|i\rangle$, i being an integer. The basis vectors satisfy

$$\langle j|i\rangle=\delta_{ij},\quad \langle 0|i\rangle=0,\quad \langle 0|0\rangle=1$$

Each state $|i\rangle$ is a superposition of multiparticle states and is generated by a creation operator $A_i^\dagger$, which is a polynomial of particle creation operators.

$$|i\rangle=A_i^\dagger|0\rangle=\sum_m\int d^mzf_i(z_1\cdots z_m)|z_1\cdots z_m\rangle,$$

$$A_i^\dagger=\sum_m\int d^mzf_i(z_1\cdots z_m)a^\dagger z_1\cdots a^\dagger z_m. \tag{40}$$

Since the $|i\rangle$ are complete, $A_k^\dagger A_l^\dagger$ can be expanded as

$$A_k^\dagger A_l^\dagger|0\rangle=P_{kl}{}^iA_i^\dagger|0\rangle$$

or

$$A_k^\dagger A_l^\dagger=P_{kl}{}^iA_i^\dagger, \tag{41}$$

with

$$P_{kl}{}^i=\langle 0|A_iA_k^\dagger A_l^\dagger|0\rangle=\langle i|kl\rangle.$$

The complex conjugate of $P_{kl}{}^i$ will be denoted by $Q_{kl}{}^i$.

In general we define $A_i^\dagger A_k^\dagger\cdots A_l^\dagger|0\rangle$ to be $|i,k\cdots l\rangle$. The P's satisfy the following conditions.

A. Expandability

Each $|i\rangle$ can be expanded in the $|kj\rangle$ in a nonunique manner. This follows since

$$|i\rangle=\sum_2^\infty\int f_i(z_1\cdots z_n)|z_1\cdots z_n\rangle+\int f_i(z)|z\rangle.$$

Expanding the single-particle contribution into two and more particles allows an expansion containing no linear terms in the $a^\dagger$.

$$A_i^\dagger=\sum_2^\infty F_i(z_1\cdots z_i,\, z_{i+1}\cdots z_m)(a_{z_1}{}^\dagger\cdots a_{z_i}{}^\dagger)\times(a_{z_{i+1}}{}^\dagger\cdots a_{z_m}{}^\dagger).$$

Each product of $a^\dagger$ in the expansion of $A_i^\dagger$ has been partitioned into two factors. By the completeness of the $|i\rangle$ each operator $a_{z_1}{}^\dagger\cdots a_{z_i}{}^\dagger$ can be written as a linear sum of $A^\dagger$'s. Hence

$$A_i^\dagger=e_i(j,k)A_j^\dagger A_k^\dagger. \tag{42}$$

The condition $\langle l|i\rangle=\delta_{il}$ gives

$$\langle 0|A_lA_i^\dagger\rangle=\langle 0|A_lA_j^\dagger A_k^\dagger|0\rangle e_i(j,k)=e_i(j,k)P_{jk}{}^l=\delta_{il}. \tag{43}$$

Let us consider $P_{jk}{}^l$ to be the l component of the (jk) column vector P_{jk}. Equation (43) says that the vectors P_{jk} are complete since any basis vector δ_{il} can be expanded. Hence there is no vector orthogonal to all P_{jk}, which means that there does not exist a vector R with components R^l such that $P_{jk}{}^lR^l=0$ for all j, k unless $R^l\equiv 0$. *Equivalently, the matrices P^l with (jk) element $P_{jk}{}^l$ are linearly independent.*

B. Symmetry

For boson systems $\langle i|jk\rangle=\langle i|kj\rangle$ so that the matrices P^l are symmetric. Also $A_i^\dagger A_j^\dagger A_k^\dagger=P_{ij}{}^lP_{lk}{}^mA_m^\dagger$. Hence

$$P_{ij}{}^lP_{kl}{}^m=P_{kj}{}^lP_{il}{}^m=P_{ki}{}^lP_{jl}{}^m. \tag{44}$$

C. X Representation

Since the $|m\rangle$ are an orthonormal basis for the space orthogonal to $|0\rangle$, we can evaluate $\langle ij|kl\rangle$ in terms of the P's:

$$\langle ij|kl\rangle=\langle ij|m\rangle\langle m|kl\rangle=Q_{ij}{}^mP_{kl}{}^m. \tag{45}$$

A second formula for the same quantity follows from the 2-cluster X representation.

$$\langle ij|kl\rangle=\sum_{ZW}\langle i|Z_{ik},Z_{il}\rangle\langle j|Z_{jk},Z_{jl}\rangle \times\langle W_{ik},W_{jk}|k\rangle\langle W_{il},W_{jl}|l\rangle I(Z_{ik};W_{ik})I(Z_{il};W_{il}) \times I(Z_{jk};W_{jk})I(Z_{lj};W_{lj}), \tag{46}$$

where the Z's run over all particle configurations including the null configuration.

The intermediate sums over $|Z\rangle I(Z;W)\langle W|$ can be replaced by $|i\rangle\langle j|\delta_{ij}+|0\rangle\langle 0|$, giving

$$\langle ij|kl\rangle=\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}+P_{mk}{}^iQ_{mi}{}^l+P_{ml}{}^iQ_{mj}{}^k +P_{km}{}^iQ_{mj}{}^l+P_{ml}{}^jQ_{mi}{}^k+P_{mn}{}^iQ_{mr}{}^lP_{rs}{}^jQ_{sn}{}^k. \tag{47}$$

Defining a set of new matrices, p^i, by

$$p_{jk}{}^i=P_{jk}{}^i,\quad \text{for } i,j,k\neq 0$$
$$p_{jk}{}^0=0,\quad \text{for } j\neq 0 \text{ or } k\neq 0$$
$$p_{00}{}^0=1,\quad p_{j0}{}^i=p_{0j}{}^i=\delta_{ij}$$

and their complex conjugates $q_{ik}{}^i$, Eqs. (47) and (45) give

$$q_{ij}{}^mp_{kl}{}^m=\mathrm{Tr}[p^iq^lp^jq^k]. \tag{48}$$

If we denote $p_{kl}{}^m(q_{ij}{}^m)$ graphically by a bubble with $p(q)$ in it and use the convention that internal lines are summed over, Eq. (48) takes the graphical form shown in Fig. 1.

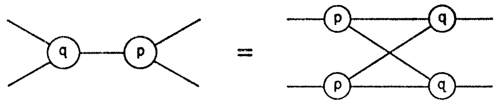

FIG. 1. The X representation for $\langle ij|kl\rangle$.

D. Existence of Space-Time Generators

The space-time group generated by α, the total longitudinal ratio; $\mathbf{k}$, the total transverse momentum; Λ, the generator of z boosts; $\mathbf{Q}$, the generator of transverse "Galilean" transformations; R_z, the rotation about the z axis; and H, the Hamiltonian, must be reflected in the space of states. The generators are to satisfy certain $\mathfrak{X}$-type representations and commutation relations.

Consider first α, R_z, $\mathbf{k}$, Λ, and $\mathbf{Q}$. When not distinguishing between these we shall call them F. The transformations generated by F act independently on the subsystems of a composite system. Suppose $e^{i\epsilon F}|i\rangle = U_{ij}|j\rangle$. Then $e^{i\epsilon F}|ik\rangle = U_{ij}U_{kl}|jl\rangle$. In terms of F itself

$$F|ik\rangle = F_{ji}|jk\rangle + F_{jk}|ij\rangle, \tag{49}$$

where

$$F_{ji} = \langle j|F|i\rangle.$$

We shall derive a representation for the matrix elements of F which reflects the special way in which F acts.

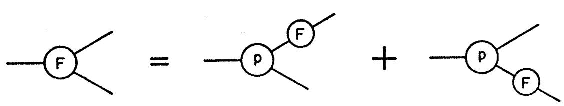

FIG. 2. Representation for $\langle i|F|jk\rangle$.

We start with the matrix element $\langle l|F|ij\rangle$:

$$\langle l|F|ij\rangle = \langle l|qj\rangle\langle q|F|i\rangle + \langle l|iq\rangle\langle q|F|j\rangle = \langle l|qj\rangle F_{qi} + \langle l|iq\rangle F_{qj}. \tag{50}$$

This is shown in Fig. 2.

Next, consider $\langle ij|F|kl\rangle$ which equals

$$\langle ij|ql\rangle F_{qk} + \langle ij|kq\rangle F_{ql}$$

as in Fig. 3. Using the X representation for $\langle ij|ql\rangle$ and $\langle ij|kq\rangle$ gives Fig. 4 for the matrix elements of F.

The combination $\langle mn|q\rangle\langle q|F|k\rangle$ appears and is equal to $\langle mn|F|k\rangle$. Using Fig. 2 gives the symmetrical representation of Fig. 5.

A reversal of the above argument leads to a representation with the F bubbles to the left analogous to Fig. 4. See Fig. 6.

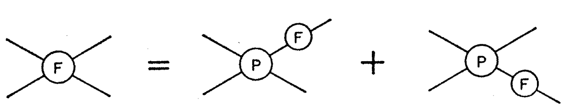

FIG. 3. Representation for $\langle ij|F|kl\rangle$.

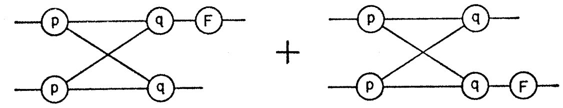

FIG. 4. Result of expanding Fig. 3 using the X representation.

In terms of the p's, Figs. 4 and 5 read

$$\begin{aligned}\langle ij|F|kl\rangle &= \mathrm{Tr}[p^i q^q p^j q^l]F_{qk} + \mathrm{Tr}[p^i q^k p^j q^q]F_{ql} \\ &= \mathrm{Tr}[p^i q^k p^j q^l F] + \mathrm{Tr}[p^i q^k p^j F q^l] \\ &\quad + \mathrm{Tr}[p^i q^k F p^j q^l] + \mathrm{Tr}[p^i F q^k p^j q^l].\end{aligned} \tag{51}$$

Consistency relations for F are obtained by equating the expression implied by Fig. 5 to $\langle ij/m\rangle F_{mn}\langle n|kl\rangle$. This is shown in Fig. 7.

Let us now evaluate the commutator of two F-type operators F and G:

$$\langle ij|[F,G]|kl\rangle = \langle ij|FG|kl\rangle - \langle ij|GF|kl\rangle.$$

The first term is evaluated by inserting a complete set of states between F and G giving the expression in Fig. 8. We use Fig. 1 in Fig. 8 to get Fig. 9.

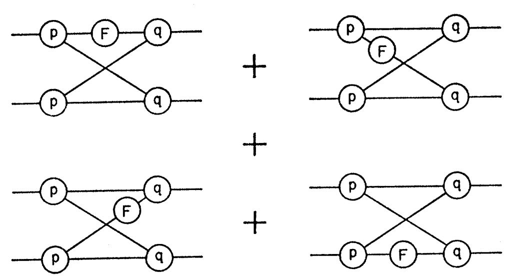

FIG. 5 Another representation for $\langle ij|F|kl\rangle$.

Using Fig. 2 for both F and G gives 16 terms, each containing a single F and G bubble on internal lines. Four such terms will have F and G on the same line with F to the left of G. One such term is indicated in Fig. 10. The other terms have F and G on different lines as in Fig. 11.

When evaluating the product GF the terms analogous to Fig. 10 will have G and F in opposite order while the terms in which F and G are on different lines are identical for both orders of multiplication. Hence only the terms like Fig. 10 will survive the commutation. The result is given in Fig. 12.

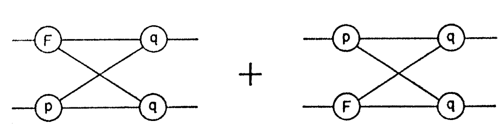

FIG. 6. Yet another representation for $\langle ij|F|kl\rangle$.

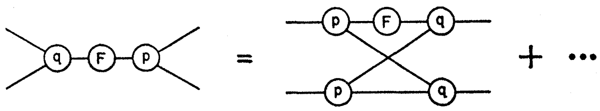

FIG. 7. Consistency requirements on the matrix elements of F.

In Fig. 12 the E bubble stands for

$$E_{ij}=F_{ik}G_{kj}-G_{ik}F_{kj}=[F,G]_{ij}.$$

Hence the commutator of two operators satisfying Eq. (51) is another such operator.

The matrix elements of H satisfy a generalization of Eq. (51) obtained directly from the $\mathfrak{X}$ representation.

Define $H_{ij}=\langle i|H|j\rangle$ to be the ij element of a matrix H. Also let $\langle i|H|jk\rangle=H_{il}P_{jk}{}^{l}=H_{jk}{}^{i}$ be the jk element

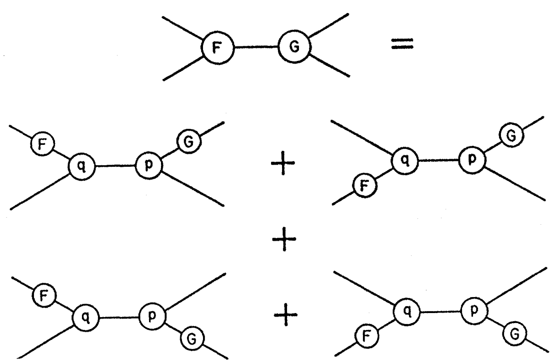

FIG. 8. The commutator of two F-type operators.

of the matrix H^i whose adjoint is $\bar{H}^i$. Then

$$\begin{aligned}\langle ij|H|kl\rangle &= \mathrm{Tr}[H^iq^kp^jq^l]+\mathrm{Tr}[p^i\bar{H}^kp^jq^l]\\ &+\mathrm{Tr}[p^iq^kH^jq^l]+\mathrm{Tr}[p^iq^kp^j\bar{H}^l]\\ &-\mathrm{Tr}[p^iq^kp^jq^lH]-\mathrm{Tr}[p^iq^kp^jHq^l]\\ &-\mathrm{Tr}[p^iq^kHp^jq^l]-\mathrm{Tr}[p^iHq^kp^jq^l], \qquad (52)\end{aligned}$$

which for consistency must equal

$$q_{ij}{}^{m}H_{mn}p_{kl}{}^{n}. \qquad (53)$$

An easy calculation shows that commuting any operator satisfying Eq. (52) with any operator satisfying Eq. (45) produces an operator satisfying Eq. (52). Since operators satisfying Eq. (51) also satisfy Figs. 4 and 6, they also satisfy Eq. (52). This is seen by adding Figs. 4 and 6 and subtracting Eq. (51). Hence commuting H with an F produces an operator with structure similar to H or F.

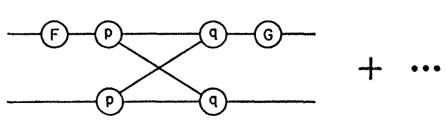

FIG. 9. Expansion of Fig. 8 using the X representation.

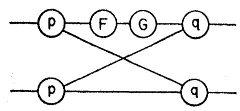

FIG. 10. A term in $[F,G]$ with F and G on the same internal line.

From Eq. (8) we see that the commutator of any pair of F's is an F and the commutator of Λ with H is an H. The last commutation relation between $\mathbf{Q}$ and H gives an F.

By working in a representation in which α and $\mathbf{k}$ are diagonal, as in the previous section, the existence and commutation relations of the F's are assured.

Hence the study of solutions of the theory may be reduced to a search for sets of *linearly independent* matrices P^i satisfying Eqs. (44) and (48) and admitting the existence of the matrices H_{ij} and F_{ij} with the correct $\mathfrak{X}$ representations and commutation relations.

An interesting question is whether the group structure puts enough constraints on the spectrum of H, α, and $\mathbf{k}$ to limit solutions of the abstract problem to those with a physically realistic particle description such as the description we started with.

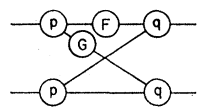

FIG. 11. A term in $[F,G]$ with F and G on different lines.

Two more generators must be added in order that the full Lorentz algebra be represented. We can take them to be the rotations R_x and R_y which mix the longitudinal direction with the transverse directions in the ORF. Invariance with respect to these rotations is a very nontrivial requirement which we shall not discuss here except to remark that R must have the $\mathfrak{X}$ structure of the Hamiltonian and not the simpler structure of the F operators. This follows from the commutation relation

$$[R_x,k_y]=\tfrac{1}{2}i(\alpha-H). \qquad (54)$$

Since $\mathbf{k}$ has the F-type structure, if R also was an F-type operator, the commutator would also be F type. However, the presence of H on the right-hand side of Eq. (54) gives the commutator the more complicated $\mathfrak{X}$ structure which can only be if R has $\mathfrak{X}$ structure. Note also that like H, R generates transformations which displace the surfaces $(x+t)=c$.

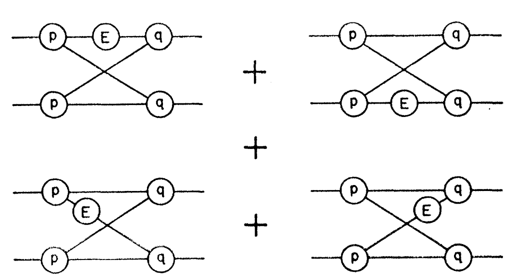

FIG. 12. Representation for $E=[F,G]$.

VI. ABSENCE OF VACUUM STRUCTURE

In Ref. (2) the presence of vacuum structure presented grave complications to the structure of the inner products and Hamiltonian. In the present model we have ignored vacuum structure altogether. This is consistent with a theorem of Weinberg,[3] which states that in field-theoretic perturbation theory, vacuum structure effects go to zero for infinite-momentum observers.

Qualitatively, the reason is that each constituent of any state in our model carries a positive longitudinal ratio corresponding to a positive fraction of the infinite momentum. It is not possible to avoid having at least one line coming from a vacuum bubble connecting into one of the lines carrying infinite momentum. If the vacuum clusters fall off to zero as the momentum of the legs becomes infinite, then the vacuum terms will not be able to connect into the X or $\mathfrak{X}$ structure. A careful analysis of the approach to infinite momentum of the equations of Ref. 2 would be worthwhile. In particular, it seems likely that the vacuum Schrödinger equation will require the vacuum clusters to fall off at least as fast as E^{-1}, where E is the sum of the kinetic energies of the particles in the cluster.

An interesting question which we are presently investigating is whether the Galilean invariance of the transverse motion is more than a curiosity. The author has found that several questions concerning the structure of currents are greatly illuminated by considering them in the limit of infinite momentum where much of the theory of nonrelativistic currents can be taken directly over into a relativistic LRF description.[4] A somewhat amusing point which will be discussed elsewhere concerns the normal and anomalous threshold singularities of relativistic form factors.[5] The anomalous singularities are usually associated with nonrelativistic or semirelativistic wave function effects while the normal singularities are generally assumed to originate in truly relativistic effects which have no analog in terms of wave functions.

We have found that by going to the LRF, both the anomalous and normal singularities can be understood very easily in terms of the simplest wave-function ideas for the transverse motion.

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[4] L. Susskind, following paper, Phys. Rev. **165**, 1547 (1968).

[5] L. Susskind and C. Frye, this issue, Phys. Rev. **165**, 1553 (1968).