Vierbein Field Theory of Gravitation

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The sixteen components of the vierbein field which factorize the metric tensor are used to construct a simple nonlinear field theory of gravitation which, although it is shown to be equivalent to Einstein's theory physically, is based on a scalar action function of first order, replacing the Riemann scalar which serves as a second-order action function in the conventional approach.

1. INTRODUCTION

IN 1928 Einstein,[1] aiming at the acquisition of additional field variables for the purpose of grasping both gravitation and electromagnetism in a unified field theory, introduced into the geometric theory of gravitation the use of a sixteen-component "vierbein field" from which the ten components of the symmetric metric tensor can be constructed as bilinear forms. Although the mathematical properties of vierbein fields were made available to physicists soon afterward in two lucid papers by Weitzenböck[2] and by Levi-Civita,[3] and although the advantage they present for the formulation of a quantum field theory of gravitation was recognized by Rosenfeld[4] in the very first paper on this subject, they are not widely employed in the current literature on general relativity, except in conjunction with attempts[5,6] at understanding gravitation as a compensating field in the sense of Yang and Mills,[7] where their use is, in fact, indispensable.

The purpose of this paper is to revive interest in these vierbein fields as variables eminently suited for the description of gravitation, even if one does not aim at any so-called unified field theory, by showing their usefulness for the distinction of true gravitational fields from pseudogravitational fields (Sec. 2); by exhibiting invariance under reorientation and under gauge transformations as the requirements that lead uniquely to the linearized theory of Einstein for the case of weak fields (Sec. 3); by extending the uniqueness proof to Einstein's full theory; and by giving a simple, properly invariant, nonlinear vierbein field theory of gravitation which, although it is equivalent to Einstein's theory physically, is based on a scalar action function of first order (Sec. 4). The transformation properties of the Weitzenböck invariants under coordinate-dependent reorientations of inertial frames, which are essential to the main argument put forth in this paper, have been collected in an Appendix for ready reference.

[1] A. Einstein, Sitzber. preuss. Akad. Wiss., Physik.-math. Kl. **1928**, 217 (1928); **1928**, 224 (1928).

[2] R. Weitzenböck, Sitzber. preuss. Akad. Wiss., Physik.-math. Kl. **1928**, 466 (1928).

[3] T. Levi-Civita, Sitzber. preuss. Akad. Wiss., Physik.-math. Kl. **1929**, 137 (1929).

[4] L. Rosenfeld, Ann. Physik **5**, 113 (1930).

[5] R. Utiyama, Phys. Rev. **101**, 1597 (1956).

[6] T. W. B. Kibble, J. Math. Phys. **2**, 212 (1961).

[7] C. N. Yang and R. L. Mills, Phys. Rev. **96**, 191 (1954).

2. TRUE GRAVITATIONAL FIELDS

The distinction between pseudogravitational fields, such as the Coriolis and centrifugal fields encountered in rotating noninertial frames, and true gravitational fields, such as the Newtonian field of the sun, classifies inertial acceleration fields into those that can and those that cannot be transformed away globally. Thus, Coriolis and centrifugal fields vanish everywhere when one undoes the rotation with respect to the Newtonian inertial frame that gave rise to their appearance, whereas one cannot find an inertial frame in which the sun's gravitational field has the value zero everywhere.

Since one can always find an inertial frame y^k in which an inertial acceleration field given in terms of the metric properties of an underlying coordinate continuum x^α has at least locally the value zero, the so-called vierbein field components, consisting of the sixteen transformation functions $h^k{}_\alpha(x)$ connecting the displacements dy^k and dx^α by

$$dy^k = h^k{}_\alpha(x) dx^\alpha\,, \tag{2.1}$$

present themselves as a most convenient device for making the distinction between pseudogravitational and true gravitational fields formally precise. Indeed, if the vierbein field components are integrable, i.e., if $h^k{}_{\alpha|\beta} = h^k{}_{\beta|\alpha}$ (where $h^k{}_{\alpha|\beta}$ means $\partial h^k{}_\alpha / \partial x^\beta$), one can transform everywhere from the underlying continuum to an inertial frame $y^k = y^k(x^\alpha)$, and the metric field described by the $h^k{}_\alpha$ can be recognized as a pseudogravitational field in this case.[8] Nonintegrability of the $h^k{}_\alpha$ is characteristic for presence of true gravitational fields. Therefore, the functions

$$G^k{}_{\alpha\beta} = h^k{}_{\alpha|\beta} - h^k{}_{\beta|\alpha}\,, \tag{2.2}$$

which vanish for pseudogravitational fields, will be given the name "true gravitational field strengths." This definition also suggests looking upon the components $h^k{}_\alpha$ of the vierbein field as "gravitational potentials," in analogy to the electromagnetic potentials A_α from which the electromagnetic field strengths $F_{\alpha\beta} = A_{\alpha|\beta} - A_{\beta|\alpha}$ are derived by differentiation. The analog to the gauge transformations $A_\alpha \to A_\alpha + \Lambda_{|\alpha}$ of electrodynamics are the "gravitational gauge

[8] This implies that the notion of distant parallelism can be retained in metric fields describing pseudogravitational effects [see Einstein (Ref. 1)].

transformations"

$$h^k{}_\alpha \rightarrow h^k{}_\alpha + \Lambda^k{}_{|\alpha}, \qquad (2.3)$$

which leave the true field strengths $G^k{}_{\alpha\beta}$ invariant.

The observation that for given vierbein field the metric tensor is uniquely determined by

$$g_{\alpha\beta} = h^k{}_\alpha h_{k\beta}, \qquad (2.4)$$

whereas for given metric tensor the factorization into vierbein field components is not uniquely possible, provides additional motivation for treating the vierbein field components as basic field variables with which to apprehend gravitation in a field theory.

The components of the vierbein field are not tensors.[9] Under transformations of the coordinates x^α they transform as a set of four covariant four-vectors with components labeled by the index α, whereas under reorientations of the local inertial frame generated by an orthogonal matrix $R^k{}_i$ so that

$$R_k{}^i R_j{}^k = \delta_j{}^i, \qquad (2.5)$$

they transform as

$$h^k{}_\alpha \rightarrow h'^k{}_\alpha = R^k{}_i h^i{}_\alpha. \qquad (2.6)$$

The reciprocals $f^\alpha{}_k$ of the vierbein field, defined by

$$f^\alpha{}_k h^k{}_\beta = \delta^\alpha{}_\beta, \quad h^i{}_\alpha f^\alpha{}_k = \delta^i{}_k, \qquad (2.7)$$

transform accordingly as a set of four contravariant four-vectors under transformations of the coordinates x^α, and permit unique construction of the contravariant components of the metric tensor by

$$g^{\alpha\beta} = f^\alpha{}_k f^{\beta k}. \qquad (2.8)$$

Similarly, the true field strengths $G^k{}_{\alpha\beta}$ are not tensors. Accordingly, the name "true gravitational field tensor" will be reserved for the mixed tensor

$$S^\gamma{}_{\alpha\beta} = f^\gamma{}_k G^k{}_{\alpha\beta}, \qquad (2.9)$$

which is antisymmetric in the indices α and β and has thus 24 components. They may be written alternatively as $S^\gamma{}_{\alpha\beta} = \Delta^\gamma{}_{\alpha\beta} - \Delta^\gamma{}_{\beta\alpha}$ with the tensor

$$\Delta^\gamma{}_{\alpha\beta} = f^\gamma{}_k h^k{}_{\alpha|\beta}, \qquad (2.10)$$

which can be used to express the derivatives of the vierbein field.

3. WEAK-FIELD APPROXIMATION

If one believes in the universal validity of the proposition that energy in any form is gravitating, then any theory of gravitation permitting introduction of the concept of field energy must recognize the gravitating effect of gravity. Since in all customary field theories that are derivable from an action principle the energy density is not a linear function of the field variables, the field equations governing the dynamics of the gravitational field ought to be essentially nonlinear, in contradistinction to the classical field equations of vacuum electrodynamics which are essentially linear because of the electric neutrality of the electromagnetic field. However, the extreme weakness of the gravitational coupling has prevented, to date, experimental verification of that proposition for the case of the gravitational field. In particular, the tests of Einstein's theory verify that theory only for gravitational fields that are described sufficiently by equations linear in the field variables.[10] Thus, any deviations $\gamma_{\alpha\beta}$ from the pseudo-Euclidean metric $\delta_{\alpha\beta}$, defined by

$$g_{\alpha\beta} = \delta_{\alpha\beta} + \gamma_{\alpha\beta}, \qquad (3.1)$$

have not been observed beyond a linear approximation permitting terms of quadratic and higher order in the small quantities $\gamma_{\alpha\beta}$ to be neglected.

All experimental knowledge about gravitation existing at present is therefore compatible with any theory that coincides for weak fields with the linearized version of Einstein's theory. In particular, a vierbein field theory of gravitation is sufficiently supported by experiment, provided the field equations written in terms of the variables $\eta^k{}_\alpha$, defined by

$$h^k{}_\alpha = \delta^k{}_\alpha + \eta^k{}_\alpha, \qquad (3.2)$$

reproduce in an approximation linear in these variables the physical content of Einstein's linearized theory.

Substitution of the definition (3.2) into the expression (2.4) for the metric tensor yields the connection

$$\gamma_{\alpha\beta} = \eta_{\alpha\beta} + \eta_{\beta\alpha} + O(\eta^2), \qquad (3.3)$$

which shows that in the weak-field approximation the sixteen functions $\eta_{\alpha\beta}$ may be treated as a tensor field, and that in the lowest order only the symmetric part of the tensor $\eta_{\alpha\beta}$ contributes to the symmetric metric tensor.

Just as Maxwell's vacuum field equations $A_{\alpha|\sigma\sigma} - A_{\sigma|\alpha\sigma} = 0$ are the only Lorentz-invariant linear equations of second order for a vector field A_α that satisfies the requirement of gauge invariance, so the linear field equations governing the tensor field $\eta_{\alpha\beta}$ follow uniquely from the requirements of invariance under the gauge transformations (2.3) and (2.6), as follows.

One can form fourteen linearly independent invariants bilinear in the fields $\eta_{\alpha\beta}$ and their first derivatives,

$$I_1 = \eta_{\alpha\beta}\eta_{\alpha\beta}; \quad I_2 = \eta_{\alpha\beta}\eta_{\beta\alpha}; \quad I_3 = \eta_{\alpha\alpha}\eta_{\beta\beta}; \quad I_4 = \eta_{\alpha\beta|\gamma}\eta_{\alpha\beta|\gamma};$$
$$I_5 = \eta_{\alpha\beta|\gamma}\eta_{\beta\alpha|\gamma}; \quad I_6 = \eta_{\alpha\alpha|\gamma}\eta_{\beta\beta|\gamma}; \quad I_7 = \eta_{\alpha\beta|\gamma}\eta_{\alpha\gamma|\beta};$$
$$I_8 = \eta_{\alpha\beta|\gamma}\eta_{\gamma\alpha|\beta}; \quad I_9 = \eta_{\alpha\beta|\gamma}\eta_{\gamma\beta|\alpha}; \quad I_{10} = \eta_{\alpha\alpha|\gamma}\eta_{\beta\gamma|\beta};$$
$$I_{11} = \eta_{\alpha\beta|\beta}\eta_{\alpha\gamma|\gamma}; \quad I_{12} = \eta_{\alpha\beta|\beta}\eta_{\gamma\alpha|\gamma};$$
$$I_{13} = \eta_{\alpha\beta|\alpha}\eta_{\gamma\beta|\gamma}; \quad I_{14} = \eta_{\alpha\alpha|\gamma}\eta_{\gamma\beta|\beta}. \qquad (3.4)$$

Since I_{11}, I_{12}, I_{13}, I_{14} differ from I_7, I_8, I_9, I_{10}, respec-

[9] B. S. DeWitt, *Dynamical Theory of Groups and Fields* (Gordon & Breach Science Publishers, Inc., New York, 1965), p. 114.

[10] R. H. Dicke, in *Gravitation and Relativity* (W. A. Benjamin, Inc., New York, 1964), Chap. 1.

tively, by divergences only, the linear combination

$$L=\sum_{i=1}^{10} c_i I_i \tag{3.5}$$

with arbitrary coefficients c_i is the most general Lagrangian that yields linear field equations of second order. Turning first to the requirement of orientation invariance it should be noted that the coefficients $R^k{}_i$ introduced in (2.5), which connect equivalent local inertial frames, must be written

$$R^k{}_i=\delta^k{}_i+\epsilon^k{}_i, \tag{3.6}$$

with $\epsilon_{ki}=-\epsilon_{ik}$ regarded as small so as not to violate the linear approximation. The transformation law for the $\eta_{\alpha\beta}$ is now obtained from that of the $h^k{}_\alpha$ by writing, keeping only terms of zero and first order in small quantities,

$$h'^k{}_\alpha=\delta^k{}_\alpha+\eta'^k{}_\alpha=R^k{}_i h^i{}_\alpha=\delta^k{}_\alpha+\eta^k{}_\alpha+\epsilon^k{}_\alpha \tag{3.7}$$

yielding the result

$$\eta'_{\alpha\beta}=\eta_{\alpha\beta}+\epsilon_{\alpha\beta}, \tag{3.8}$$

which shows, on account of the antisymmetry of the $\epsilon_{\alpha\beta}$, that rotation of the local inertial frame changes only the skew part of the $\eta_{\alpha\beta}$. The scalars I_3 and I_6 are obviously invariant under (3.8), and so is $\frac{1}{2}(I_{10}+I_{14})$ which differs from I_{10} only by a divergence. Requiring the remainder of L to be invariant under (3.8) yields the condition

$$2(c_1-c_2)\epsilon_{\alpha\beta}\eta_{\alpha\beta}+[2(c_4-c_5)\epsilon_{\alpha\beta|\gamma} +(2c_7-c_8)\epsilon_{\alpha\gamma|\beta}+(c_8-2c_9)\epsilon_{\beta\gamma|\alpha}]\eta_{\alpha\beta|\gamma}=0, \tag{3.9}$$

leading to the constraints

$$c_1=c_2;\quad c_4=c_5;\quad 2c_7=c_8=2c_9, \tag{3.10}$$

which have the effect of letting $\eta_{\alpha\beta}$ appear in L only in the symmetric combination $\gamma_{\alpha\beta}=\eta_{\alpha\beta}+\eta_{\beta\alpha}$. The most general orientation invariant Lagrangian is therefore

$$L=c_1I_1+c_3I_3+c_4I_4+c_6I_6+c_7I_7+c_{10}I_{10}, \tag{3.11}$$

with the understanding that only the symmetric part of the tensor $\eta_{\alpha\beta}$ is considered. Apparently, the skew part of the vierbein field can play no dynamical role in the linear approximation.

The Lagrangian (3.11) is identical with the one considered by Wyss[11] who showed that imposition of invariance under the gauge transformations (2.3) reduces (up to an unimportant common factor) the Lagrangian to the linear combination

$$L=I_4-I_6-2(I_7-I_{10}), \tag{3.12}$$

yielding the field equations

$$\eta_{\alpha\beta|\sigma\sigma}+\eta_{\sigma\sigma|\alpha\beta}-\eta_{\alpha\sigma|\beta\sigma}-\eta_{\beta\sigma|\alpha\sigma} +\delta_{\alpha\beta}(\eta_{\gamma\sigma|\gamma\sigma}-\eta_{\gamma\gamma|\sigma\sigma})=0, \tag{3.13}$$

which reduce in the Fierz[12] gauge to wave equations, and which have, on account of the identification (3.3), the same physical content as those obtained by linearization of Einstein's theory. By contraction of (3.13) one obtains the additional relation

$$\eta_{\gamma\sigma|\gamma\sigma}-\eta_{\gamma\gamma|\sigma\sigma}=0, \tag{3.14}$$

which can be used to cast (3.13) into the form

$$\eta_{\alpha\beta|\sigma\sigma}-\eta_{\alpha\sigma|\beta\sigma}+\eta_{\sigma\sigma|\alpha\beta}-\eta_{\beta\sigma|\alpha\sigma}=0. \tag{3.15}$$

These equations contain information about true gravitational fields only, as they may be written

$$G_{\alpha,\beta\sigma|\sigma}+G_{\sigma,\sigma\beta|\alpha}=0. \tag{3.16}$$

4. FIELD EQUATIONS

In keeping with the traditional aim of field theory, the field equations governing gravitation ought to be derivable from an action principle

$$\delta\int \mathfrak{L}dx=0, \tag{4.1}$$

with a Lagrangian density

$$\mathfrak{L}=hL \tag{4.2}$$

composed of the determinant $h=\det|h^k{}_\alpha|$ and an invariant action L which is a function of the field variables $h^k{}_\alpha$ and their derivatives. When casting for a suitable action among the invariants that can be formed with these variables one is constrained by the desire to land an expression which is also invariant under coordinate-dependent reorientations of the local inertial frames, and which in the limit of weak fields will again yield the field equations (3.13).

Now, there exist[13] three invariants of first order containing the true gravitational field tensor (2.9) bilinearly, namely,

$$W_1=g^{\alpha\beta}S^\gamma{}_{\alpha\sigma}S^\sigma{}_{\beta\gamma}, \tag{4.3}$$

$$W_2=g_{\alpha\beta}g^{\gamma\sigma}g^{\rho\tau}S^\alpha{}_{\gamma\rho}S^\beta{}_{\sigma\tau}, \tag{4.4}$$

$$W_3=g^{\alpha\beta}S^\gamma{}_{\alpha\gamma}S^\sigma{}_{\beta\sigma}, \tag{4.5}$$

and one invariant of second order containing the derivatives of the true gravitational field tensor linearly, namely,

$$W_4=(g^{\alpha\beta}S^\gamma{}_{\beta\gamma})_{|\alpha}+g^{\alpha\beta}\Delta^\sigma{}_{\alpha\sigma}S^\gamma{}_{\beta\gamma}. \tag{4.6}$$

Compatibility with the weak-field approximation restricts the most general Lagrangian to the linear combination[14]

$$L=\sum_{i=1}^{4} c_i W_i \tag{4.7}$$

with arbitrary coefficients c_i.

[11] W. Wyss, Helv. Phys. Acta 38, 469 (1965).

[12] M. Fierz, Helv. Phys. Acta 12, 3 (1939).

[13] R. Weitzenböck, Ref. 2, Eqs. (20) and (24).

[14] Adding a constant, which is the only invariant of order zero, leads to field equations containing the so-called cosmological term.

Insistence on invariance under coordinate-dependent reorientation transformations (2.6) leads by a straightforward calculation (see Appendix) to the conditions

$$4c_1+c_4=0;\quad 2c_3+c_4=0;\quad 2c_2-c_1=0;\quad 3c_1-2c_2+c_3+c_4=0, \tag{4.8}$$

which select, up to an unimportant common factor, the linear combination

$$L_R=2W_4-W_3-\tfrac{1}{4}W_2-\tfrac{1}{2}W_1 \tag{4.9}$$

as the only Lagrangian with the desired invariance properties that leads to field equations of not higher than second order. This expression happens to be identical with the Riemann scalar R, as can be shown easily by computation[15] using the representation of the affinities in terms of the vierbein field

$$\Gamma^{\gamma}{}_{\alpha\beta}=\tfrac{1}{2}f^{\gamma}{}_{k}[f^{\sigma k}(h_{n\beta}G^{n}{}_{\sigma\alpha}+h_{n\alpha}G^{n}{}_{\sigma\beta})+h^{k}{}_{\beta|\alpha}+h^{k}{}_{\alpha|\beta}], \tag{4.10}$$

and thus this derivation of the Lagrangian (4.9) amounts to a proof for the contention that restriction to reorientation invariant tensor field equations of no higher than second order leads uniquely to Einstein's theory.

Another benefit stems from the observation that hW_4 differs from hW_3 only by a divergence,

$$hW_4=(hg^{\alpha\beta}S^{\gamma}{}_{\beta\gamma})_{|\alpha}+hW_3, \tag{4.11}$$

so that the field equations flowing from the Lagrangian density

$$\mathfrak{L}=hL=h[W_3-\tfrac{1}{4}W_2-\tfrac{1}{2}W_1] \tag{4.12}$$

are equivalent to Einstein's field equations[16] which flow from $\mathfrak{L}_R=hL_R$. The Lagrangian L is a proper scalar[17] and has the advantage of being of first order, resulting in considerable simplification. In particular, there is no need here to cast the action principle in Palatini form, which is such an encumbrance when one wants to translate the conventional approach, based on L_R using the metric tensor and the affinities as variables, into the language of quantum field theory.

As an example of the ease provided by the Lagrangian (4.12) consider the case of a vierbein field described completely by two variables,

$$f^1{}_1=f^2{}_2=f^3{}_3=A(r);\quad f^4{}_4=B(r);\quad \text{all other } f^{\alpha}{}_{k}=0;\quad h=A^{-3}B^{-1} \tag{4.13}$$

corresponding to a metric

$$g^{11}=g^{22}=g^{33}=A^2(r);\quad g^{44}=B^2(r);\quad g^{\alpha\beta}=0 \text{ for } \alpha\neq\beta. \tag{4.14}$$

The Lagrangian density (4.12) becomes

$$\mathfrak{L}=A^{-1}B^{-1}[A^{-2}(\nabla A)^2+2A^{-1}B^{-1}(\nabla A\,\nabla B)] \tag{4.15}$$

and the field equations are

$$\begin{aligned}\nabla^2A-(3/2A)(\nabla A)^2&=0;\\ \nabla^2B-(2/B)(\nabla B)^2-(1/A)(\nabla A\,\nabla B)&=0.\end{aligned} \tag{4.16}$$

They are easily solved and yield, with the boundary conditions $(A,B)\rightarrow 1$ as $r\rightarrow\infty$, the well-known Schwarzschild metric in rectangular coordinates[18]

$$\begin{aligned}A&=[1+(M/2r)]^{-2};\\ B&=[1+(M/2r)]/[1-(M/2r)].\end{aligned} \tag{4.17}$$

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APPENDIX

When the coefficients $R^{i}{}_{k}$ which describe reorientations of the local inertial frame are dependent on the coordinates, the true gravitational field tensors $S^{\gamma}{}_{\alpha\beta}$ introduced in (2.9) do not remain frame-invariant and transform as

$$S'^{\gamma}{}_{\alpha\beta}=S^{\gamma}{}_{\alpha\beta}+R^{i}{}_{k}f^{\gamma}{}_{i}(R^{k}{}_{m|\beta}h^{m}{}_{\alpha}-R^{k}{}_{m|\alpha}h^{m}{}_{\beta}). \tag{A1}$$

Consequently, the Weitzenböck invariants (4.3)–(4.6), which are true scalars under coordinate transformations, are not frame-invariant either and transform as

$$W_1'=W_1+2A-B+3C, \tag{A2}$$

$$W_2'=W_2+2B-2C, \tag{A3}$$

$$W_3'=W_3+C+2D, \tag{A4}$$

$$W_4'=W_4+\tfrac{1}{2}A+D, \tag{A5}$$

where

$$A=R^{i}{}_{m}R^{m}{}_{n|\beta}f^{\sigma}{}_{i}f^{\alpha n}S^{\beta}{}_{\alpha\sigma}, \tag{A6}$$

$$B=g^{\alpha\beta}(2R^{i}{}_{m}R^{m}{}_{k|\beta}f^{\sigma}{}_{i}G^{k}{}_{\alpha\sigma}+R^{n}{}_{k|\alpha}R^{k}{}_{n|\beta}), \tag{A7}$$

$$C=R^{i}{}_{k|\alpha}R^{k}{}_{n|\beta}f^{\alpha}{}_{i}f^{\beta n}, \tag{A8}$$

$$D=R^{i}{}_{m}R^{m}{}_{n|\beta}f^{\beta}{}_{i}f^{\alpha n}S^{\sigma}{}_{\alpha\sigma}. \tag{A9}$$

The condition that the linear combination (4.7) be frame-invariant leads now at once to the relations (4.8).

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[15] R. Weitzenböck, Ref. 2, Eq. (23). The field tensor used here differs from that used by Weitzenböck by a factor 2.

[16] That this linear combination leads to symmetric field equations was known to Einstein [Sitzber. preuss. Akad. Wiss., Physik.-math. Kl. **1929**, 157 (1929)]. For that very reason he rejected it at the time, because he was intent on acquiring more than ten equations for the purpose of obtaining a unified field theory. *Note added in proof*. Dr. P. Rastall kindly pointed out two papers by C. Møller, Kgl. Danske Videnskab. Selskab, Mat.-Fys. Medd. **31**, No. 10 (1961); **34**, No. 3 (1964); who uses the same Lagrangian, expressed in terms of covariant derivatives, as basis for a discussion of the energy-momentum complex.

[17] The corresponding quantity G, obtained by partial integration from the Riemann scalar in the conventional approach, is not a scalar. See L. D. Landau and E. M. Lifshitz, *Classical Theory of Fields* (Addison-Wesley Publishing Co., Reading, Mass., 1962), Sec. 93.

[18] R. C. Tolman, *Relativity Thermodynamics and Cosmology* (Oxford University Press, Oxford, England, 1934), Eq. (82.14).