We note that small-angle elastic scatterings can easily be missed by scanners. This effect, which is a serious one for two-prong final states, has been studied in detail by Jacobs.[7] Using the result of his analysis, we have made a correction to the total number of interactions (from the cross-section scan); this correction amounts to about 8% at 3.2 GeV/c, 7% at 4.2 GeV/c.

In order to obtain any reliable cross sections, one must also correct for the scanning efficiency of the scanners. Based on two separate second scans of 15 rolls (about 3000 events) of film each, the scanning efficiency was found to be $(96\pm2)\%$ for the first scan.

In addition, for partial-cross-section calculations, we have corrected for the contamination in each category resulting from erroneously assigned hypotheses (see Sec. D3).

The resulting cross sections, after all these corrections have been made, are shown in Table II for both 3.2 and 4.2 GeV/c. Of course, only the 3.2-GeV/c normal sample was used to calculate the cross sections at that momentum.

We point out here that cross sections were calculated from the data that had no cutoff based on the fiducial-volume criterion. For subsequent analysis in Secs. IV through VIII, however, the rigid fiducial-volume criterion was applied. The events failing to satisfy the criterion (about 11% of the total) showed a poor resolution, based on the width of ω from this sample. This is, of course, because these events are largely from the periphery of the bubble chamber and they tend to have short tracks; this results in poor measurements.

We have also applied a cutoff at $\pm2°$ for the dip angle of the beam evaluated at the interaction vertex, thereby eliminating about 2% of the total events. The number of events shown in Table I is that obtained after these cutoffs were applied.

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New Low-Energy Theorems for Nucleon Compton Scattering*

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Two new low-energy theorems for Compton scattering from spin-$\frac{1}{2}$ targets, giving some terms of order ω^2, are derived using a recently obtained lemma. One of these theorems is a generalization to the spin-$\frac{1}{2}$ case of a similar theorem for the spin-0 case. The other theorem involves "magnetic moment radius," i.e., $[dG_M(t)/dt]_{t=0}$, which does not occur in any of the low-energy theorems obtained earlier.

I. INTRODUCTION

We report here two new low-energy theorems for nucleon Compton scattering, giving coefficients of the $\omega^2(1-\cos\theta)$ terms, where ω is the incident lab photon energy and θ the lab angle of scattering, in two of the amplitudes. It is obvious that the possibility of writing down such theorems depends on being able to deal with excited-state contributions to $T_{00}{}^{\alpha\beta}(p',k';p,k)$, where

$$i(2\pi)^4\delta^4(p'+k'-p-k)\frac{1}{(2\pi)^3}(m^2/E_{p'}E_p)^{1/2}$$
$$\times T_{\mu\nu}{}^{\alpha\beta}(p',k';pk)=\int d^4x d^4y\, e^{i(k'\cdot x-k\cdot y)}$$
$$\times\langle p'|[T\{J_\mu{}^\alpha(x),J_\nu{}^\beta(y)\}-i\rho_{\mu\nu}{}^{\alpha\beta}(x)\delta^4(x-y)]|p\rangle,\quad(1)$$

and p, k, $E_p(p',k',E_{p'})$ are, respectively, initial (final) nucleon and photon four-momenta and nucleon energy. The "charge labels" for final and initial photons are denoted by α and β, respectively. The general form of the excited-state contributions was discussed recently using current conservation and was used to derive a new low-energy theorem for pion Compton scattering.[1] The use of this information for the Compton scattering from systems with spin $S\geq1$ leads to a number of new theorems; in particular, for the spin-1 case one obtains a "quadrupole moment" theorem.[2]

2. LOW-ENERGY THEOREMS

Let us write the nucleon Compton scattering in the lab frame ($\mathbf{p}=0$) as

$$\epsilon_m{}'T_{mn}{}^{\alpha\beta}(\omega',\mathbf{k}';\omega,\mathbf{k})\epsilon_n=\epsilon_m{}'U_{mn}{}^{\alpha\beta}(\omega',\mathbf{k}';\omega,\mathbf{k})\epsilon_n$$
$$+\boldsymbol{\varepsilon}'\cdot\boldsymbol{\varepsilon}E_1{}^{\alpha\beta}+\tfrac{1}{2}[\boldsymbol{\sigma}\cdot\boldsymbol{\varepsilon}'\boldsymbol{\sigma}\cdot\boldsymbol{\varepsilon}]E_2{}^{\alpha\beta}$$
$$+(\boldsymbol{\varepsilon}'\cdot\mathbf{k}\boldsymbol{\varepsilon}\cdot\mathbf{k}'-\mathbf{k}'\cdot\mathbf{k}\boldsymbol{\varepsilon}'\cdot\boldsymbol{\varepsilon})E_3{}^{\alpha\beta}$$
$$+\tfrac{1}{2}([\boldsymbol{\sigma}\cdot\mathbf{k}',\boldsymbol{\sigma}\cdot\mathbf{k}]\boldsymbol{\varepsilon}'\cdot\boldsymbol{\varepsilon}-\mathbf{k}'\cdot\mathbf{k}[\boldsymbol{\sigma}\cdot\boldsymbol{\varepsilon}',\boldsymbol{\sigma}\cdot\boldsymbol{\varepsilon}])E_4{}^{\alpha\beta}$$
$$+\tfrac{1}{2}([\boldsymbol{\sigma}\cdot\boldsymbol{\varepsilon}',\boldsymbol{\sigma}\cdot\mathbf{k}]\boldsymbol{\varepsilon}\cdot\mathbf{k}'-\boldsymbol{\varepsilon}'\cdot\mathbf{k}[\boldsymbol{\sigma}\cdot\boldsymbol{\varepsilon},\boldsymbol{\sigma}\cdot\mathbf{k}']$$
$$-2\mathbf{k}'\cdot\mathbf{k}[\boldsymbol{\sigma}\cdot\boldsymbol{\varepsilon}',\boldsymbol{\sigma}\cdot\boldsymbol{\varepsilon}])E_5{}^{\alpha\beta}$$
$$+\tfrac{1}{2}([\boldsymbol{\sigma}\cdot\boldsymbol{\varepsilon}',\boldsymbol{\sigma}\cdot\mathbf{k}]\boldsymbol{\varepsilon}\cdot\mathbf{k}'+\boldsymbol{\varepsilon}'\cdot\mathbf{k}[\boldsymbol{\sigma}\cdot\boldsymbol{\varepsilon},\boldsymbol{\sigma}\cdot\mathbf{k}'])E_6{}^{\alpha\beta}$$
$$+\tfrac{1}{2}([\boldsymbol{\sigma}\cdot\boldsymbol{\varepsilon}',\boldsymbol{\sigma}\cdot\mathbf{k}']\boldsymbol{\varepsilon}\cdot\mathbf{k}'-\boldsymbol{\varepsilon}'\cdot\mathbf{k}[\boldsymbol{\sigma}\cdot\boldsymbol{\varepsilon},\boldsymbol{\sigma}\cdot\mathbf{k}])E_7{}^{\alpha\beta}$$
$$+\tfrac{1}{2}([\boldsymbol{\sigma}\cdot\boldsymbol{\varepsilon}',\boldsymbol{\sigma}\cdot\mathbf{k}']\boldsymbol{\varepsilon}\cdot\mathbf{k}'+\boldsymbol{\varepsilon}'\cdot\mathbf{k}[\boldsymbol{\sigma}\cdot\boldsymbol{\varepsilon},\boldsymbol{\sigma}\cdot\mathbf{k}])E_8{}^{\alpha\beta},\quad(2)$$

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[1] V. Singh, Phys. Rev. Letters 19, 730 (1967).
[2] A. Pais, Phys. Rev. Letters 19, 544 (1967).

where $\boldsymbol{\varepsilon}$, ω ($\boldsymbol{\varepsilon}'^*,\omega'$) are, respectively, the initial (final) polarization vector and lab energy of the photon; $E_i^{\alpha\beta}=E_i^{\alpha\beta}(\omega',\omega)$. The $U_{mn}^{\alpha\beta}(\omega',\mathbf{k}';\omega,\mathbf{k})$, with m, n =1, 2, 3, is the explicit nucleon intermediate-state contribution given by

$$U_{mn}^{\alpha\beta}(\omega',\mathbf{k}';\omega,\mathbf{k})=\frac{[(2k_m-k_m')F_1^\alpha-i(\mathbf{k}'\times\boldsymbol{\sigma})_m(F_1^\alpha+2mF_2^\alpha)][k_nF_1^\beta+i(\mathbf{k}'\times\boldsymbol{\sigma})_n(F_1+2mF_2^\beta)]}{(4m^2)(m+\omega-E_\mathbf{k})}$$
$$+\frac{[(2k_n'-k_n)F_1^\beta-i(\mathbf{k}\times\boldsymbol{\sigma})_n(F_1^\beta+2mF_2^\beta)][k_m'F_1^\alpha+i(\mathbf{k}'\times\boldsymbol{\sigma})_m(F_1^\alpha+2mF_2^\alpha)]}{(4m^2)(m-\omega'-E_{-\mathbf{k}'})}+O(\omega^3), \quad (3)$$

where $F_i^\alpha=F_i^\alpha(0)$, and $F_{1,2}^\alpha(t)$ are the usual Dirac-Pauli nucleon-electromagnetic form factors for the photons with the charge index α. The time-reversal invariance reduces the number of the linearly independent amplitudes by two. We shall, however, not exhibit this reduction explicitly.

Further let us define

$$E_i^{\{\alpha\beta\}}(\omega',\omega)\equiv\tfrac{1}{2}[E_i^{\alpha\beta}(\omega',\omega)+E_i^{\beta\alpha}(\omega',\omega)],$$
$$E_i^{[\alpha\beta]}(\omega',\omega)\equiv\tfrac{1}{2}[E_i^{\alpha\beta}(\omega',\omega)-E_i^{\beta\alpha}(\omega',\omega)], \quad (4)$$

and let

$$E_1^{\{\alpha\beta\}}(\omega',\omega)=E_1^{\{\alpha\beta\}}(0,0)+m(\omega'-\omega)e_{11}^{\{\alpha\beta\}}+(\omega+\omega')^2\times e_{12}^{\{\alpha\beta\}}+O(\omega^3),$$
$$E_2^{[\alpha\beta]}(\omega',\omega)=E_2^{[\alpha\beta]}(0,0)+m(\omega'-\omega)e_{21}^{[\alpha\beta]}+(\omega'+\omega')^2\times e_{22}^{[\alpha\beta]}+O(\omega^3). \quad (5)$$

The two new theorems can be stated as follows:

Theorem 1:

$$e_{11}^{\{\alpha\beta\}}=0. \quad (6)$$

Theorem 2:

$$e_{21}^{[\alpha\beta]}=2if^{\alpha\beta\gamma}\left(F_2'+\frac{F_1'}{2m}\right)^\gamma-\left(\frac{1}{4m^2}\right)if^{\alpha\beta\gamma}F_2^\gamma-\frac{1}{4m^2}\times[\{F_2^\alpha,F_1^\beta\}-\{F_2^\beta,F_1^\alpha\}], \quad (7)$$

where $F_i'\equiv[d/dt\,F_i(t)]_{t=0}$ and $f^{\alpha\beta\gamma}$ are the structure constants of the symmetry group. We may emphasize here that these theorems give only the coefficients of $m(\omega'-\omega)=-\omega^2(1-\cos\theta)+O(\omega^3)$ term. We do not have any theorems for pure ω^2 terms, and thus we do not have anything to say about Rayleigh scattering. This is as it should be, since Rayleigh scattering is known to be shape-dependent and involves a knowledge of electric polarizability.[3] It therefore cannot be calculated in terms of only the electromagnetic form factors.

3. PROOF

We shall now indicate briefly the ingredients used in proving these theorems.

(i) Using the current conservation $\partial^\mu J_\mu^\alpha(x)=0$, and the equal-time commutators

$$[J_0^\alpha(x),J_0^\beta(y)]\delta(x_0-y_0)=if^{\alpha\beta\gamma}J_0^\gamma(x)\delta^4(x-y), \quad (8)$$
$$[J_0^\alpha(x),J_n^\beta(y)]\delta(x_0-y_0)=if^{\alpha\beta\gamma}J_n^\gamma(x)\delta^4(x-y)+i\partial_m[\rho_{mn}^{\alpha\beta}(x)\delta^4(x-y)], \quad (9)$$

one can obtain the divergence conditions[4]

$$k'^\mu T_{\mu\nu}^{\alpha\beta}=T_{\mu\lambda}^{\alpha\beta}k^\lambda=i(2\pi)^3(E_{p'}E_p/m^2)^{1/2}f^{\alpha\beta\gamma}\times\langle p'|J_\nu^\gamma(0)|p|\rangle. \quad (10)$$

From the conditions (10) one can derive the identity

$$k'^mT_{mn}^{\alpha\beta}k^n=k_0k_0'T_{00}^{\alpha\beta}-i(2\pi)^3(E_{p'}E_p/m^2)^{1/2}f^{\alpha\beta\gamma}\times\langle p'|[\tfrac{1}{2}(k_0+k_0')J_0^\gamma(0)+\tfrac{1}{2}(k_m+k_m')J_m^\gamma(0)]|p\rangle. \quad (11)$$

(ii) We have

$$\rho_{00}^{\alpha\beta}(x)=0, \quad (12)$$

and therefore

$$i(2\pi)^4\delta^4(p'+k'-p-k)\frac{1}{(2\pi)^3}(m^2/E_{p'}E_p)^{1/2}\times T_{00}^{\alpha\beta}(p',k';p,k)=\int d^4x\,d^4y\,e^{ik'\cdot x-ik\cdot y}\times\langle p'|T\{J_0^\alpha(x),J_0^\beta(y)\}|p\rangle. \quad (13)$$

Let us write

$$T_{00}^{\alpha\beta}=U_{00}^{\alpha\beta}+E_{00}^{\alpha\beta}, \quad (14)$$

where $U_{00}^{\alpha\beta}$ is the explicit one-nucleon intermediate-state contribution to $T_{00}^{\alpha\beta}$, and $E_{00}^{\alpha\beta}$ is the contribution of all other states which are not degenerate with the nucleon in energy, i.e., the excited-state contribution. We retain as intermediate states only the states with no photons present, and therefore all our new results will be exact to all orders in strong interactions, but only to second order in electromagnetism. We also assume that there is no other state degenerate with the nucleon (except, of course, those corresponding to the internal symmetry).

As is well known, $E_{00}^{\alpha\beta}$ is of second order in ω^2. Using current conservation we have the lemma[1]

$$E_{00}^{\alpha\beta}=k_m'k_n\Lambda_{mn}^{\alpha\beta}(\omega',\mathbf{k}';\omega,\mathbf{k})+k_mk_n'\Lambda_{mn}^{\beta\alpha}\times(-\omega,-\mathbf{k},-\omega',-\mathbf{k}'), \quad (15)$$

where $\Lambda_{mn}^{\alpha\beta}(\omega',\mathbf{k}';\omega,\mathbf{k})$ is a three-tensor of second rank which is regular around $\omega=0$ and is a rational function of $\mathbf{k}$ and $\mathbf{k}'$. Basically, this result is due to the fact that none of the energy denominators vanish and all the numerators are rational functions of $\mathbf{k}$ and $\mathbf{k}'$. We therefore have

$$E_{00}^{\alpha\beta}=k_m'k_n[\Lambda_{mn}^{\alpha\beta}(0,0;0,0)+\Lambda_{mn}^{\beta\alpha}(0,0;0,0)]+O(\omega^3). \quad (16)$$

[3] A. M. Baldin, Nucl. Phys. 18, 310 (1960).

[4] M. A. B. Bég, Phys. Rev. Letters 17, 333 (1966); and M. A. B. Bég, SINBI Lectures, Copenhagen, 1967 (unpublished), where one may find a good discussion and references to the literature on low-energy theorems for Compton scattering.

Since the only pure numerical tensors available are δ_{mn} and $\frac{1}{2}[\sigma_m,\sigma_n]$, we obtain

$$E_{00}{}^{\{\alpha\beta\}}=\mathbf{k}'\cdot\mathbf{k}\xi^{\alpha\beta}+O(\omega^3),$$
$$E_{00}{}^{[\alpha\beta]}=\tfrac{1}{2}[\boldsymbol{\sigma}\cdot\mathbf{k}',\boldsymbol{\sigma}\cdot\mathbf{k}]\eta^{\alpha\beta}+O(\omega^3), \tag{17}$$

where the superscripts $\{\alpha\beta\}$, $[\alpha\beta]$ respectively denote the parts of $E_{00}{}^{\alpha\beta}$ which are symmetric and antisymmetric under the exchange of $\alpha\leftrightarrow\beta$. The $\xi^{\alpha\beta}$, $\eta^{\alpha\beta}$ are unknown constants.

(iii) We also need the tensor decomposition of $T_{mn}{}^{\alpha\beta}$. To first order in ω one has

$$T_{mn}{}^{\alpha\beta}=U_{mn}{}^{\alpha\beta}+\delta_{mn}E_1{}^{\alpha\beta}(\omega',\omega)+\tfrac{1}{2}[\sigma_m,\sigma_n]\times E_2{}^{\alpha\beta}(\omega',\omega)+O(\omega^2), \tag{18}$$

where $U_{mn}{}^{\alpha\beta}$ is the explicit nucleon intermediate-state contribution [Eq. (3)], and $E_1{}^{\alpha\beta}(\omega',\omega)$, $E_2{}^{\alpha\beta}(\omega',\omega)$ are regular at $\omega=0$. We need, however, to retain also terms of order ω^2. We then have

$$\begin{aligned}T_{mn}=&U_{mn}+\delta_{mn}E_1+\tfrac{1}{2}[\sigma_m,\sigma_n]E_2+[k_mk_n'-\mathbf{k}'\cdot\mathbf{k}\delta_{mn}]E_3\\
&+[\tfrac{1}{2}\delta_{mn}[\boldsymbol{\sigma}\cdot\mathbf{k}',\boldsymbol{\sigma}\cdot\mathbf{k}]-\tfrac{1}{2}\mathbf{k}'\cdot\mathbf{k}[\sigma_m,\sigma_n]]E_4\\
&+[\tfrac{1}{2}[\sigma_m,\boldsymbol{\sigma}\cdot\mathbf{k}]k_n'-\tfrac{1}{2}k_m[\sigma_n,\boldsymbol{\sigma}\cdot\mathbf{k}']-\mathbf{k}'\cdot\mathbf{k}[\sigma_m,\sigma_n]]E_5\\
&+[\tfrac{1}{2}[\sigma_m,\boldsymbol{\sigma}\cdot\mathbf{k}]k_n'+\tfrac{1}{2}k_m[\sigma_n,\boldsymbol{\sigma}\cdot\mathbf{k}']]E_6\\
&+[\tfrac{1}{2}[\sigma_m,\boldsymbol{\sigma}\cdot\mathbf{k}']k_n'-\tfrac{1}{2}[\sigma_n,\boldsymbol{\sigma}\cdot\mathbf{k}]k_m]E_7\\
&+[\tfrac{1}{2}[\sigma_m,\boldsymbol{\sigma}\cdot\mathbf{k}']k_n'+\tfrac{1}{2}[\sigma_n,\boldsymbol{\sigma}\cdot\mathbf{k}]k_m]E_8\\
&+(k_mk_n+k_m'k_n')E_9+(k_mk_n-k_m'k_n')E_{10}+k_m'k_nE_{11}\\
&+[\tfrac{1}{2}[\sigma_m,\boldsymbol{\sigma}\cdot\mathbf{k}]k_n+k_m'\tfrac{1}{2}[\sigma_n,\boldsymbol{\sigma}\cdot\mathbf{k}']]E_{12}\\
&+[\tfrac{1}{2}[\sigma_m,\boldsymbol{\sigma}\cdot\mathbf{k}]k_n-k_m'\tfrac{1}{2}[\sigma_n,\boldsymbol{\sigma}\cdot\mathbf{k}']]E_{13}\\
&+[\tfrac{1}{2}[\sigma_m,\boldsymbol{\sigma}\cdot\mathbf{k}']k_n+k_m'\tfrac{1}{2}[\sigma_n,\boldsymbol{\sigma}\cdot\mathbf{k}]]E_{14}\\
&+[\tfrac{1}{2}[\sigma_m,\boldsymbol{\sigma}\cdot\mathbf{k}']k_n-k_m'\tfrac{1}{2}[\sigma_n,\boldsymbol{\sigma}\cdot\mathbf{k}]]E_{15}\\
&+O(\omega^3),\end{aligned} \tag{19}$$

where $E_i(\omega',\omega)$ are regular at $\omega=0$. The crossing-symmetry requirements, which we use quite often, are given by

$$E_i{}^{\alpha\beta}(\omega',\omega)=\eta_iE_i{}^{\beta\alpha}(-\omega,-\omega'), \tag{20}$$

where

$$\begin{aligned}\eta_i&=+1\quad\text{for}\quad i=1,3,6,8,9,11,12,14\\&=-1\quad\text{for}\quad i=2,4,5,7,10,13,15.\end{aligned}$$

Using Eqs. (3), (11), (14), (17), (19), (20), together with an explicit calculation of $U_{00}{}^{\alpha\beta}$, we obtain the two new low-energy theorems given above by Eqs. (6) and (7).

Some remarks on why we have chosen this particular derivation of the low-energy theorems may be in order. The present method generalizes with ease to the case of Compton scattering of the "physical" as well as "charged" photons on higher-spin targets. It is conceivable that it may be possible, at least for the case of the "physical" photons, to give a pure S-matrix derivation of these theorems, provided one is able to write down a set of linearly independent invariant amplitudes which are free of both the kinematic singularities and zeros. This, however, is a highly nontrivial problem, and is still unsolved in general.

4. CONCLUDING REMARKS

Theorem 1 is a generalization to spin-$\frac{1}{2}$ Compton scattering of a similar theorem for spin-0 Compton scattering. We conjecture that the following generalization of this theorem is true for the higher-spin case.

Conjecture: Let

$$\mathrm{Tr}(\boldsymbol{\varepsilon}_m'T_{mn}{}^{\{\alpha\beta\}}\boldsymbol{\varepsilon}_n)/\mathrm{Tr}(1)=T_1{}^{\{\alpha\beta\}}(\omega',\omega)\boldsymbol{\varepsilon}'\cdot\boldsymbol{\varepsilon}+T_2{}^{\{\alpha\beta\}}(\omega',\omega)(\boldsymbol{\varepsilon}'\cdot\mathbf{k}\boldsymbol{\varepsilon}\cdot\mathbf{k}'-\mathbf{k}'\cdot\mathbf{k}\boldsymbol{\varepsilon}'\cdot\boldsymbol{\varepsilon}), \tag{21}$$

where the trace is over spin states, and let

$$T_1{}^{\{\alpha\beta\}}(\omega',\omega)=T_1{}^{\{\alpha\beta\}}(0,0)+t_1{}^{\{\alpha\beta\}}m(\omega'-\omega)+(\omega+\omega')^2t_2{}^{\{\alpha\beta\}}+O(\omega^2); \tag{22}$$

then

$$t_1{}^{\{\alpha\beta\}}=0.$$

One can convert these low-energy theorems into sum rules if one assumes unsubtracted dispersion relations for the relevant amplitudes. One can then try to saturate them with low-lying baryon resonances. This aspect of the problem and further details will be discussed in a later paper.

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