PHYSICAL REVIEW VOLUME 165, NUMBER 5 25 JANUARY 1968

Electromagnetic Interaction of the Bargmann-Wigner Fields*

SURAJ N. GUPTA AND WAYNE W. REPKO†
Department of Physics, Wayne State University, Detroit, Michigan
(Received 30 August 1967)

It is proposed that the electromagnetic interaction in the Lagrangian formalism for the Bargmann-Wigner fields should be introduced in such a way that the number of linearly independent components of the auxiliary fields is minimized in the resulting field equations. In this way a consistent formulation of the electromagnetic interaction of the Bargmann-Wigner field with spin $\frac{3}{2}$ is obtained, and its equivalence with the Rarita-Schwinger formulation is established.

I. INTRODUCTION

By using the generalized Bargmann-Wigner field equations,[1] it is possible to combine the baryon octet of spin $\frac{1}{2}$ with the baryon decuplet of spin $\frac{3}{2}$ and the pseudoscalar-meson nonet with the vector-meson nonet in a simple manner.[2] Although the $U(6,6)$ hadron model, obtained by the above procedure, is still in a preliminary stage, there seems little doubt that the Bargmann-Wigner form of field equations offers a very promising approach for the description of strong-interaction symmetries. While these multispinor field equations are equivalent to other forms of field equations in the absence of interaction, such equivalence is no longer apparent in the presence of interaction. Indeed, it is known that the usual prescription of replacing ∂_μ by $D_\mu=\partial_\mu-ieA_\mu$ for the electromagnetic interaction, when applied to the multispinor field equations, leads to inconsistencies.

Unless a consistent theory of the electromagnetic interaction of the multispinor fields is developed, the use of such fields in any theory of strong interactions will be looked upon with suspicion. Therefore, in an earlier paper[3] we have investigated the electromagnetic interaction of the multispinor field with spin $\frac{1}{2}$, and we shall now extend our treatment to the field with spin $\frac{3}{2}$. For this purpose we shall first discuss the Lagrangian formalism for the multispinor field with spin $\frac{3}{2}$ in the absence of interaction, which requires the use of auxiliary fields. We shall then show that the usual prescription of replacing ∂_μ by D_μ for the electromagnetic interaction leads to undesirable results, and propose an alternative prescription. Finally, we shall discuss the equivalence between the Bargmann-Wigner and Rarita-Schwinger[4] formulations in the presence of the electromagnetic interaction.

The present investigation shows that the usual treatment of the electromagnetic interaction by means of the replacement of ∂_μ by D_μ is suitable only when the Lagrangian formalism does not involve auxiliary fields. But, when auxiliary fields appear in the Lagrangian formalism, the electromagnetic interaction should be introduced in such a way that it is not only gauge invariant but also minimizes the number of linearly independent components of the auxiliary fields in the presence of interaction.

We shall, as before,[3] denote the space-time coordinates as $x_\mu=(x_i,ix_0)$, and take the γ_μ as Hermitian matrices with $\gamma_\mu\gamma_\nu+\gamma_\nu\gamma_\mu=2\delta_{\mu\nu}$ and $\gamma_\mu\gamma_\nu-\gamma_\nu\gamma_\mu=2i\sigma_{\mu\nu}$. An asterisk will be used to denote the complex conjugate of a number or the Hermitian conjugate of an operator.

2. LAGRANGIAN FORMALISM FOR THE BARGMANN-WIGNER FIELD WITH SPIN $\frac{3}{2}$

According to Bargmann and Wigner,[1] a field with spin $\frac{3}{2}$ is described by a totally symmetric multispinor $\psi_{\alpha\beta\gamma}$ satisfying the field equations

$$(\gamma\partial)_\alpha{}^{\alpha'}\psi_{\alpha'\beta\gamma}=(\gamma\partial)_\beta{}^{\beta'}\psi_{\alpha\beta'\gamma}=(\gamma\partial)_\gamma{}^{\gamma'}\psi_{\alpha\beta\gamma'}=-m\psi_{\alpha\beta\gamma}. \quad (1)$$

In order to obtain the above field equations from the variational principle, it proves necessary to introduce two auxiliary fields $\chi_{\alpha\beta\gamma}$ and $\Omega_{\alpha\beta\gamma}$, where the field $\chi_{\alpha\beta\gamma}$ of mixed symmetry satisfies the relations

$$\chi_{\alpha\beta\gamma}=-\chi_{\beta\alpha\gamma},\quad \chi_{\alpha\beta\gamma}+\chi_{\beta\alpha\gamma}+\chi_{\gamma\alpha\beta}=0, \quad (2)$$

while $\Omega_{\alpha\beta\gamma}$ is totally antisymmetrical. As originally presented by Guralnik and Kibble,[5] the Lagrangian formalism for the Bargmann-Wigner field with spin $\frac{3}{2}$ contained an error which, when corrected, makes the derivation of the field equations excessively complicated. This complication can be avoided by the reformulation presented below.

By choosing the Lagrangian density as

$$\begin{aligned}L=&-\bar{\psi}[(\gamma\partial)_1+m]\psi+\tfrac{2}{3}\bar{\chi}[(\gamma\partial)_1-(\gamma\partial)_3+3m]\chi\\&+\bar{\Omega}[(\gamma\partial)_3-m]\Omega-(\sqrt{3})^{-1}[\bar{\chi}(\gamma\partial)_1\psi+\bar{\psi}(\gamma\partial)_1\chi]\\&+\tfrac{1}{2}[\bar{\chi}(\gamma\partial)_3\Omega+\bar{\Omega}(\gamma\partial)_3\chi], \quad (3)\end{aligned}$$

* Supported in part by the National Science Foundation.
† National Aeronautics and Space Administration Predoctoral Fellow.
[1] V. Bargmann and E. P. Wigner, Proc. Natl. Acad. Sci. (U. S.), **34**, 211 (1948).
[2] A. Salam, R. Delbourgo, and J. Strathdee, Proc. Roy. Soc. (London) **A284**, 146 (1965); B. Sakita and K. C. Wali, Phys. Rev. **139**, B1355 (1965).
[3] S. N. Gupta and W. W. Repko, Phys. Rev. **159**, 1082 (1967).
[4] W. Rarita and J. Schwinger, Phys. Rev. **60**, 61 (1941).
[5] G. S. Guralnik and T. W. B. Kibble, Phys. Rev. **139**, B712 (1965); **150**, 1406 (E) (1966).

with

$$[(\gamma\partial)_1\psi]_{\alpha\beta\gamma}=(\gamma\partial)_\alpha{}^{\alpha'}\psi_{\alpha'\beta\gamma},$$
$$\bar{\psi}^{\alpha\beta\gamma}=\psi^{*\alpha'\beta'\gamma'}(\gamma_4)_{\alpha'}{}^{\alpha}(\gamma_4)_{\beta'}{}^{\beta}(\gamma_4)_{\gamma'}{}^{\gamma},\ \text{etc.,} \quad (4)$$

we obtain by the variations of $\bar{\psi}$, $\bar{\chi}$, and $\bar{\Omega}$ subject to the appropriate symmetry constraints:

$$(\alpha+m)\psi+\tfrac{1}{2}(\beta\chi'+\epsilon\chi)=0, \quad (5)$$

$$-\tfrac{1}{2}(\beta\chi+\epsilon\chi')+m\chi-\tfrac{1}{2}\epsilon\psi+\tfrac{1}{2}\beta\Omega=0, \quad (6)$$

$$\tfrac{1}{2}(\beta\chi'-\epsilon\chi)+m\chi'-\tfrac{1}{2}\beta\psi-\tfrac{1}{2}\epsilon\Omega=0, \quad (7)$$

$$(\alpha-m)\Omega+\tfrac{1}{2}(\beta\chi-\epsilon\chi')=0, \quad (8)$$

where $\chi'_{\alpha\beta\gamma}$ is related to $\chi_{\alpha\beta\gamma}$ by

$$\chi'_{\alpha\beta\gamma}=-(\sqrt{3})^{-1}(\chi_{\beta\gamma\alpha}-\chi_{\gamma\alpha\beta})$$
or
$$\chi_{\alpha\beta\gamma}=(\sqrt{3})^{-1}(\chi'_{\beta\gamma\alpha}-\chi'_{\gamma\alpha\beta}), \quad (9)$$

which enables us to express the field equation, obtained by the variation of $\bar{\chi}$, in the form (6) or (7). We have also introduced the operators

$$\alpha=\tfrac{1}{3}[(\gamma\partial)_1+(\gamma\partial)_2+(\gamma\partial)_3],$$
$$\beta=\tfrac{1}{6}[2(\gamma\partial)_3-(\gamma\partial)_1-(\gamma\partial)_2], \quad (10)$$
$$\epsilon=(2\sqrt{3})^{-1}[(\gamma\partial)_1-(\gamma\partial)_2],$$

which commute with each other and satisfy the relations

$$\alpha\epsilon=\beta\epsilon,\quad \beta^2-\epsilon^2=-2\alpha\beta,\quad \beta^2+\epsilon^2=-\tfrac{1}{2}(\alpha^2-\Box^2) \quad (11)$$

and

$$\beta(9\alpha^2-\Box^2)=0,\quad \epsilon(9\alpha^2-\Box^2)=0. \quad (12)$$

Elimination of Ω from (6) and (7) yields

$$-\beta\epsilon\chi+\tfrac{1}{2}(\beta^2-\epsilon^2)\chi'+m(\epsilon\chi+\beta\chi')-\tfrac{1}{2}(\beta^2+\epsilon^2)\psi=0,$$

or, in view of (11),

$$(\alpha-m)(\beta\chi'+\epsilon\chi)=\tfrac{1}{4}(\alpha^2-\Box^2)\psi,$$

which gives, on using (5),

$$(9\alpha^2-\Box^2)\psi=8m^2\psi. \quad (13)$$

It follows from (12) and (13) that not only

$$\beta\psi=0,\quad \epsilon\psi=0, \quad (14)$$

but (5) reduces, after multiplication by $(9\alpha^2-\Box^2)$, to

$$(\alpha+m)\psi=0. \quad (15)$$

The field equations (14) and (15) are identical with the Bargmann-Wigner equations (1).

Similarly, elimination of ψ from (6) and (7) gives, with the help of (11),

$$(\alpha+m)(\beta\chi-\epsilon\chi')=\tfrac{1}{4}(\alpha^2-\Box^2)\Omega,$$

or, on using (8),

$$(9\alpha^2-\Box^2)\Omega=8m^2\Omega. \quad (16)$$

It then follows from (12) and (16) that

$$\beta\Omega=0,\quad \epsilon\Omega=0, \quad (17)$$

while, after multiplication by $(9\alpha^2-\Box^2)$, (8) reduces to

$$(\alpha-m)\Omega=0, \quad (18)$$

so that, according to (17) and (18),

$$(\gamma\partial)_1\Omega=(\gamma\partial)_2\Omega=(\gamma\partial)_3\Omega=m\Omega, \quad (19)$$

which cannot be satisfied by the totally antisymmetrical Ω unless[6]

$$\Omega=0. \quad (20)$$

In view of (14), (15), and (20), the field equations (5) to (8) reduce to

$$\beta\chi'+\epsilon\chi=0, \quad (21)$$

$$\beta\chi+\epsilon\chi'=2m\chi, \quad (22)$$

$$\beta\chi'-\epsilon\chi=-2m\chi', \quad (23)$$

$$\beta\chi-\epsilon\chi'=0. \quad (24)$$

Since (21) and (24) yield the relation

$$(\beta^2+\epsilon^2)\chi=0,$$

while (22) and (23) give

$$(\beta^2+\epsilon^2)\chi=2m(\beta\chi+\epsilon\chi')=4m^2\chi,$$

we conclude that

$$\chi=0. \quad (25)$$

3. TRANSFORMATION OF THE LAGRANGIAN DENSITY TO THE SPINOR-TENSOR FORM

We can carry out a transformation of the multispinor Lagrangian density (3) to the more familiar spinor-tensor form. By using the symmetrical matrices $(\gamma_\mu C)_{\alpha\beta}$ and $(\sigma_{\mu\nu}C)_{\alpha\beta}$, the multispinor $\psi_{\alpha\beta\gamma}$ can be expressed in terms of the spinor-tensors ψ_μ and $\psi_{\mu\nu}=-\psi_{\nu\mu}$ as[7]

$$\psi_{\alpha\beta\gamma}=\tfrac{1}{2}[(\gamma_\mu C)_{\alpha\beta}(\psi_\mu)_\gamma+\tfrac{1}{2}(\sigma_{\mu\nu}C)_{\alpha\beta}(\psi_{\mu\nu})_\gamma], \quad (26)$$

where the further requirement that the right side of (26) be also symmetrical with respect to the indices β and γ gives

$$\psi_\mu=i\gamma_\nu\psi_{\mu\nu},\quad \sigma_{\mu\nu}\psi_{\mu\nu}=0. \quad (27)$$

Similarly, $\chi_{\alpha\beta\gamma}$ and $\Omega_{\alpha\beta\gamma}$ can be expressed by means of the antisymmetrical matrices $(C)_{\alpha\beta}$, $(i\gamma_\mu\gamma_5C)_{\alpha\beta}$, and $(\gamma_5C)_{\alpha\beta}$ as

$$\chi_{\alpha\beta\gamma}=\tfrac{1}{4}\sqrt{3}[(C)_{\alpha\beta}(\eta)_\gamma+i(\gamma_\mu\gamma_5C)_{\alpha\beta}(\chi_\mu)_\gamma+(\gamma_5C)_{\alpha\beta}(\chi)_\gamma], \quad (28)$$

$$\Omega_{\alpha\beta\gamma}=\tfrac{1}{4}\sqrt{3}[(C)_{\alpha\beta}(\omega)_\gamma+i(\gamma_\mu\gamma_5C)_{\alpha\beta}(\Omega_\mu)_\gamma+(\gamma_5C)_{\alpha\beta}(\Omega)_\gamma], \quad (29)$$

[6] With the usual representation of the Dirac matrices in which γ_4 is diagonal, the vanishing of Ω due to (19) becomes especially transparent in the rest frame.

[7] The matrix C satisfies the relations $C^{-1}=C^*$ and $C^{-1}\gamma_\mu C=-\gamma_\mu^T$, where γ_μ^T is the transpose of γ_μ.

where the symmetry properties of $\chi_{\alpha\beta\gamma}$ and $\Omega_{\alpha\beta\gamma}$ imply

$$\eta=\gamma_5(\chi-i\gamma_\nu\chi_\nu)\,, \tag{30}$$

$$\omega=-\gamma_5\Omega\,,\quad \Omega_\mu=i\gamma_\mu\Omega\,. \tag{31}$$

According to (27), (30), and (31), ψ_μ, η, ω, and Ω_μ can be expressed in terms of $\psi_{\mu\nu}$, χ_μ, χ, and Ω, which are to be regarded as independent field variables except that the components of $\psi_{\mu\nu}$ are interrelated through the condition $\sigma_{\mu\nu}\psi_{\mu\nu}=0$.

The above relations enable us to transform the various terms in (3) to the spinor-tensor form as

$$\bar\psi^{\alpha\beta\gamma}(\gamma\partial)_\alpha{}^{\alpha'}\psi_{\alpha'\beta\gamma}=i\bar\psi_\mu\partial_\nu\psi_{\mu\nu}-\tfrac{1}{2}i\bar\psi_{\mu\nu}(\partial_\mu\psi_\nu-\partial_\nu\psi_\mu)\,, \tag{32}$$
$$\bar\psi^{\alpha\beta\gamma}\psi_{\alpha\beta\gamma}=\bar\psi_\mu\psi_\mu-\tfrac{1}{2}\bar\psi_{\mu\nu}\psi_{\mu\nu}\,;$$

$$\bar\chi^{\alpha\beta\gamma}(\gamma\partial)_\alpha{}^{\alpha'}\chi_{\alpha'\beta\gamma}=\tfrac{3}{4}i(\bar\chi_\mu\partial_\mu\chi+\bar\chi\partial_\mu\chi_\mu)\,,$$
$$\bar\chi^{\alpha\beta\gamma}(\gamma\partial)_\gamma{}^{\gamma'}\chi_{\alpha\beta\gamma'}=-\tfrac{3}{4}[(\bar\chi-i\bar\chi_\mu\gamma_\mu)\gamma\partial(\chi-i\gamma_\nu\chi_\nu)+\bar\chi_\mu\gamma\partial\chi_\mu-\bar\chi\gamma\partial\chi]\,, \tag{33}$$
$$\bar\chi^{\alpha\beta\gamma}\chi_{\alpha\beta\gamma}=\tfrac{3}{4}[(\bar\chi-i\bar\chi_\mu\gamma_\mu)(\chi-i\gamma_\nu\chi_\nu)-\bar\chi_\mu\chi_\mu+\bar\chi\chi]\,;$$

$$\bar\Omega^{\alpha\beta\gamma}(\gamma\partial)_\gamma{}^{\gamma'}\Omega_{\alpha\beta\gamma'}=-\tfrac{3}{2}\bar\Omega\gamma\partial\Omega\,, \tag{34}$$
$$\bar\Omega^{\alpha\beta\gamma}\Omega_{\alpha\beta\gamma}=\tfrac{9}{2}\bar\Omega\Omega\,;$$

$$\bar\psi^{\beta\gamma\alpha}(\gamma\partial)_\alpha{}^{\alpha'}\chi_{\alpha'\beta\gamma}=\tfrac{1}{2}\sqrt{3}(\bar\psi_\mu\gamma_5\partial_\mu\chi-2i\bar\psi_\mu\gamma_5\partial_\mu\gamma_\nu\chi_\nu+i\bar\psi_\mu\gamma_5\gamma\partial\chi_\mu+\bar\psi_{\mu\nu}\gamma_5\partial_\mu\chi_\nu)\,, \tag{35}$$

$$\bar\chi^{\alpha\beta\gamma}(\gamma\partial)_\alpha{}^{\alpha'}\psi_{\beta\gamma\alpha'}=\tfrac{1}{2}\sqrt{3}(\bar\chi\gamma_5\partial_\mu\psi_\mu-2i\bar\chi_\mu\gamma_\mu\gamma_5\partial_\nu\psi_\nu-i\bar\chi_\mu\gamma_5\gamma\partial\psi_\mu-\bar\chi_\mu\gamma_5\partial_\nu\psi_{\mu\nu})\,;$$
$$\bar\Omega^{\alpha\beta\gamma}(\gamma\partial)_\gamma{}^{\gamma'}\chi_{\alpha\beta\gamma'}=\tfrac{3}{2}(\bar\Omega\gamma\partial\chi-i\bar\Omega\partial_\mu\chi_\mu)\,, \tag{36}$$
$$\bar\chi^{\alpha\beta\gamma}(\gamma\partial)_\gamma{}^{\gamma'}\Omega_{\alpha\beta\gamma'}=\tfrac{3}{2}(\bar\chi\gamma\partial\Omega-i\bar\chi_\mu\partial_\mu\Omega)\,,$$

where we have permuted the indices of $\psi_{\alpha\beta\gamma}$ and $\bar\psi^{\alpha\beta\gamma}$ in (35) to obtain simpler terms in the spinor-tensor form.

It is convenient to combine the auxiliary fields χ and χ_μ with $\psi_{\mu\nu}$ and ψ_μ by defining

$$\phi_{\mu\nu}=\psi_{\mu\nu}-(1/18)\sigma_{\mu\nu}\gamma_5\chi\,, \tag{37}$$
$$\phi_\mu=\psi_\mu+\tfrac{1}{2}i\gamma_5\chi_\mu+\tfrac{1}{6}\gamma_\mu\gamma_5\chi\,,$$

which give

$$\psi_{\mu\nu}=\phi_{\mu\nu}-\tfrac{1}{12}\sigma_{\mu\nu}\sigma_{\lambda\rho}\phi_{\lambda\rho}\,,$$
$$\gamma_5\chi_\mu=-2(i\phi_\mu+\gamma_\nu\phi_{\mu\nu})\,, \tag{38}$$
$$\gamma_5\chi=-\tfrac{3}{2}\sigma_{\lambda\rho}\phi_{\lambda\rho}\,.$$

It should be observed that, although $\phi_{\mu\nu}$ shares the antisymmetrical property of $\psi_{\mu\nu}$, its components are not interrelated through any condition, and therefore $\phi_{\mu\nu}$ and ϕ_μ contain the same number of independent components as $\psi_{\mu\nu}$, χ_μ, and χ.

By substituting the relations (32) to (36) into (3), and applying (38), the Lagrangian density can be expressed in terms of the independent variables $\phi_{\mu\nu}$, ϕ_μ, and Ω as

$$\begin{aligned}L={}&2\bar\phi_\mu(\gamma\partial\delta_{\mu\nu}-\gamma_\mu\gamma\partial\gamma_\nu+3m\delta_{\mu\nu}+3m\gamma_\mu\gamma_\nu)\phi_\nu-i(\bar\phi_\mu\partial_\nu\phi_{\mu\nu}+\bar\phi_{\mu\nu}\partial_\nu\phi_\mu)-i(\bar\phi_\mu\gamma\partial\gamma_\nu\phi_{\mu\nu}+\bar\phi_{\mu\nu}\gamma_\nu\gamma\partial\phi_\mu)\\&-(13/12)(\bar\phi_\mu\partial_\mu\sigma_{\lambda\rho}\phi_{\lambda\rho}-\bar\phi_{\lambda\rho}\sigma_{\lambda\rho}\partial_\mu\phi_\mu)-(7/6)(\bar\phi_\mu\gamma_\mu\gamma\partial\sigma_{\lambda\rho}\phi_{\lambda\rho}-\bar\phi_{\lambda\rho}\sigma_{\lambda\rho}\gamma\partial\gamma_\mu\phi_\mu)-2i(\bar\phi_\mu\gamma_\mu\partial_\lambda\gamma_\rho\phi_{\lambda\rho}+\bar\phi_{\lambda\rho}\gamma_\rho\partial_\lambda\gamma_\mu\phi_\mu)\\&+\tfrac{1}{2}m\bar\phi_{\mu\nu}\phi_{\mu\nu}+5m\bar\phi_{\mu\nu}\gamma_\mu\gamma_\lambda\phi_{\nu\lambda}-(97/24)m\bar\phi_{\mu\nu}\sigma_{\mu\nu}\sigma_{\lambda\rho}\phi_{\lambda\rho}-6im(\bar\phi_\mu\gamma_\nu\phi_{\mu\nu}+\bar\phi_{\mu\nu}\gamma_\nu\phi_\mu)+\tfrac{3}{2}m(\bar\phi_\mu\gamma_\mu\sigma_{\lambda\rho}\phi_{\lambda\rho}-\bar\phi_{\lambda\rho}\sigma_{\lambda\rho}\gamma_\mu\phi_\mu)\\&-\tfrac{3}{2}\bar\Omega(\gamma\partial+3m)\Omega-\tfrac{3}{2}(\bar\Omega\gamma_5\partial_\mu\phi_\mu+\bar\phi_\mu\partial_\mu\gamma_5\Omega)+\tfrac{3}{2}i(\bar\Omega\gamma_5\partial_\mu\gamma_\nu\phi_{\mu\nu}+\bar\phi_{\mu\nu}\gamma_\nu\partial_\mu\gamma_5\Omega)+(9/8)(\bar\Omega\gamma_5\gamma\partial\sigma_{\mu\nu}\phi_{\mu\nu}-\bar\phi_{\mu\nu}\sigma_{\mu\nu}\gamma\partial\gamma_5\Omega)\,.\end{aligned} \tag{39}$$

The field equations, obtained from (39), can be reduced to the form

$$(\gamma\partial+m)\phi_\mu=0\,,\quad \phi_{\mu\nu}=-(i/m)(\partial_\mu\phi_\nu-\partial_\nu\phi_\mu)\,, \tag{40}$$

$$\gamma_\mu\phi_\mu=0\,,\quad \sigma_{\mu\nu}\phi_{\mu\nu}=0\,,\quad \phi_\mu-i\gamma_\nu\phi_{\mu\nu}=0\,, \tag{41}$$

$$\Omega=0\,, \tag{42}$$

where (41), when substituted into (38), also implies the vanishing of χ_μ and χ.

We observe that ϕ_μ satisfies the Rarita-Schwinger field equation

$$(\gamma\partial+m)\phi_\mu=0\,, \tag{43}$$

with

$$\gamma_\mu\phi_\mu=0\,. \tag{44}$$

It follows from (43) and (44) that

$$\partial_\mu\phi_\mu=0\,, \tag{45}$$

and, for $\mu=4$, (43) gives on multiplication by γ_4

$$(m-\boldsymbol{\gamma}\cdot\boldsymbol{\nabla})\gamma_4\phi_4+\partial_4\phi_4=0$$

or, in view of (44) and (45),

$$(m-\boldsymbol{\gamma}\cdot\boldsymbol{\nabla})\gamma_k\phi_k+\partial_k\phi_k=0\,. \tag{46}$$

Since the relations (44) and (46) do not involve time derivatives, they are to be regarded as the interdependence relations, which reduce the independent components of ϕ_μ to eight as required for the field of spin $\tfrac{3}{2}$.

4. INTRODUCTION OF ELECTROMAGNETIC INTERACTION

If we ignore the Lagrangian formalism and introduce the electromagnetic interaction merely by replacing ∂_μ by D_μ in the field equations (1), we immediately arrive at an inconsistency, because then in the identity

$$[(\gamma D)_\alpha{}^{\alpha'}(\gamma D)_\beta{}^{\beta'}-(\gamma D)_\beta{}^{\beta'}(\gamma D)_\alpha{}^{\alpha'}]\psi_{\alpha'\beta'\gamma}=-ieF_{\mu\nu}(\gamma_\mu)_\alpha{}^{\alpha'}(\gamma_\nu)_\beta{}^{\beta'}\psi_{\alpha'\beta'\gamma}\,,$$

the left side vanishes owing to the field equations, while the right side remains nonvanishing unless $\psi_{\alpha\beta\gamma}=0$.

This inconsistency can be avoided by using the Lagrangian formalism.

When the electromagnetic interaction is introduced in the Lagrangian density (39) by the usual replacement of ∂_μ by D_μ, we find that corresponding to (40) and (41) we now have

$$(\gamma D+m)\phi_\mu=-3m(\phi_\mu-i\gamma_\nu\phi_{\mu\nu})-(\gamma_\mu\gamma D+D_\mu)\gamma_\lambda\phi_\lambda+4m\gamma_\mu\gamma_\lambda\phi_\lambda-\tfrac{3}{4}(\gamma_\mu\gamma D+2D_\mu)\sigma_{\lambda\rho}\phi_{\lambda\rho}+6m\gamma_\mu\sigma_{\lambda\rho}\phi_{\lambda\rho}, \quad (47)$$

$$\phi_{\mu\nu}=-(i/m)(D_\mu\phi_\nu-D_\nu\phi_\mu)-2i[\gamma_\mu(\phi_\nu-i\gamma_\lambda\phi_{\nu\lambda})-\gamma_\nu(\phi_\mu-i\gamma_\lambda\phi_{\mu\lambda})]+(i/m)(D_\mu\gamma_\nu-D_\nu\gamma_\mu)\gamma_\lambda\phi_\lambda-2\sigma_{\mu\nu}\gamma_\lambda\phi_\lambda+\tfrac{3}{4}(i/m)(D_\mu\gamma_\nu-D_\nu\gamma_\mu)\sigma_{\lambda\rho}\phi_{\lambda\rho}-(8/3)\sigma_{\mu\nu}\sigma_{\lambda\rho}\phi_{\lambda\rho}, \quad (48)$$

with

$$\gamma_\mu\phi_\mu=(1/27)(e/m^2)[(9/4)\sigma_{\lambda\rho}F_{\lambda\rho}\gamma_\mu\phi_\mu-\tfrac{7}{2}i\gamma_\nu\phi_\mu F_{\mu\nu}-4\gamma_\lambda F_{\mu\lambda}\gamma_\nu\phi_{\mu\nu}-2F_{\mu\nu}\phi_{\mu\nu}+(25/24)\sigma_{\lambda\rho}F_{\lambda\rho}\sigma_{\mu\nu}\phi_{\mu\nu}], \quad (49)$$

$$\sigma_{\mu\nu}\phi_{\mu\nu}=(1/27)(e/m^2)[2i\gamma_\nu\phi_\mu F_{\mu\nu}+\gamma_\lambda F_{\mu\lambda}\gamma_\nu\phi_{\mu\nu}+\tfrac{1}{2}F_{\mu\nu}\phi_{\mu\nu}+(7/12)\sigma_{\lambda\rho}F_{\lambda\rho}\sigma_{\mu\nu}\phi_{\mu\nu}], \quad (50)$$

$$\phi_\mu-i\gamma_\nu\phi_{\mu\nu}=\tfrac{1}{6}(e/m^2)[iF_{\mu\nu}\phi_\nu+\tfrac{1}{2}\sigma_{\lambda\rho}F_{\lambda\rho}\phi_\mu-\tfrac{1}{6}\gamma_\mu\gamma_\rho F_{\nu\rho}\gamma_\lambda\phi_{\nu\lambda}-\tfrac{1}{12}\gamma_\mu F_{\lambda\rho}\phi_{\lambda\rho}+\tfrac{1}{2}\gamma_\mu\gamma_\rho\phi_\lambda F_{\lambda\rho}+\tfrac{3}{4}i\gamma_\nu F_{\mu\nu}\sigma_{\lambda\rho}\phi_{\lambda\rho}+(5/12)\gamma_\mu\sigma_{\lambda\rho}F_{\lambda\rho}\sigma_{\xi\eta}\phi_{\xi\eta}]. \quad (51)$$

Moreover, the auxiliary fields no longer vanish, and while χ_μ and χ are given by (38), Ω is given by

$$\gamma_5\Omega=-\tfrac{3}{2}\sigma_{\lambda\rho}\phi_{\lambda\rho}, \quad (52)$$

which shows that $\Omega=\chi$, but no other linear relation is found to exist between the auxiliary fields. Thus, not only the field equations become excessively complicated, but the interdependence relations are also destroyed by the appearance of $\phi_{\mu\nu}$, which contains the time derivative of ϕ_μ, in (49).

In order to overcome the above difficulties, we propose the following prescription, which is suggested by our earlier treatment[3] of the spin-$\frac{1}{2}$ case: *The electromagnetic interaction in the Lagrangian formalism for the Bargmann-Wigner fields should be introduced in such a way that the number of linearly independent components of the auxiliary fields is minimized in the resulting field equations.*

For this purpose, we take the Lagrangian density for the spin-$\frac{3}{2}$ field interacting with the electromagnetic field as

$$L_{\text{total}}=L_0+L'+a_1\bar\phi_\mu\sigma_{\lambda\rho}\phi_\mu F_{\lambda\rho}+ia_2\bar\phi_\mu\phi_\nu F_{\mu\nu}+a_3\bar\phi_\mu\gamma_\mu\sigma_{\lambda\rho}\gamma_\nu\phi_\nu F_{\lambda\rho}+ia_4(\bar\phi_\mu\gamma_\mu\gamma_\rho\phi_\lambda-\bar\phi_\lambda\gamma_\rho\gamma_\mu\phi_\mu)F_{\lambda\rho}+b_1(\bar\phi_\mu\gamma_\mu\sigma_{\lambda\rho}\gamma_5\Omega+\bar\Omega\gamma_5\sigma_{\lambda\rho}\gamma_\mu\phi_\mu)F_{\lambda\rho}+ib_2(\bar\phi_\mu\gamma_\nu\gamma_5\Omega-\bar\Omega\gamma_5\gamma_\nu\phi_\mu)F_{\mu\nu}+c\bar\Omega\sigma_{\mu\nu}\Omega F_{\mu\nu}, \quad (53)$$

where L_0 is the Lagrangian density of the photon field, L' is obtained from (39) on replacing ∂_μ by D_μ, and we have added all possible magnetic-moment-type interaction terms involving ϕ_μ and Ω. The real parameters a_i, b_j, and c will be chosen in a manner which produces the largest number of linear relations with constant coefficients between the auxiliary field components.

The field equations, obtained from (51) by the variations of $\bar\phi_{\mu\nu}$, $\bar\phi_\mu$, and $\bar\Omega$, are

$$\tfrac{1}{2}m\phi_{\mu\nu}+\tfrac{5}{2}m(\gamma_\mu\gamma_\lambda\phi_{\nu\lambda}-\gamma_\nu\gamma_\lambda\phi_{\mu\lambda})-(97/24)m\sigma_{\mu\nu}\sigma_{\lambda\rho}\phi_{\lambda\rho}+\tfrac{1}{2}i(D_\mu\phi_\nu-D_\nu\phi_\mu)+\tfrac{1}{2}i(\gamma_\mu\gamma D\phi_\nu-\gamma_\nu\gamma D\phi_\mu)+(13/12)\sigma_{\mu\nu}D_\lambda\phi_\lambda+(7/6)\sigma_{\mu\nu}\gamma D\gamma_\lambda\phi_\lambda-i(D_\mu\gamma_\nu-D_\nu\gamma_\mu)\gamma_\lambda\phi_\lambda+3im(\gamma_\mu\phi_\nu-\gamma_\nu\phi_\mu)-\tfrac{3}{2}m\sigma_{\mu\nu}\gamma_\lambda\phi_\lambda+\tfrac{3}{4}i(D_\mu\gamma_\nu-D_\nu\gamma_\mu)\gamma_5\Omega-(9/8)\sigma_{\mu\nu}\gamma D\gamma_5\Omega=0, \quad (54)$$

$$2\gamma D\phi_\mu-2\gamma_\mu\gamma D\gamma_\nu\phi_\nu+6m\phi_\mu+6m\gamma_\mu\gamma_\nu\phi_\nu-iD_\nu\phi_{\mu\nu}-i\gamma D\gamma_\nu\phi_{\mu\nu}-(13/12)D_\mu\sigma_{\lambda\rho}\phi_{\lambda\rho}-(7/6)\gamma_\mu\gamma D\sigma_{\lambda\rho}\phi_{\lambda\rho}-2i\gamma_\mu D_\lambda\gamma_\rho\phi_{\lambda\rho}-6im\gamma_\nu\phi_{\mu\nu}+\tfrac{3}{2}m\gamma_\mu\sigma_{\lambda\rho}\phi_{\lambda\rho}-\tfrac{3}{2}D_\mu\gamma_5\Omega+a_1\sigma_{\lambda\rho}F_{\lambda\rho}\phi_\mu+ia_2F_{\mu\nu}\phi_\nu+a_3\gamma_\mu\sigma_{\lambda\rho}F_{\lambda\rho}\gamma_\nu\phi_\nu+ia_4\gamma_\mu\gamma_\rho\phi_\lambda F_{\lambda\rho}-ia_4\gamma_\nu F_{\mu\nu}\gamma_\lambda\phi_\lambda+b_1\gamma_\mu\sigma_{\lambda\rho}F_{\lambda\rho}\gamma_5\Omega+ib_2\gamma_\nu F_{\mu\nu}\gamma_5\Omega=0, \quad (55)$$

$$-\tfrac{3}{2}(\gamma D+3m)\Omega-\tfrac{3}{2}\gamma_5(D_\mu\phi_\mu-iD_\mu\gamma_\nu\phi_{\mu\nu}-\tfrac{3}{4}\gamma D\sigma_{\mu\nu}\phi_{\mu\nu})+b_1\gamma_5\sigma_{\lambda\rho}F_{\lambda\rho}\gamma_\mu\phi_\mu-ib_2\gamma_5\gamma_\nu\phi_\mu F_{\mu\nu}+c\sigma_{\mu\nu}F_{\mu\nu}\Omega=0. \quad (56)$$

Multiplying (54) by $-(5/12)i\gamma_\mu\gamma_\nu$, (55) by $\frac{1}{2}\gamma_\mu$, and (56) by $3\gamma_5$, and then adding, we obtain

$$-(27/2)m\gamma_5\Omega-(81/4)m\sigma_{\mu\nu}\phi_{\mu\nu}+(\tfrac{1}{2}a_1+2a_3+\tfrac{1}{2}a_4+3b_1)\sigma_{\lambda\rho}F_{\lambda\rho}\gamma_\mu\phi_\mu-i(2a_1+a_2-2a_4+3b_2)\gamma_\nu\phi_\mu F_{\mu\nu}+(2b_1-\tfrac{1}{2}b_2+3c)\sigma_{\mu\nu}F_{\mu\nu}\gamma_5\Omega=0, \quad (57)$$

which is free from the space-time derivatives of $\phi_{\mu\nu}$, ϕ_μ, and Ω. Moreover, the interaction terms in (57) vanish if the a_i, b_j, and c satisfy

$$(\tfrac{1}{2}a_1+2a_3+\tfrac{1}{2}a_4+3b_1)=0, \quad (2a_1+\tfrac{1}{2}a_2-2a_4+3b_2)=0, \quad (2b_1-\tfrac{1}{2}b_2+3c)=0, \quad (58)$$

which reduces (57) to the form

$$\gamma_5\Omega=-\tfrac{3}{2}\sigma_{\mu\nu}\phi_{\mu\nu}. \quad (59)$$

Further, by multiplying (54) by

$$2i\delta_{\mu\lambda}D_\nu+\tfrac{4}{3}\delta_{\mu\lambda}\gamma D\gamma_\nu+(17/36)iD_\lambda\gamma_\mu\gamma_\nu-(11/36)i\gamma_\lambda\gamma D\gamma_\mu\gamma_\nu-\tfrac{1}{2}im\gamma_\lambda\gamma_\mu\gamma_\nu,$$

(55) by $m\delta_{\mu\lambda}$, and (56) by $-2m\gamma_\lambda\gamma_5$, and then adding and simplifying with the help of (59), another relation without space-time derivatives is obtained:

$$6m^2\phi_\lambda-6im^2\gamma_\nu\phi_{\lambda\nu}-12m^2\gamma_\lambda\gamma_\mu\phi_\mu-(87/2)m^2\gamma_\lambda\sigma_{\mu\nu}\phi_{\mu\nu}-i(e-ma_2)F_{\lambda\nu}\phi_\nu+(ma_1-\tfrac{1}{2}e)\sigma_{\mu\nu}F_{\mu\nu}\phi_\lambda$$
$$+i(ma_4+2mb_2+e)\gamma_\lambda\gamma_\nu\phi_\mu F_{\mu\nu}-ima_4\gamma_\nu F_{\lambda\nu}\gamma_\mu\phi_\mu+(ma_3-2mb_1+e)\gamma_\lambda\sigma_{\mu\nu}F_{\mu\nu}\gamma_\rho\phi_\rho$$
$$-\tfrac{3}{2}i(\tfrac{1}{2}e+mb_2)\gamma_\rho F_{\lambda\rho}\sigma_{\mu\nu}\phi_{\mu\nu}-\tfrac{3}{2}(mb_1-2mc-\tfrac{3}{4}e)\gamma_\lambda\sigma_{\mu\nu}F_{\mu\nu}\sigma_{\xi\eta}\phi_{\xi\eta}=0. \quad (60)$$

The interaction terms in (60) vanish, provided we take

$$a_1=\tfrac{1}{2}(e/m),\quad a_2=e/m,\quad a_4=0,\quad b_2=-\tfrac{1}{2}(e/m), \quad (61)$$
$$a_3-2b_1=-(e/m),\quad b_1-2c=\tfrac{3}{4}(e/m),$$

which reduces (60) to

$$\phi_\mu-i\gamma_\nu\phi_{\mu\nu}=2\gamma_\mu\gamma_\nu\phi_\nu+(29/4)\gamma_\mu\sigma_{\lambda\rho}\phi_{\lambda\rho}. \quad (62)$$

This relation can be simplified by multiplying by γ_μ, which yields

$$\sigma_{\mu\nu}\phi_{\mu\nu}=-\tfrac{1}{4}\gamma_\mu\phi_\mu, \quad (63)$$

and thus enables us to express (60) as

$$\phi_\mu-i\gamma_\nu\phi_{\mu\nu}=-\tfrac{3}{4}\gamma_\mu\sigma_{\lambda\rho}\phi_{\lambda\rho}. \quad (64)$$

The relations (56) and (59) are compatible and found to determine a_i, b_j, and c uniquely as

$$a_1=\tfrac{1}{2}(e/m),\quad a_2=(e/m),\quad a_3=-\tfrac{1}{2}(e/m),$$
$$a_4=0,\quad b_1=\tfrac{1}{4}(e/m),\quad b_2=-\tfrac{1}{2}(e/m), \quad (65)$$
$$c=-\tfrac{1}{4}(e/m),$$

while it follows from (38), (59), and (64) that χ_μ, χ, and Ω are now related by

$$\chi_\mu=i\gamma_\mu\chi,\quad \chi=\Omega. \quad (66)$$

Thus, by introducing the electromagnetic interaction by means of the Lagrangian density, given by (53) and (65), we are able to reduce the number of linearly independent components of the auxiliary fields χ_μ, χ, and Ω to four, which appears to be the minimum possible number according to our formulation.

5. FIELD EQUATIONS WITH THE ELECTROMAGNETIC INTERACTION

We shall now consider the simplification of the field equations resulting from the treatment of the electromagnetic interaction given in the preceding section. Multiplication of (54) by $\tfrac{2}{3}i\gamma_\nu$ gives, in view of (59), (63), and (64), a field equation solely in terms of ϕ_μ as

$$(\gamma D+m)\phi_\mu+\tfrac{5}{8}D_\mu\gamma_\nu\phi_\nu-\tfrac{3}{2}\gamma_\mu D_\nu\phi_\nu-(43/32)\gamma_\mu\gamma D\gamma_\nu\phi_\nu$$
$$+(50/16)m\gamma_\mu\gamma_\nu\phi_\nu=0, \quad (67)$$

while a further multiplication by γ_μ yields

$$D_\nu\phi_\nu=-(23/16)\gamma D\gamma_\mu\phi_\mu+(27/8)m\gamma_\mu\phi_\mu. \quad (68)$$

It then follows from (67) and (68) that

$$(\gamma D+m)\phi_\mu+\tfrac{5}{8}D_\mu\gamma_\nu\phi_\nu+(13/16)\gamma_\mu\gamma D\gamma_\nu\phi_\nu$$
$$-(31/16)m\gamma_\mu\gamma_\nu\phi_\nu=0, \quad (69)$$

while (54), when simplified with the help of (59), (63), (64), (68), and (69), becomes

$$\phi_{\mu\nu}=-(i/m)(D_\mu\phi_\nu-D_\nu\phi_\mu)$$
$$+(13/16)(i/m)(D_\mu\gamma_\nu-D_\nu\gamma_\mu)\gamma_\lambda\phi_\lambda$$
$$-(7/12)\sigma_{\mu\nu}\gamma_\lambda\phi_\lambda. \quad (70)$$

By means of the transformation

$$\phi_\mu\to\phi_\mu-(13/36)\gamma_\mu\gamma_\nu\phi_\nu, \quad (71)$$
$$\phi_{\mu\nu}\to\phi_{\mu\nu}+(7/27)\sigma_{\mu\nu}\gamma_\lambda\phi_\lambda,$$

the field equations (69) and (70) can be expressed in the simpler form

$$(\gamma D+m)\phi_\mu-D_\mu\gamma_\nu\phi_\nu+\tfrac{1}{2}m\gamma_\mu\gamma_\nu\phi_\nu=0, \quad (72)$$

$$\phi_{\mu\nu}=-(i/m)(D_\mu\phi_\nu-D_\nu\phi_\mu), \quad (73)$$

and the interdependence relations, which follow from (72), are given by

$$\gamma_\mu\phi_\mu=-(e/3m^2)(2i\gamma_\nu F_{\mu\nu}-\sigma_{\lambda\rho}F_{\lambda\rho}\gamma_\mu)\phi_\mu, \quad (74)$$

$$(m-\boldsymbol{\gamma}\cdot\mathbf{D})\gamma_k\phi_k+D_k\phi_k=0. \quad (75)$$

Finally, our field equations with the electromagnetic interaction are in agreement with those obtained by Moldauer and Case[8] by means of the Rarita-Schwinger formulation, which does not involve any auxiliary fields. This agreement confirms that while the usual procedure for introducing the electromagnetic interaction is adequate in the absence of auxiliary fields, it is necessary to follow the procedure of Sec. 4 when the Lagrangian density involves auxiliary fields.

---

[8] P. A. Moldauer and K. M. Case, Phys. Rev. **102**, 279 (1956).