Eq. (8) yields the following values for the crystal-field parameters:

$$b_4=(0.268\pm0.014)\times10^{-4}\mathrm{cm}^{-1},$$

$$b_6=(-0.019\pm0.009)\times10^{-4}\mathrm{cm}^{-1}.$$

Table I compares values of b_4 and b_6 for the Eu^{++} ion in hosts of two different cubic structures. The oxides have the NaCl structure, as does EuS, and the fluorides have the fluorite structure. The host lattices are listed in order in increasing lattice parameter. The strong dependence of the crystal-field parameters on interatomic separation, illustrated in Table I, is consistent with the small values inferred for EuS on the basis of our results. In particular, the change in sign of b_4 in the oxides, occurring at SrO, suggests a possible explanation for the small values of b_4 and b_6 exhibited by EuS. Table II lists the available data concerning K_1/M and K_2/M in the europium chalcogenides, which indicate that b_4 may be going through a minimum as a function of lattice parameter, rather than changing sign. The variation in b_4 in the fluoride series reinforces this opinion. One cannot easily extend this comparison to EuTe, the last number of the chalcogenide series, because it is an antiferromagnetic with a complicated spin arrangement.

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Theory of Toeplitz Determinants and the Spin Correlations of the Two-Dimensional Ising Model. III

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We study the asymptotic behavior, for large separations, of the spin-spin correlation function $\langle\sigma_{0,0}\sigma_{M,N}\rangle$ in the two-dimensional Ising model, where the two spins are not necessarily on the same row. Besides the limiting value for infinite separation, which is the square of the spontaneous magnetization, we evaluate the two leading terms in the asymptotic expression in each of the two cases $T<T_c$ and $T>T_c$. It is found that the nearest singularity of the generating function for the correlation is quite simple in the case $T>T_c$, but much more complicated for $T<T_c$. In an Appendix, we also give exactly in a very simple form the correlation $\langle\sigma_{0,0}\sigma_{N,N}\rangle$ for symmetrical Ising lattice at the critical temperature T_c.

1. INTRODUCTION

IN a previous paper on the two-dimensional Ising model,[1] the asymptotic form for large separation of the spin-spin correlation function $\langle\sigma_{0,0}\sigma_{0,N}\rangle$ was given for two spins in the same row. In this paper, we shall give the asymptotic form of the correlation function $\langle\sigma_{0,0}\sigma_{M,N}\rangle$ for arbitrary M and N, when M^2+N^2 is large. Since the case $M=0$ or $N=0$ is already treated in I, we shall, without loss of generality, assume both M and N to be positive. As in I, we have to treat the three cases $T<T_c$, $T>T_c$, $T=T_c$, separately. We shall, however, give the asymptotic form of the correlation function only for the cases $T<T_c$ and $T>T_c$, where the results in I can be regarded as a special case of our results here. These results are summarized in Sec. 5. For the case $T=T_c$, we shall give in Appendix A the correlation function $\langle\sigma_{0,0}\sigma_{N,N}\rangle$ for a symmetrical Ising model. The asymptotic form of $\langle\sigma_{0,0}\sigma_{M,N}\rangle$ for arbitrary M and N at $T=T_c$ has not been obtained. In other words, we carry out the program outlined in Sec. 8(Aa) of I.

2. CORRELATION $\langle\sigma_{0,0}\sigma_{M,N}\rangle$

Let us consider a two-dimensional Ising lattice with $2\mathfrak{M}\times2\mathfrak{N}$ lattice sites. The lattice sites at the boundary are assumed to join in such a way that $(0,-\mathfrak{N}+1)$ and $(M,\mathfrak{N})$ are nearest neighbors. More precisely, we assume $\mathfrak{M}$ to be multiple of M, and the Hamiltonian is taken to be

$$-E_1\sum_{m=-\mathfrak{M}+1}^{\mathfrak{M}}\Big[\sigma_{m,-\mathfrak{N}+1}\sigma_{m+M,\mathfrak{N}}+\sum_{n=-\mathfrak{N}+1}^{\mathfrak{N}-1}\sigma_{m,n}\sigma_{m,n+1}\Big]$$

$$-E_2\sum_{m=-\mathfrak{M}+1}^{\mathfrak{M}}\sum_{n=-\mathfrak{N}+1}^{\mathfrak{N}}\sigma_{m,n}\sigma_{m+1,n}, \qquad (2.1)$$

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[1] T. T. Wu, Phys. Rev. **149**, 380 (1966). The paper is hereafter referred to as I. For a related article, see B. M. McCoy and T. T. Wu, *ibid.* **155**, 438 (1967), which is II in the series.

where the positive constants E_1 and E_2 are the two interaction energies between pairs of neighboring spins. In (2.1), $\sigma_{m,n}$ is to be interpreted as $\sigma_{m-2\mathfrak{M},n}$ when $m>\mathfrak{M}$.

Note that our Hamiltonian as given by (2.1) is *dependent* on M. That is, we are here proposing to calculate the spin-spin correlation function $\langle\sigma_{0,0}\sigma_{M,N}\rangle$ with a Hamiltonian which varies with M. Strictly speaking this is not the Ising model. However, the dependence on M comes from the boundary terms only. It is our hope that, as $\mathfrak{M}$, $\mathfrak{N}\to\infty$, the "boundary effects" would vanish, and the correlation function we obtain agrees with that of the Ising model. It is inconceivable to us that such artificial alteration of the Hamiltonian is inherently necessary for the analysis of the Ising model. In other words, the present difficulty with the boundary effects is believed to be entirely due to imperfections in our calculational procedure. Unfortunately, despite all of our efforts, a simpler method cannot be found.

In order to understand this particular way of joining the boundaries, we evaluate the free energy for the Ising model on the basis of (2.1). Let $K_1=(kT)^{-1}E_1$ and $K_2=(kT)^{-1}E_2$, then the partition function Z is given by[2]

$$Z^2=(2\cosh K_1\cosh K_2)^{8\mathfrak{M}\mathfrak{N}}\det A, \tag{2.2}$$

where A is the same matrix as the one given by Montroll, Potts, and Ward,[3] except that here

$$A(m+M,\mathfrak{N};m,-\mathfrak{N}+1)=\begin{bmatrix}0&z_1&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}, \tag{2.3}$$

$$A(m,-\mathfrak{N}+1;m+M,\mathfrak{N})=\begin{bmatrix}0&0&0&0\\-z_1&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}, \tag{2.4}$$

and

$$A(m,-\mathfrak{N}+1;m,\mathfrak{N})=A(m,\mathfrak{N};m,-\mathfrak{N}+1)=0, \tag{2.5}$$

where the notation of Montroll, Potts, and Ward[3] is used, in particular,

$$z_i=\tanh K_i, \qquad i=1,2.$$

[2] Strictly speaking, because of boundary effects, Z and $Z\langle\sigma_{0,0}\sigma_{M,N}\rangle$ are each to be expressed in terms of four Pfaffians, P. W. Kasteleyn, J. Math. Phys. **4**, 287 (1963); see p. 293]. However, in view of the difficulty discussed in the predecing paragraph, we here ignore all problems associated with boundary effects.

[3] E. W. Montroll, R. B. Potts, and J. C. Ward, J. Math. Phys. **4**, 308 (1963).

The value of $\det A$ can be evaluated in the following way. Define the $4M\times4M$ matrices $\bar{A}(\bar{m},n;\bar{m}',n')$, where $\bar{m},\bar{m}'=-M^{-1}\mathfrak{M}+1,-M^{-1}\mathfrak{M}+2,\cdots,M^{-1}\mathfrak{M}-1,M^{-1}\mathfrak{M}$ and $n,n'=-\mathfrak{N}+1,\cdots,\mathfrak{N}-1,\mathfrak{N}$, by the matrix elements, with $j,j'=0,1,\cdots,M-1$,

$$[\bar{A}(\bar{m},n;\bar{m}',n')]_{j,j'}=A(M\bar{m}+j,n;M\bar{m}'+j',n'). \tag{2.6}$$

Note that both sides of (2.6) are 4×4 matrices. We shall order these $4M\times4M$ matrices $\bar{A}$ by the index

$$l=n-2\mathfrak{N}\bar{m}, \tag{2.7}$$

which runs from $-2M^{-1}\mathfrak{M}\mathfrak{N}-\mathfrak{N}+1$ to $2M^{-1}\mathfrak{M}\mathfrak{N}-\mathfrak{N}$. Because of these limits, it is convenient to extend l periodically. With this convention, by (2.1) $\bar{A}(\bar{m},n;\bar{m}',n')$ really depends only on $l-l'$; i.e.,

$$\bar{A}(\bar{m},n;\bar{m}',n',)=\bar{A}_{l-l'}, \tag{2.8}$$

with

$$\bar{A}_{l+P}=\bar{A}_l, \tag{2.9}$$

where $P=4M^{-1}\mathfrak{M}\mathfrak{N}$. By rearranging the rows and columns, the original matrix A can be written as

$$\begin{bmatrix}\bar{A}_0&\bar{A}_1&\bar{A}_2&\cdots&\bar{A}_{P-1}\\\bar{A}_{-1}&\bar{A}_0&\bar{A}_1&\cdots&\bar{A}_{P-2}\\\bar{A}_{-2}&\bar{A}_{-1}&\bar{A}_0&\cdots&\bar{A}_{P-3}\\\cdot&\cdot&\cdot&\cdots&\cdot\\\cdot&\cdot&\cdot&\cdots&\cdot\\\cdot&\cdot&\cdot&\cdots&\cdot\\\bar{A}_{-P+1}&\bar{A}_{-P+2}&\bar{A}_{-P+3}&\cdots&\bar{A}_0\end{bmatrix}. \tag{2.10}$$

By (2.9), this is a cyclic matrix. Thus its determinant is easily found to be

$$\det A=\prod_{l=0}^{P-1}\det\lambda^{(l)}, \tag{2.11}$$

where $\lambda^{(l)}$ is the $4M\times4M$ matrix

$$\lambda^{(l)}=\sum_{l'=0}^{P-1}A_{l'}\exp(i2\pi ll'/P). \tag{2.12}$$

It remains to evaluate $\det\lambda^{(l)}$. Equation (2.12) is more explicitly

$$(\lambda^{(l)})_{jj'}=\sum_{n'=-\mathfrak{N}+1}^{\mathfrak{N}}\sum_{m'=-M^{-1}\mathfrak{M}+1}^{M^{-1}\mathfrak{M}}A(M\bar{m}'+j,n';j',0)\times\exp[2\pi il(n'-2\mathfrak{N}\bar{m}')/P]. \tag{2.13}$$

Since $A(\alpha,\beta;\alpha',\beta')$ depends only on $\alpha-\alpha'$ and $\beta-\beta'$, $\lambda^{(l)}$ is a Toeplitz matrix. Therefore, we may define

$$(\lambda^{(l)})_{jj'}=\lambda_{j-j'}{}^{(l)}, \tag{2.14}$$

Furthermore, from (2.13), we have

$$\lambda_{-j}{}^{(l)}=\lambda_{M-j}{}^{(l)}\exp(-i\theta), \tag{2.15}$$

where

$$\theta=\pi Ml/\mathfrak{M}. \tag{2.16}$$

The matrix $\lambda^{(l)}$ can therefore be written as

$$\begin{bmatrix} \lambda_0^{(l)} & \lambda_1^{(l)} & \lambda_2^{(l)} & \cdots & \lambda_{M-1}^{(l)} \\ e^{-i\theta}\lambda_{M-1}^{(l)} & \lambda_0^{(l)} & \lambda_1^{(l)} & \cdots & \lambda_{M-2}^{(l)} \\ e^{-i\theta}\lambda_{M-2}^{(l)} & e^{-i\theta}\lambda_{M-1}^{(l)} & \lambda_0^{(l)} & \cdots & \lambda_{M-3}^{(l)} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ e^{-i\theta}\lambda_1^{(l)} & e^{-i\theta}\lambda_2^{(l)} & e^{-i\theta}\lambda_3^{(l)} & \cdots & \lambda_0^{(l)} \end{bmatrix}. \tag{2.17}$$

The determinant of $\lambda^{(l)}$ is found to be

$$\det\lambda^{(l)} = \prod_{j=0}^{M-1} \det\lambda^{(l)}(j), \tag{2.18}$$

where

$$\lambda^{(l)}(j) = \sum_{j'=0}^{M-1} \lambda_{j'}^{(l)} \exp[i(2\pi j-\theta)j'/M]. \tag{2.19}$$

Substituting (2.13) and (2.14) into (2.19), and taking care of the fact that $A(\alpha\,\beta;\alpha',\beta')$ is nonzero only when (α,β) and (α',β') are either the same or nearest neighbors, we get

$$\begin{aligned}\lambda^{(l)}(j) &= \sum_{j'=0}^{M-1} \sum_{n'=-\mathfrak{N}+1}^{\mathfrak{N}} \sum_{\bar{m}'=-M^{-1}\mathfrak{M}+1}^{M^{-1}\mathfrak{M}} A(M\bar{m}'+j', n'; 0, 0) \exp[2\pi i l(n'-2\mathfrak{N}\bar{m}')/P + i(2\pi j-\theta)j'/M] \\ &= A(0,0;0,0) + A(1,0;0,0)\exp[i(2\pi j-\theta)/M] + A(-1,0;0,0)\exp[-i(2\pi j-\theta)/M] \\ &\quad + A(0,1;0,0)\exp(2\pi i l/P) + A(0,-1;0,0)\exp(-2\pi i l/P).\end{aligned} \tag{2.20}$$

From (2.20), we may make the explicit evaluation

$$\det\lambda^{(l)}(j) = (1+z_1^2)(1+z_2^2) - 2z_2(1-z_1^2)\cos(2\pi j/M - \pi l/\mathfrak{M}) - 2z_1(1-z_2^2)\cos(2\pi l/P). \tag{2.21}$$

From (2.2), (2.11), (2.18), and (2.21), the free energy per spin is given by

$$\begin{aligned} F &= -kT(4\mathfrak{M}\mathfrak{N})^{-1}\ln Z \\ &= -kT\ln(2\cosh K_1 \cosh K_2) \\ &\quad -kT(8\mathfrak{M}\mathfrak{N})^{-1}\sum_{l=0}^{P-1}\sum_{j=0}^{M-1}\ln[(1+z_1^2)(1+z_2^2) - 2z_2(1-z_1^2)\cos(2\pi j/M-\pi l/\mathfrak{M}) - 2z_1(1-z_2^2)\cos(2\pi l/P)]. \end{aligned} \tag{2.22}$$

In the limit $\mathfrak{M}, \mathfrak{N}\to\infty$, (2.22) becomes

$$F = -kT\ln(2\cosh K_1\cosh K_2) - kT(8\pi^2)^{-1}\int_{-\pi}^{\pi} d\phi_1 \int_{-\pi}^{\pi} d\phi_2 \ln[(1+z_1^2)(1+z_2^2) - 2z_2(1-z_1^2)\cos\phi_1 - 2z_1(1-z_2^2)\cos\phi_2]. \tag{2.23}$$

Note that M does not appear in (2.23).

Having verified that the particular way of joining the boundaries does not effect the thermodynamic properties of the system, we turn our attention to the correlation $\langle\sigma_{0,0}\sigma_{M,N}\rangle$. From the lattice site (0, 0), we may arrive at the lattice site (M, N) by the following sequence: (0, 0), (0, −1), (0, −2), ⋯, $(0, -\mathfrak{N}+1)$, $(M, \mathfrak{N})$, $(M, \mathfrak{N}-1)$, ⋯, $(M, N+1)$, (M, N). Again using the notation of Montroll, Potts, and Ward,[3] it is convenient to designate

$$(0,0)L,\ (0,-1)L,\ \cdots,\ (0,-\mathfrak{N}+2)L,\ (0,-\mathfrak{N}+1)L \quad \text{as} \quad ①,$$

$$(0,-1)R,\ (0,-2)R,\ \cdots,\ (0,-\mathfrak{N}+1)R,\ (0,-\mathfrak{N})R \quad \text{as} \quad ②,$$

$$(M,N)R,\ (M,N+1)R,\ \cdots,\ (M,\mathfrak{N}-2)R,\ (M,\mathfrak{N}-1)R \quad \text{as} \quad ③,$$

and

$$(M,N+1)L,\ (M,N+2)L,\ \cdots,\ (M,\mathfrak{N}-1)L,\ (M,\mathfrak{N})L \quad \text{as} \quad ④, \tag{2.24}$$

where $(0, -\mathfrak{N})R$ is defined to be $(0, \mathfrak{N})R$ for the sake of symmetry. With the choice of such a sequence of lattice sites, we can follow the known procedure to express $\langle\sigma_{0,0}\sigma_{M,N}\rangle$ in terms of Pfaffians[2]

$$\langle\sigma_{0,0}\sigma_{M,N}\rangle = \pm \lim_{\mathfrak{M},\mathfrak{N}\to\infty} z_1^{2\mathfrak{N}-N} P(y^{-1}+Q)P(y), \quad (2.25)$$

where $P(y)$ denotes the Pfaffian of y, and the sign on the right-hand side should be chosen so that the correlation is positive. In (2.25),

$$y = (z_1^{-1}-z_1) \begin{array}{c} ① \\ ② \\ ③ \\ ④ \end{array} \overset{\begin{array}{cccc} ① & ② & ③ & ④ \end{array}}{\begin{bmatrix} 0 & -I & 0 & 0 \\ I & 0 & 0 & 0 \\ 0 & 0 & 0 & I \\ 0 & 0 & -I & 0 \end{bmatrix}}, \quad (2.26)$$

$$y^{-1}+Q = (1-z_1^2)^{-1} \begin{array}{c} ① \\ ② \\ ③ \\ ④ \end{array} \overset{\begin{array}{cccc} ① & ② & ③ & ④ \end{array}}{\begin{bmatrix} 0 & \bar{S} & \bar{T} & \bar{U} \\ -\bar{S}^T & 0 & -\bar{U} & \bar{V} \\ -\bar{T} & \bar{U} & 0 & -\bar{S} \\ -\bar{U} & -\bar{V} & \bar{S}^T & 0 \end{bmatrix}}, \quad (2.27)$$

where $\bar{S}^T$ is the transpose of $\bar{S}$. In (2.27), the elements of the finite matrices $\bar{S}$, $\bar{T}$, $\bar{U}$, and $\bar{V}$ have the following limiting values, called S_{mn} etc., as $\mathfrak{M}\to\infty$ and $\mathfrak{N}\to\infty$ for fixed $m, n=0, 1, 2\cdots$:

$$S_{mn} = (1-z_1^2)[0, m-n-1]_{LR} + z_1\delta_{mn}, \quad (2.28)$$

$$T_{mn} = (1-z_1^2)[M, N+m+n]_{LR}, \quad (2.29)$$

$$U_{mn} = (1-z_1^2)[M, N+m+n+1]_{LL}, \quad (2.30)$$

and

$$V_{mn} = -(1-z_1^2)[M, -N-m-n-2]_{LR}, \quad (2.31)$$

with

$$[l_1, l_2]_{LR} = -(2\pi)^{-2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} d\phi_1 d\phi_2 \times \exp(-il_1\phi_1 - il_2\phi_2)[\Delta(\phi_1,\phi_2)]^{-1} \times[1-z_2^2 - z_1(1+z_2^2+2z_2\cos\phi_1)\exp(-i\phi_2)], \quad (2.32)$$

$$[l_1, l_2]_{LL} = -(2\pi)^{-2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} d\phi_1 d\phi_2 \exp(-il_1\phi_1 - il_2\phi_2) \times[\Delta(\phi_1,\phi_2)]^{-1}(2iz_2\sin\phi_1), \quad (2.33)$$

and

$$\Delta(\phi_1,\phi_2) = (1+z_1^2)(1+z_2^2) - 2z_2(1-z_1^2)\cos\phi_1 - 2z_1(1-z_2^2)\cos\phi_2. \quad (2.34)$$

Note that S_{mn} is the same as the a_{m-n} of I.

From (2.26) we have

$$P(y) = (z_1^{-1}-z_1)^{2\mathfrak{N}-N};$$

thus (2.25) becomes

$$\langle\sigma_{0,0}\sigma_{M,N}\rangle = \pm \lim_{\mathfrak{M},\mathfrak{N}\to\infty} (1-z_1^2)^{2\mathfrak{N}-N} P(y^{-1}+Q) = \lim_{\mathfrak{M},\mathfrak{N}\to\infty} \begin{vmatrix} 0 & \bar{S} & \bar{T} & \bar{U} \\ -\bar{S}^T & 0 & -\bar{U} & \bar{V} \\ -\bar{T} & \bar{U} & 0 & -\bar{S} \\ -\bar{U} & -\bar{V} & \bar{S}^T & 0 \end{vmatrix}^{1/2}. \quad (2.35)$$

We are therefore left with the evaluation of the determinant on the right-hand side of (2.35).

3. SPIN CORRELATIONS BELOW THE CRITICAL TEMPERATURE

To obtain the asymptotic form of the correlation function $\langle\sigma_{0,0}\sigma_{M,N}\rangle$ for $T<T_c$, we first consider $f_{MN}(\mathfrak{M}, \mathfrak{N})$, which is the ratio of the expectation value of $\sigma_{0,0}\sigma_{M,N+1}$ for finite $\mathfrak{M}$ and $\mathfrak{N}$ to that of $\sigma_{0,0}\sigma_{M,N}$ for the same $\mathfrak{M}$ and $\mathfrak{N}$. More precisely

$$f_{\mathfrak{M}\mathfrak{N}}{}^2(\mathfrak{M},\mathfrak{N}) = \frac{\begin{vmatrix} 0 & \bar{S} & \bar{T}\sim & \bar{U}\sim \\ -\bar{S}^T & 0 & -\bar{U}\sim & \bar{V}\sim \\ -\sim\bar{T} & \sim\bar{U} & 0 & -\sim\bar{S}\sim \\ -\sim\bar{U} & -\sim\bar{V} & (\sim\bar{S}\sim)^T & 0 \end{vmatrix}}{\begin{vmatrix} 0 & \bar{S} & \bar{T} & \bar{U} \\ -\bar{S}^T & 0 & -\bar{U} & \bar{V} \\ -\bar{T} & \bar{U} & 0 & -\bar{S} \\ -\bar{U} & -\bar{V} & \bar{S}^T & 0 \end{vmatrix}}, \quad (3.1)$$

where the left (right) $\sim$ signifies the deletion of the first row (column) of the matrix. As in I, consider the linear equations

$$\begin{bmatrix} 0 & \bar{S} & \bar{T} & \bar{U} \\ -\bar{S}^T & 0 & -\bar{U} & \bar{V} \\ -\bar{T} & \bar{U} & 0 & -\bar{S} \\ -\bar{U} & -\bar{V} & \bar{S}^T & 0 \end{bmatrix} \begin{bmatrix} \bar{x}_1 & \bar{x}_1' \\ \bar{x}_2 & \bar{x}_2' \\ \bar{x}_3 & \bar{x}_3' \\ \bar{x}_4 & \bar{x}_4' \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \bar{\delta} & 0 \\ 0 & \bar{\delta} \end{bmatrix}, \quad (3.2)$$

where

$$\bar{\delta}=\begin{bmatrix}1\\0\\ \cdot\\ \cdot\\ \cdot\\0\end{bmatrix}, \tag{3.3}$$

the number of elements being $\mathfrak{N}-N$. Application of Jacobi's theorem[4] to (3.1) and (3.2) gives

$$f_{MN}^2(\mathfrak{M},\mathfrak{N})=\bar{x}_{30}\bar{x}_{40}'-\bar{x}_{30}'\bar{x}_{40}. \tag{3.4}$$

where $\bar{x}_{30}$, for example, denotes the first element of the column matrix $\bar{x}_3$.

At this stage, we take the limit $\mathfrak{M}\to\infty$ and $\mathfrak{N}\to\infty$. Let

$$f_{MN}=\lim_{\mathfrak{M},\mathfrak{N}\to\infty} f_{MN}(\mathfrak{M},\mathfrak{N}) \tag{3.5}$$

or

$$f_{MN}=\langle\sigma_{0,0}\sigma_{M,N+1}\rangle/\langle\sigma_{0,0}\sigma_{M,N}\rangle. \tag{3.6}$$

By (2.28)–(2.31) consider the infinite system of linear equations

$$\begin{bmatrix}0 & S & T & U\\ -S^T & 0 & -U & V\\ -T & U & 0 & -S\\ -U & -V & S^T & 0\end{bmatrix}\begin{bmatrix}x_1 & x_1'\\ x_2 & x_2'\\ x_3 & x_3'\\ x_4 & x_4'\end{bmatrix}=\begin{bmatrix}0 & 0\\ 0 & 0\\ \delta & 0\\ 0 & \delta\end{bmatrix}, \tag{3.7}$$

when δ is the infinite column matrix

$$\delta=\begin{bmatrix}1\\0\\0\\ \cdot\\ \cdot\\ \cdot\end{bmatrix}. \tag{3.8}$$

It then follows from (3.4) and (3.5) that

$$f_{MN}^2=x_{30}x_{40}'-x_{30}'x_{40}. \tag{3.9}$$

Once f_{MN} is known, the correlation function, in the limit $\mathfrak{M}\to\infty$ and $\mathfrak{N}\to\infty$ can be expressed as

$$\langle\sigma_{0,0}\sigma_{M,N}\rangle=S_\infty\Big[\prod_{n=N}^{\infty}f_{Mn}\Big]^{-1}, \tag{3.10}$$

where[3,5]

$$S_\infty=[1-(\sinh 2K_1\sinh 2K_2)^{-2}]^{1/4} \tag{3.11}$$

is the square of the spontaneous magnetization.

We now need to evaluate x_3, x_3', x_4, and x_4' when M^2+N^2 is large. We observe that, for large M^2+N^2, the elements of S are of the order of unity while those of T, U, and V are exponentially small. Series expansion in T, U, and V gives

$$\begin{bmatrix}0 & S & T & U\\ -S^T & 0 & -U & V\\ -T & U & 0 & -S\\ -U & -V & S^T & 0\end{bmatrix}^{-1}=\begin{bmatrix}0 & -(S^T)^{-1} & 0 & 0\\ S^{-1} & 0 & 0 & 0\\ 0 & 0 & 0 & (S^T)^{-1}\\ 0 & 0 & -S^{-1} & 0\end{bmatrix}-\begin{bmatrix}0 & -(S^T)^{-1} & 0 & 0\\ S^{-1} & 0 & 0 & 0\\ 0 & 0 & 0 & (S^T)^{-1}\\ 0 & 0 & -S^{-1} & 0\end{bmatrix}$$

$$\times\begin{bmatrix}0 & 0 & T & U\\ 0 & 0 & -U & V\\ -T & U & 0 & 0\\ -U & -V & 0 & 0\end{bmatrix}\begin{bmatrix}0 & -(S^T)^{-1} & 0 & 0\\ S^{-1} & 0 & 0 & 0\\ 0 & 0 & 0 & (S^T)^{-1}\\ 0 & 0 & -S^{-1} & 0\end{bmatrix}+\cdots. \tag{3.12}$$

[4] See, for instance, A. C. Aitken, *Determinants and Matrices* (Interscience Publishers, Inc., New York, 1951), p. 99.
[5] C. N. Yang, Phys. Rev. **85**, 808 (1952).

In particular

$$\begin{bmatrix} x_{30} & x_{30}' \\ x_{40} & x_{40}' \end{bmatrix} = \begin{bmatrix} 0 & 0 & \delta^T & 0 \\ 0 & 0 & 0 & \delta^T \end{bmatrix} \begin{bmatrix} 0 & S & T & U \\ -S^T & 0 & -U & V \\ -T & U & 0 & -S \\ -U & -V & S^T & 0 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \delta & 0 \\ 0 & \delta \end{bmatrix}$$

$$= \begin{bmatrix} 0 & \delta^T S^{-1}\delta \\ -\delta^T S^{-1}\delta & 0 \end{bmatrix}$$

$$+ \begin{bmatrix} \delta^T (S^T)^{-1}[VS^{-1}U - U(S^T)^{-1}V]S^{-1}\delta & -\delta^T (S^T)^{-1}[VS^{-1}T + U(S^T)^{-1}U](S^T)^{-1}\delta \\ \delta^T S^{-1}[US^{-1}U + T(S^T)^{-1}V]S^{-1}\delta & -\delta^T S^{-1}[US^{-1}T - T(S^T)^{-1}U](S^T)^{-1}\delta \end{bmatrix} + \cdots. \quad (3.13)$$

The substitution of (3.13) into (3.9) gives

$$f_{MN}^2 \doteqdot (\delta^T S^{-1}\delta)^2 \{1 - 2(\delta^T S^{-1}\delta)^{-1} \delta^T S^{-1}[US^{-1}U + T(S^T)^{-1}V]S^{-1}\delta\}, \quad (3.14)$$

where, as in I, $\doteqdot$ means that the right- and left-hand sides have the same asymptotic expansion in the limit $M^2+N^2 \to \infty$ and fixed T ($<T_c$ in this case). The terms neglected in (3.14) are smaller than those retained by an exponential factor. We proceed to calculate the right-hand side of (3.14) asymptotically.

From (2.28) and (2.32), we may get

$$S_{mn} = (2\pi)^{-1} \int_0^{2\pi} \psi(\theta) e^{-i(n-m)\theta} d\theta, \quad (3.15)$$

with

$$\psi(\theta) = [(1-\alpha_1 e^{i\theta})(1-\alpha_2 e^{-i\theta})/(1-\alpha_1 e^{-i\theta})(1-\alpha_2 e^{i\theta})]^{1/2}, \quad (3.16)$$

$$\alpha_1 = z_1 z_2^*, \qquad \alpha_2 = z_2^*/z_1, \quad (3.17)$$

and

$$z_2^* = (1-z_2)/(1+z_2). \quad (3.18)$$

The method of Wiener-Hopf, as discussed in I, can be applied to obtain the matrix elements of S^{-1}:

$$(S^{-1})_{mn} = -(2\pi)^{-2} \oint d\xi\, \xi^{-n-1} (1-\alpha^2\xi)^{1/2} (1-\alpha_1\xi)^{-1/2} \oint d\xi'\, \xi'^m (\xi'-\xi)^{-1} (1-\alpha_1\xi'^{-1})^{1/2} (1-\alpha_2\xi'^{-1})^{-1/2}, \quad (3.19)$$

where the contours of integration are the unit circles, except that the one for ξ' is to be indented outward near $\xi'=\xi$. In particular

$$\delta^T S^{-1} \delta = (S^{-1})_{00} = 1, \quad (3.20)$$

$$(S^{-1})_{m0} = (2\pi i)^{-1} \oint d\xi\, \xi^{m-1} (1-\alpha_1\xi^{-1})^{1/2} (1-\alpha_2\xi^{-1})^{-1/2}, \quad (3.21)$$

and

$$(S^{-1})_{0n} = (2\pi i)^{-1} \oint d\xi\, \xi^{-n-1} (1-\alpha_2\xi)^{1/2} (1-\alpha_1\xi)^{-1/2}. \quad (3.22)$$

Substituting (2.29)–(2.33) and (3.19)–(3.22) into (3.14), we get, after performing the matrix multiplication,

$$f_{MN}^2 \doteqdot 1 + (8\pi^4)^{-1}(1-z_1^2)^2 \int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} d\phi_1 d\phi_2 d\phi_3 d\phi_4 \exp[-iM(\phi_1+\phi_3) - i(N+1)(\phi_2+\phi_4)]$$

$$\times \Big\{ [(1-z_2^2)e^{-i\phi_2} - z_1(1+z_2^2+2z_2\cos\phi_1)][(1-z_2^2)e^{-i\phi_4} - z_1(1+z_2^2+2z_2\cos\phi_3)]$$

$$\times \frac{(1-\alpha_2 e^{-i\phi_2})(1-\alpha_1 e^{-i\phi_4})}{(1-\alpha_2 e^{-i\phi_4})(1-\alpha_1 e^{-i\phi_2})} + 4z_2^2 \sin\phi_1 \sin\phi_3 \Big\} [1 - \exp(-i\phi_2 - i\phi_4)]^{-1} [\Delta(\phi_1,\phi_2)\Delta(\phi_3,\phi_4)]^{-1}. \quad (3.23)$$

Equation (3.23) can be simplified considerably if we substitute

$$\cos\phi_1=\tfrac{1}{2}z_2^{-1}(1-z_1^2)^{-1}[(1+z_1^2)(1+z_2^2)-2z_1(1-z_2^2)\cos\phi_2] \tag{3.24}$$

and

$$\cos\phi_3=\tfrac{1}{2}z_2^{-1}(1-z_1^2)^{-1}[(1+z_1^2)(1+z_2^2)-2z_1(1-z_2^2)\cos\phi_4] \tag{3.25}$$

into the brackets. These substitutions are justified since the region where $\Delta(\phi_1, \phi_2)$ and $\Delta(\phi_3, \phi_4)$ are zero is exactly the region which contributes to the integral. We then obtain

$$f_{MN}{}^2 \doteqdot 1-\pi^{-4}(1-z_1^2)^2 z_2^2\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} d\phi_1 d\phi_2 d\phi_3 d\phi_4 \exp[-iM(\phi_1+\phi_3)-iN(\phi_2+\phi_4)]$$
$$\times[e^{i(\phi_2+\phi_4)}-1]^{-1}\sin^2\tfrac{1}{2}(\phi_1-\phi_3)[a-\gamma_1\cos\phi_1-\gamma_1\cos\phi_2]^{-1}[a-\gamma_1\cos\phi_3-\gamma_2\cos\phi_4]^{-1}, \tag{3.26}$$

where

$$a=(1+z_1^2)(1+z_2^2), \qquad \gamma_1=2z_2(1-z_1^2),$$

and

$$\gamma_2=2z_1(1-z_2^2). \tag{3.27}$$

Substituting (3.26) into (3.10), we get

$$\langle\sigma_{0,0}\sigma_{M,N}\rangle \doteqdot S_\infty\Big\{1-\tfrac{1}{2}(2\pi)^{-4}\gamma_1^2\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} d\phi_1 d\phi_2 d\phi_3 d\phi_4 \exp[-iM(\phi_1+\phi_3)-iN(\phi_2+\phi_4)]$$
$$\times[\sin\tfrac{1}{2}(\phi_2+\phi_4)]^{-2}[\sin\tfrac{1}{2}(\phi_1-\phi_3)]^2[a-\gamma_1\cos\phi_1-\gamma_2\cos\phi_2]^{-1}[a-\gamma_1\cos\phi_3-\gamma_2\cos\phi_4]^{-1}\Big\}. \tag{3.28}$$

Equation (3.28) is the desired result. However, the form as it stands is not symmetrical under the exchange of M and N together with that of γ_1 and γ_2. It is trivial to recover this symmetry by observing that if we carry out the integrations over ϕ_2 and ϕ_4 the multiple integral in (3.28) is equal to the residue at

$$a-\gamma_1\cos\phi_1-\gamma_2\cos\phi_2=0, \quad \text{and} \quad a-\gamma_1\cos\phi_3-\gamma_2\cos\phi_4=0. \tag{3.29}$$

From (3.29), we obtain

$$\gamma_1\sin\tfrac{1}{2}(\phi_1-\phi_3)\sin\tfrac{1}{2}(\phi_1+\phi_3)+\gamma_2\sin\tfrac{1}{2}(\phi_2-\phi_4)\sin\tfrac{1}{2}(\phi_2+\phi_4)=0. \tag{3.30}$$

Making use of (3.30), we may write (3.28) in the symmetrical form

$$\langle\sigma_{0,0}\sigma_{M,N}\rangle \doteqdot S_\infty\Big\{1+\tfrac{1}{2}(2\pi)^{-4}\gamma_1\gamma_2\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} d\phi_1 d\phi_2 d\phi_3 d\phi_4$$
$$\times\exp[-iM(\phi_1+\phi_3)-iN(\phi_2+\phi_4)]\sin\tfrac{1}{2}(\phi_1-\phi_2)\sin\tfrac{1}{2}(\phi_2-\phi_4)$$
$$\times[\sin\tfrac{1}{2}(\phi_1+\phi_3)\sin\tfrac{1}{2}(\phi_2+\phi_4)]^{-1}[a-\gamma_1\cos\phi_1-\gamma_2\cos\phi_2]^{-1}[a-\gamma_1\cos\phi_3-\gamma_2\cos\phi_4]^{-1}\Big\}. \tag{3.31}$$

The asymptotic evaluation of the right-hand side of (3.31) is rather tedious and carried out in Appendix B. The first few terms of the result is given explicitly in (5.2).

4. SPIN CORRELATIONS ABOVE THE CRITICAL TEMPERATURE

We next turn our attention to the case $T>T_c$. Even in the special case treated in I, it is necessary to modify the Toeplitz determinant. We accordingly define

$$D(M,N;\mathfrak{M},\mathfrak{N})=\begin{array}{c} \\ \textcircled{1} \\ \textcircled{2}' \\ \textcircled{3} \\ \textcircled{4}' \end{array}\begin{array}{cccc} \textcircled{1} & \textcircled{2}' & \textcircled{3}' & \textcircled{4}' \\ \end{array}\left|\begin{array}{cccc} 0 & \bar{S}\approx & \bar{T} & \bar{U}\approx \\ -\approx\bar{S}^T & 0 & -\approx U & \approx V\approx \\ -\bar{T} & \bar{U}\approx & 0 & -\bar{S}\approx \\ -\approx\bar{U} & -\approx V\approx & \approx\bar{S}^T & 0 \end{array}\right|, \tag{4.1}$$

where the right (left) $\approx$ signifies the addition of a column (row) to the matrix, making use of the points $(0, 0)R$

and $(M, N)L$ in addition to those of (2.24); i.e., we designate

$$(0,0)R,\ (0,-1)R,\ (0,-2)R,\ \cdots,\ (0,-\mathfrak{N}+1)R,\ (0,-\mathfrak{N})R \quad \text{as} \quad ②',$$

and

$$(M,N)L,\ (M,N+1)L,\ (M,N+2)L,\ \cdots,\ (M,\mathfrak{N}-1)L,\ (M,\mathfrak{N})L \quad \text{as} \quad ④'. \tag{4.2}$$

We consider the ratio

$$r(M,N;\mathfrak{M},\mathfrak{N}) = [D(M,N;\mathfrak{M},\mathfrak{N})]^{-1} \begin{array}{c|cccc|} & ① & ② & ③ & ④ \\ ① & 0 & \bar{S} & \bar{T} & \bar{U} \\ ② & -\bar{S}^T & 0 & -\bar{U} & \bar{V} \\ ③ & -\bar{T} & \bar{U} & 0 & -\bar{S} \\ ④ & -\bar{U} & -\bar{V} & \bar{S}^T & 0 \end{array}. \tag{4.3}$$

Analogous to (3.3), we consider the linear equations

$$\begin{bmatrix} 0 & \bar{S}\approx & \bar{T} & U\approx \\ -\approx\bar{S}^T & 0 & -\approx\bar{U} & \approx\bar{V}\approx \\ -\bar{T} & \bar{U}\approx & 0 & -\bar{S}\approx \\ -\approx\bar{U} & -\approx\bar{V}\approx & \approx\bar{S}^T & 0 \end{bmatrix} \begin{bmatrix} \bar{x}_1 & \bar{x}_1' \\ \bar{x}_2 & \bar{x}_2' \\ \bar{x}_3 & \bar{x}_3' \\ \bar{x}_4 & \bar{x}_4' \end{bmatrix} \begin{bmatrix} 0 & 0 \\ \bar{\delta}' & 0 \\ 0 & 0 \\ 0 & \bar{\delta}'' \end{bmatrix}, \tag{4.4}$$

where

$$\bar{\delta}' = \begin{bmatrix} 1 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix}, \qquad \bar{\delta}'' = \begin{bmatrix} 1 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix} \tag{4.5}$$

the number of elements being $\mathfrak{N}+1$ for $\bar{\delta}'$, and $\mathfrak{N}-N+1$ for $\bar{\delta}''$ [compare (3.4)]. Again application of Jacobi's theorem[3] to (4.3) and (4.4) gives

$$r(M,N;\mathfrak{M},\mathfrak{N}) = \bar{x}_{20}\bar{x}_{40}' - \bar{x}_{20}'\bar{x}_{40}. \tag{4.6}$$

Consider the limit $\mathfrak{M}, \mathfrak{N} \to \infty$. First, the ratio

$$D(M,N+1;\mathfrak{M},\mathfrak{N})/D(M,N;\mathfrak{M},\mathfrak{N})$$

can be obtained by solving a system of linear equations. Since the index of the kernel that generates the matrix

$$\bar{S} = \lim_{\mathfrak{M},\mathfrak{N}\to\infty} \bar{S}\approx \tag{4.7}$$

is zero, the procedure of Sec. 3 can be applied, to show that the quantity

$$1 - \lim_{\mathfrak{M},\mathfrak{N}\to\infty} [D(M,N+1;\mathfrak{M},\mathfrak{N})/D(M,N;\mathfrak{M},\mathfrak{N})] \tag{4.8}$$

is exponentially small as M^2+N^2 is large. Thus

$$D(M) = \lim_{N\to\infty} \lim_{\mathfrak{M},\mathfrak{N}\to\infty} D(M,N;\mathfrak{M},\mathfrak{N}) \tag{4.9}$$

exists and by (4.3) and (2.35)

$$\langle \sigma_{0,0}\sigma_{M,N} \rangle \doteqdot [D(M)r(M,N)]^{1/2}, \tag{4.10}$$

where

$$r(M,N) = \lim_{\mathfrak{M},\mathfrak{N}\to\infty} r(M,N;\mathfrak{M},\mathfrak{N}). \tag{4.11}$$

In order to compute $r(M, N)$, consider the infinite system of linear equations

$$\begin{bmatrix} 0 & \tilde{S} & T & \tilde{U} \\ -\tilde{S}^T & 0 & -\tilde{U} & \tilde{V} \\ -T & \tilde{U} & 0 & -\tilde{S} \\ -\tilde{U} & \tilde{V} & \tilde{S}^T & 0 \end{bmatrix} \begin{bmatrix} x_1 & x_1' \\ x_2 & x_2' \\ x_3 & x_3' \\ x_4 & x_4' \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ \delta & 0 \\ 0 & 0 \\ 0 & \delta \end{bmatrix}, \tag{4.12}$$

where

$$\tilde{U} = \lim_{\mathfrak{M},\mathfrak{N}\to\infty} \bar{U}\approx,$$

and

$$\tilde{V} = \lim_{\mathfrak{M},\mathfrak{N}\to\infty} \approx\bar{V}\approx. \tag{4.13}$$

Both (4.7) and (4.13) hold for each fixed matrix element. With (4.12), it follows from (4.6) and (4.11) that

$$r(M,N) = x_{20}x_{40}' - x_{20}'x_{40}. \tag{4.14}$$

As before, we may obtain the asymptotic expansions of x_{20}, x_{20}', x_{40}, and x_{30}' by expanding the inverse matrix in a perturbation series (3.12):

$$x_2 = x_4' = 0,$$

and

$$x_2' = -x_4 = -\tilde{S}^{-1}T(\tilde{S}^T)^{-1}\delta, \tag{4.15}$$

to first order in T, $\tilde{U}$, and $\tilde{V}$. The terms neglected in

(4.15) are exponentially smaller than those retained as $M^2+N^2\to\infty$. The substitution of (4.15) into (4.14) gives

$$r(M,N)\doteq[\delta^T\tilde{S}^{-1}T(\tilde{S}^T)^{-1}\delta]^2. \qquad (4.16)$$

To obtain the matrix $\tilde{S}^{-1}$, we solve the equations

$$\sum_{l=0}^{\infty}b_{n-l}(\tilde{S}^{-1})_{lm}=\delta_{nm} \qquad (4.17)$$

for $n\geq 0$ (see Sec. 2 of I). The solution is

$$(\tilde{S}^{-1})_{lm}=-(2\pi)^{-2}\oint[(1-\alpha_1\xi)(1-\alpha_2^{-1}\xi)]^{-1/2}\xi^{-m-1}d\xi$$

$$\times\oint[(1-\alpha_1\xi'^{-1})]^{1/2}(\xi'-\xi)^{-1}\xi'^{l}d\xi', \qquad (4.18)$$

where the contours of integration are the unit circles, except that the one for ξ' is to be indented outward near $\xi'=\xi$. In particular,

$$(\tilde{S}^{-1})_{0m}=(2\pi i)^{-1}\oint[(1-\alpha_1\xi)(1-\alpha_2^{-1}\xi)]^{-1/2}\xi^{-m-1}d\xi. \qquad (4.19)$$

From (2.29), (4.16), and (4.19), we obtain

$$r(M,N)\doteq\bigg\{(2\pi)^{-2}(1-z_1^2)\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}d\phi_1d\phi_2$$

$$\times\exp(-iM\phi_1-iN\phi_2)(a-\gamma_1\cos\phi_1-\gamma_2\cos\phi_2)^{-1}$$

$$\times[1-z_2^2-z_1(1+z_2^2+2z_2\cos\phi_1)e^{-i\phi_2}]$$

$$\times(1-\alpha_1e^{-i\phi_2})^{-1}(1-\alpha_2^{-1}e^{-i\phi_2})^{-1}\bigg\}^2. \qquad (4.20)$$

By setting

$$\cos\phi_1=\gamma_1^{-1}(a-\gamma_2\cos\phi_2),$$

(4.20) takes the form

$$r(M,N)\doteq(1-z_2^2)^2F_{M,N}^2, \qquad (4.21)$$

where

$$F_{M,N}=(2\pi)^{-2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}d\phi_1d\phi_2$$

$$\times\exp(-iM\phi_1-iN\phi_2)(a-\gamma_1\cos\phi_1-\gamma_2\cos\phi_2)^{-1}. \qquad (4.22)$$

It follows from (4.10) and (4.21) that

$$\langle\sigma_{0,0}\sigma_{M,N}\rangle\doteq DF_{M,N}, \qquad (4.23)$$

where

$$D=[D(M)]^{1/2}(1-z_2^2)$$

is independent of M, as $\langle\sigma_{0,0}\sigma_{M,N}\rangle$ should be symmetrical with respect to the exchange of M, z_1 and N, z_2. The values of D can be obtained by comparing (4.23) with the known asymptotic form of $\langle\sigma_{0,0}\sigma_{0,N}\rangle$ given in I. The asymptotic form of $F_{M,N}$ is derived in Appendix C. The asymptotic form of $\langle\sigma_{0,0}\sigma_{M,N}\rangle$ will be explicitly given in the next section. The value of D is obtained as

$$(\gamma_1\gamma_2)^{-1/2}[(\sinh 2K_1\sinh 2K_2)^{-1}-1]^{1/4}.$$

5. SUMMARY AND DISCUSSIONS

Our results may be summarized as follows.[6] When M^2+N^2 is large,

$$\langle\sigma_{0,0}\sigma_{M,N}\rangle\sim(2\pi)^{-1/2}[(\sinh 2K_1\sinh 2K_2)^{-2}-1]^{1/4}(M\sinh\theta_1\cosh\theta_2+N\cosh\theta_1\sinh\theta_2)^{-1/2}\exp(-M\theta_1-N\theta_2)$$
$$\times\{1-(24)^{-1}(M\tanh\theta_1+N\tanh\theta_2)^{-3}[3M^2\tanh^2\theta_1(1+\tanh^2\theta_2)$$
$$+3N^2\tanh^2\theta_2(1+\tanh^2\theta_1)-\tanh^3\theta_1\tanh\theta_2(3-5\tanh^2\theta_2)M^3/N-\tanh^3\theta_2\tanh\theta_1(3-5\tanh^2\theta_1)N^3/M$$
$$+2\tanh\theta_1\tanh\theta_2MN(3+3\tanh^2\theta_1+3\tanh^2\theta_2-5\tanh^2\theta_1\tanh^2\theta_2)]+\cdots\} \qquad (5.1)$$

for $T>T_c$, and

$$\langle\sigma_{0,0}\sigma_{M,N}\rangle\sim[1-(\sinh 2K_1\sinh 2K_2)^{-2}]^{1/4}(1+(8\pi)^{-1}(M\sinh\theta_1\cosh\theta_2+N\cosh\theta_1\sinh\theta_2)^{-2}\exp(-2M\theta_1-2N\theta_2)$$
$$\times\{1-\tfrac{1}{12}(M\tanh\theta_1+N\tanh\theta_2)^{-3}[21M^2\tanh^2\theta_1(1+2\tanh^2\theta_2)$$
$$+21N^2\tanh^2\theta_2(1+2\tanh^2\theta_1)-\tanh^3\theta_1\tanh\theta_2(3-17\tanh^2\theta_2)M^3/N-\tanh^3\theta_2\tanh\theta_1(3-17\tanh^2\theta_1)N^3/M$$
$$+2MN\tanh\theta_1\tanh\theta_2(21+12\tanh^2\theta_1+12\tanh^2\theta_2-17\tanh^2\theta_1\tanh^2\theta_2)]+\cdots\}) \qquad (5.2)$$

for $T<T_c$. In (5.1) and (5.2), θ_1 and θ_2 are defined by

$$\cosh\theta_1=\{aM^2+[M^2N^2a^2+(M^2-N^2)(M^2\gamma_2^2-N^2\gamma_1^2)]^{1/2}\}^{-1}[M^2\gamma_1^{-1}(a^2-\gamma_2^2)+N^2\gamma_1],$$

and

$$\cosh\theta_2=\{aN^2+[M^2N^2a^2+(M^2-N^2)(M^2\gamma_2^2-N^2\gamma_1^2)]^{1/2}\}^{-1}[N^2\gamma_2^{-1}(a^2-\gamma_1^2)+M^2\gamma_2], \qquad (5.3)$$

where a, γ_1, and γ_2 are given by (3.27). Note that θ_1 and θ_2 are both positive when M and N are positive.

[6] The leading term of the correlation function $\langle\sigma_{0,0}\ \sigma_{M,N}\rangle$ has been discussed previously by G. V. Ryazanov {Zh. Eksperim. i Teor. Fiz. **49**, 1134 (1965) [English transl.: Soviet Phys.—JETP **22**, 789 (1966)]} and L. P. Kadanoff [Nuovo Cimento **44**, 276 (1966)] in the special case where $E_1=E_2$ and T does not differ too much from T_c. Their results are not in agreement with ours.

Even though these results are rather complicated, (5.1) actually comes from the very simple formula

$$\langle\sigma_{0,0}\sigma_{M,N}\rangle \doteqdot (\gamma_1\gamma_2)^{-1/2}[(\sinh 2K_1 \sinh 2K_2)^{-2}-1]^{1/4} \times(2\pi)^{-2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} d\phi_1 d\phi_2 \exp(-iM\phi_1-iN\phi_2) \times[a-\gamma_1\cos\phi_1-\gamma_2\cos\phi_2]^{-1}. \quad (5.4)$$

If Σ is the generating function

$$\Sigma(\phi_1,\phi_2)=\sum_{M,N=-\infty}^{\infty}\langle\sigma_{0,0}\sigma_{M,N}\rangle \exp(iM\phi_1+iN\phi_2), \quad (5.5)$$

then

$$\Sigma(\phi_1,\phi_2)-(\gamma_1\gamma_2)^{-1/2}[(\sinh 2K_1\sinh 2K_2)^{-2}-1]^{1/4} \times[a-\gamma_1\cos\phi_1-\gamma_2\cos\phi_2]^{-1} \quad (5.6)$$

is analytic in the tube

$$a-\gamma_1\cosh(\tfrac{1}{2}\,\mathrm{Im}\phi_1)-\gamma_2\cosh(\tfrac{1}{2}\,\mathrm{Im}\phi_2)>0. \quad (5.7)$$

The formula corresponding to (5.4) for $T<T_c$ is much more complicated, namely (3.31). In this case, the generating function

$$\Sigma(\phi_1,\phi_2)=\sum_{M,N=-\infty}^{\infty}(\langle\sigma_{0,0}\sigma_{M,N}\rangle-S_\infty)\exp(iM\phi_1+iN\phi_2) \quad (5.8)$$

has much more complicated analytic properties, which are the same as

$$\tfrac{1}{2}(2\pi)^{-2}\gamma_1\gamma_2 S_\infty(\sin\tfrac{1}{2}\phi_1\sin\tfrac{1}{2}\phi_2)^{-1} \times\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}d\delta_1 d\delta_2\sin\delta_1\sin\delta_2 \times[a-\gamma_1\cos(\delta_1+\tfrac{1}{2}\phi_1)-\gamma_2\cos(\delta_2+\tfrac{1}{2}\phi_2)]^{-1} \times[a-\gamma_1\cos(\delta_1-\tfrac{1}{2}\phi_1)-\gamma_2\cos(\delta_2-\tfrac{1}{2}\phi_2)]^{-1} \quad (5.9)$$

in the tube

$$a-\gamma_1\cosh(\tfrac{1}{4}\,\mathrm{Im}\phi_1)-\gamma_2\cosh(\tfrac{1}{4}\,\mathrm{Im}\phi_2)>0. \quad (5.10)$$

The function defined by (5.9) is studied in some more detail in Appendix D.

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APPENDIX A

At the critical temperature T_c, the matrix elements of T, U, and V do not vanish sufficiently fast for the perturbation method to hold, and we have not been able to obtain the asymptotic form of $\langle\sigma_{0,0}\sigma_{M,N}\rangle$ in general. However, we shall give here in closed form the correlation function $\langle\sigma_{0,0}\sigma_{N,N}\rangle$ for a symmetrical lattice ($z_1=z_2$) at the critical temperature T_c.

To arrive at the lattice site (N, N) from the lattice site $(0, 0)$, let us take the path $(0, 0)$, $(0, 1)$, $(1, 1)$, $(1, 2)\cdots(n, n)$, $(n, n+1)$, $\cdots$, $(N-1, N)$, (N, N). The correlation function is then given by the standard formula[3]

$$\langle\sigma_{0,0}\sigma_{N,N}\rangle=\pm z^{2N}P(y^{-1}+Q)P(y), \quad (A1)$$

with the proper sign chosen to make this correlation positive. In the above, y is the $4N\times4N$ matrix

$$y=(z_1^{-1}-z_1)\begin{bmatrix}0 & I & 0 & 0\\ -I & 0 & 0 & 0\\ 0 & 0 & 0 & I\\ 0 & 0 & -I & 0\end{bmatrix}, \quad (A2)$$

and

$$y^{-1}+Q=\begin{bmatrix}[j-i,j-i]_{RR} & [j-i,j-i+1]_{RL}-c\delta_{ij} & [j-i,j-i+1]_{RU} & [j-i+1,j-i+1]_{RD}\\ [j-i,j-i-1]_{LR}+c\delta_{ij} & [j-i,j-i]_{LL} & [j-i,j-i]_{LU} & [j-i+1,j-i]_{LD}\\ [j-i,j-i-1]_{UR} & [j-i,j-i]_{UL} & [j-i,j-i]_{UU} & [j-i+1,j-i]_{UD}-c\delta_{ij}\\ [j-i-1,j-i-1]_{DR} & [j-i-1,j-i]_{DL} & [j-i-1,j-i]_{DU}+c\delta_{ij} & [j-i,j-i]_{DD}\end{bmatrix}, \quad (A3)$$

where each of the elements in (A3) is a $N\times N$ matrix, with i and j running from 1 to N, and where

$$c=(z_1^{-1}-z_1)^{-1}. \quad (A4)$$

For a symmetrical lattice, at T_c, we have

$$z_1=z_2=z=\sqrt{2}-1, \qquad c=\tfrac{1}{2}. \quad (A5)$$

Since $[i,j]_{RR}=-[i,j]_{LL}$, we may make use of (A5) and (2.33) to obtain the recursive relation

$$[n+1,n+1]_{RR}=[\pi z(2n+1)]^{-1}-(\delta_{n,0}-\delta_{n+1,0})/(4z)-[n,n]_{RR}, \quad (A6)$$

It follows from (A6) that for $n \geq 0$,

$$[n, n]_{RR} = (\pi z)^{-1} \sum_{l=0}^{n-1} (-1)^{n-l+1}/(2l+1) + \frac{(-1)^n}{4z} - (4z)^{-1}\delta_{n0}. \tag{A7}$$

For $n<0$, we have

$$[-n, -n]_{RR} = -[n, n]_{RR}.$$

When (A5) holds, all other matrix elements in (A3) can be expressed in terms of $[n, n]$. We shall not go into the algebra and shall quote only the final results. We found that (A3) can be written as

$$y^{-1}+Q = z^{-1} \begin{bmatrix} X_1 & -X_1-\sqrt{2}X_2 & -X_1-X_2+X_3 & X_1+X_2-X_3/z \\ -X_1+\sqrt{2}X_2 & -X_1 & -X_1-X_2-X_3 z & -X_1-X_2+X_3 \\ -X_1+X_2-X_3^T & -X_1+X_2+X_3^T z & -X_1 & -X_1-\sqrt{2}X_2 \\ X_1-X_2+X_3^T/z & -X_1+X_2-X_3^T & -X_1+\sqrt{2}X_2 & X_1 \end{bmatrix}, \tag{A8}$$

where X_1, X_2, and X_3 are $N\times N$ matrices given by

$$(X_1)_{ij} = z[j-i, j-i]_{RR}, \tag{A9}$$

$$(X_2)_{ij} = \delta_{ij}/4, \tag{A10}$$

$$(X_3)_{ij} = (2\pi)^{-1}(2j-2i+1)^{-1}. \tag{A11}$$

The Pfaffian of $y^{-1}+Q$ can be expressed very simply in terms of X_1, X_2, and X_3. Let us denote

$$W = \frac{1}{\sqrt{2}} \begin{bmatrix} \mathrm{I} & 0 & 0 & \mathrm{II} \\ 0 & \mathrm{I} & \mathrm{II} & 0 \\ 0 & \mathrm{II} & -\mathrm{I} & 0 \\ \mathrm{II} & 0 & 0 & -\mathrm{I} \end{bmatrix}, \tag{A12}$$

where II is the $N\times N$ matrix

$$\mathrm{II} = \begin{bmatrix} 0 & & & & & 1 \\ & & & & 1 & \\ & & & 1 & & \\ & & \cdot & & & \\ & \cdot & & & & \\ \cdot & & & & & \\ & 1 & & & & \\ 1 & & & & & 0 \end{bmatrix}$$

and I the unit matrix. We first observe here that if A is any Toeplitz matrix, $\mathrm{II}A\mathrm{II} = A^T$. The skew-symmetrical matrix $W(y^{-1}+Q)W$ can then be evaluated to be as shown in Eqs. (A13)–(A14) on p. 731.

The right side of Eq. (A14) is obtained from the left side by first adding the first column to the second column and then substracting the resulting second column from the first column. From (A1), (A2), (A4), and (A14) we get

$$\langle \sigma_{0,0}\sigma_{N,N} \rangle = 2^{3N} \det(A), \tag{A15}$$

where

$$A = \begin{bmatrix} X_1(\mathrm{I}-\mathrm{II}) + X_2(\mathrm{I}-\mathrm{II}/\sqrt{2}) - z^{-1}X_3/\sqrt{2} & -X_2\mathrm{II}/\sqrt{2} + X_3/\sqrt{2} \\ -X_2\mathrm{II}/\sqrt{2} + X_3/\sqrt{2} & X_1(\mathrm{I}-\mathrm{II}) + X_2(\mathrm{I}+\mathrm{II}/\sqrt{2}) - zX_3/\sqrt{2} \end{bmatrix}. \tag{A16}$$

Let us split the matrix X_1 into two parts,

$$X_1 = X_1' + X_1'', \tag{A17}$$

where, for $j-i\geq 0$, the skew-symmetrical matrices X_1', X_1'' are given by

$$(X_1')_{ij}=\pi^{-1}\sum_{l=0}^{j-i}(-1)^{j-i-l+1}/(2l+1), \tag{A18}$$

$$(X_1'')_{ij}=\tfrac{1}{4}(-1)^{j-i}-\tfrac{1}{4}\delta_{ij}. \tag{A19}$$

We observe that X_1' and X_3 are proportional to π^{-1}, while X_1'' and X_2, have no π^{-1} factor. Let us therefore split A into two parts:

$$A=A'+A'', \tag{A20}$$

where

$$A'=\begin{bmatrix} X_1'(\mathrm{I}-\mathrm{II})-z^{-1}X_3/\sqrt{2} & X_3/\sqrt{2} \\ X_3/\sqrt{2} & X_1'(\mathrm{I}-\mathrm{II})-zX_3/\sqrt{2} \end{bmatrix}, \tag{A21}$$

and

$$A''=\begin{bmatrix} X_1''(\mathrm{I}-\mathrm{II})+X_2(\mathrm{I}-\mathrm{II}/\sqrt{2}) & -X_2\mathrm{II}/\sqrt{2} \\ -X_2\mathrm{II}/\sqrt{2} & X_1''(\mathrm{I}-\mathrm{II})+X_2(\mathrm{I}+\mathrm{II}/\sqrt{2}) \end{bmatrix}$$

$$=(X_1''-X_1''\mathrm{II}+X_2)\begin{bmatrix} \mathrm{I} & 0 \\ 0 & \mathrm{I} \end{bmatrix}+(2)^{-1/2}X_2\mathrm{II}\begin{bmatrix} -\mathrm{I} & -\mathrm{I} \\ -\mathrm{I} & \mathrm{I} \end{bmatrix}. \tag{A22}$$

To diagonalize A'', we note that the first matrix in the right-hand side of (A22) is diagonal and will remain so under any similarity transformation. Thus we only need to diagonalize the matrix

$$\begin{bmatrix} -\mathrm{I} & -\mathrm{I} \\ -\mathrm{I} & \mathrm{I} \end{bmatrix}.$$

This can be done by the operation

$$T(A'')=(1+z^2)^{-1}\begin{bmatrix} \mathrm{I} & z \\ -z & \mathrm{I} \end{bmatrix}A''\begin{bmatrix} \mathrm{I} & -z \\ z & \mathrm{I} \end{bmatrix}$$

$$=\begin{bmatrix} X_1''(\mathrm{I}-\mathrm{II})+X_2(\mathrm{I}-\mathrm{II}) & 0 \\ 0 & X_1''(\mathrm{I}-\mathrm{II})+X_2(\mathrm{I}+\mathrm{II}) \end{bmatrix}. \tag{A23}$$

Under the T operation A' becomes

$$T(A')=\begin{bmatrix} X_1'(\mathrm{I}-\mathrm{II})-X_3 & X_3 \\ X_3 & X_1'(\mathrm{I}-\mathrm{II})-X_3 \end{bmatrix}. \tag{A24}$$

In the following discussion we shall assume that N is even, as the case N is odd can be done in the same way with but slight modification. For N even we may write

$$\mathrm{II}=\begin{bmatrix} 0 & \mathrm{II} \\ \mathrm{II} & 0 \end{bmatrix}, \tag{A25}$$

where II in the right side of (A25) is actually of the order of $\frac{1}{2}N\times\frac{1}{2}N$, and is hence not quite the same as the II matrix on the left. However, we shall use the same notation for them as there is little chance for confusion. Now we may diagonalize II by the transformation T':

$$T'(\mathrm{II})=\frac{1}{2}\begin{bmatrix} \mathrm{I} & \mathrm{II} \\ -\mathrm{II} & \mathrm{I} \end{bmatrix}\begin{bmatrix} 0 & \mathrm{II} \\ \mathrm{II} & 0 \end{bmatrix}\begin{bmatrix} \mathrm{I} & -\mathrm{II} \\ \mathrm{II} & \mathrm{I} \end{bmatrix}$$

$$=\begin{bmatrix} \mathrm{I} & 0 \\ 0 & -\mathrm{I} \end{bmatrix}. \tag{A26}$$

From (A26) we have

$$T'(\mathrm{I}-\mathrm{II})=\begin{bmatrix} 0 & 0 \\ 0 & 2\mathrm{I} \end{bmatrix}, \tag{A27}$$

$$T'(\mathrm{I}+\mathrm{II})=\begin{bmatrix} 2\mathrm{I} & 0 \\ 0 & 0 \end{bmatrix}. \tag{A28}$$

$$W(y^{-1}+Q)W$$

$$=z^{-1}\begin{bmatrix} 0 & 0 & X_1(\mathrm{I}-\mathrm{II})+X_2(\mathrm{I}-\sqrt{2}\mathrm{II})-X_3 & -X_1(\mathrm{I}-\mathrm{II})-X_2+z^{-1}X_3 \\ 0 & 0 & X_1(\mathrm{I}-\mathrm{II})+X_2+X_3z & X_1(\mathrm{I}-\mathrm{II})+X_2(\mathrm{I}+\sqrt{2}\mathrm{II})-X_3 \\ -(\mathrm{I}-\mathrm{II})X_1^T-(\mathrm{I}-\sqrt{2}\mathrm{II})X_2+X_3^T & -(\mathrm{I}-\mathrm{II})X_1^T-X_2-X_3^Tz & 0 & 0 \\ (\mathrm{I}-\mathrm{II})X_1^T+X_2-z^{-1}X_3^T & -(\mathrm{I}-\mathrm{II})X_1^T-(\mathrm{I}+\sqrt{2}\mathrm{II})X_2+X_3^T & 0 & 0 \end{bmatrix}. \quad \text{(A13)}$$

Now

$$\begin{vmatrix} X_1(\mathrm{I}-\mathrm{II})+X_2(\mathrm{I}-\sqrt{2}\mathrm{II})-X_3 & -X_1(\mathrm{I}-\mathrm{II})-X_2+(1/z)X_3 \\ X_1(\mathrm{I}-\mathrm{II})+X_2+X_3z & X_1(\mathrm{I}-\mathrm{II})+X_2(\mathrm{I}+\sqrt{2}\mathrm{II})-X_3 \end{vmatrix} = 2^N \begin{vmatrix} X_1(\mathrm{I}-\mathrm{II})+X_2(\mathrm{I}-(\mathrm{II}/\sqrt{2}))-(1/z\sqrt{2})X_3 & -(X_2/\sqrt{2})\mathrm{II}+(1/\sqrt{2})X_3 \\ -(X_2/\sqrt{2})\mathrm{II}+(1/\sqrt{2})X_3 & X_1(\mathrm{I}-\mathrm{II})+X_2(\mathrm{I}+(\mathrm{II}/\sqrt{2}))-(zX_3/\sqrt{2}) \end{vmatrix}. \quad \text{(A14)}$$

It follows from (A19) that we may write

$$X_1''=\begin{bmatrix} X_1'' & H \\ -H^T & X_1'' \end{bmatrix}, \qquad \text{(A29)}$$

where X_1'' in the right side of (A29) again is of order $\frac{1}{2}N\times\frac{1}{2}N$,

$$X_1''=\frac{1}{4}\begin{bmatrix} 0 & -1 & 1 & \cdot & \cdot \\ 1 & 0 & -1 & \cdot & \cdot \\ -1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}, \qquad H=X_1''+\tfrac{1}{4}\mathrm{I}. \qquad \text{(A30)}$$

Under the transformation T', X_1'' becomes

$$T'(X_1'')=\begin{bmatrix} 0 & H+\mathrm{II}X_1'' \\ -H^T-\mathrm{II}X_1'' & 0 \end{bmatrix}. \qquad \text{(A31)}$$

From (A27), (A28), and (A31) we get

$$T'(X_1''-X_1''\mathrm{II}+X_2-X_2\mathrm{II})=\begin{bmatrix} 0 & 2(H+\mathrm{II}X_1'') \\ 0 & \frac{1}{2}\mathrm{I} \end{bmatrix},$$

$$T'(X_1''-X_1''\mathrm{II}+X_2+X_2\mathrm{II})=\begin{bmatrix} \frac{1}{2}\mathrm{I} & 2(H+\mathrm{II}X_1'') \\ 0 & 0 \end{bmatrix}. \qquad \text{(A32)}$$

And from (A23) and (A32), we obtain

$$T'T(A'')=\begin{bmatrix} 0 & 2(H+\mathrm{II}X_1'') & 0 & 0 \\ 0 & \frac{1}{2}\mathrm{I} & 0 & 0 \\ 0 & 0 & \frac{1}{2}\mathrm{I} & 2(H+\mathrm{II}X_1'') \\ 0 & 0 & 0 & 0 \end{bmatrix}. \qquad \text{(A33)}$$

Similarly, if we express X_1' in the form

$$X_1'=\begin{bmatrix} X_1' & B \\ -B^T & X_1' \end{bmatrix}, \qquad \text{(A34)}$$

then

$$T'(X_1'-X_1'\mathrm{II})=\begin{bmatrix} 0 & 2(B+\mathrm{II}X_1') \\ 0 & 0 \end{bmatrix}. \qquad \text{(A35)}$$

Let us denote

$$T'(X_3)=\begin{bmatrix} C & E \\ F & D \end{bmatrix}. \qquad \text{(A36)}$$

From (A23), (A24), (A31), (A33), (A35), and (A36), we obtain

$$T'T(A)=\begin{bmatrix}0 & 2(H+\mathrm{II}X_1'') & 0 & 0\\ 0 & \frac{1}{2}\mathrm{I} & 0 & 0\\ 0 & 0 & \frac{1}{2}\mathrm{I} & 2(H+\mathrm{II}X_1'')\\ 0 & 0 & 0 & 0\end{bmatrix}+\begin{bmatrix}-C & -E+2(B+\mathrm{II}X_1') & C & E\\ -F & -D & F & D\\ C & E & -C & -E+2(B+\mathrm{II}X_1')\\ F & D & -F & -D\end{bmatrix}. \tag{A37}$$

We observe now that both matrices in the right side of (A37) are of the order $2N\times 2N$ and the rank N. The determinant of $T'TA$ is therefore equal to

$$\sum \det(M')\det(M''),$$

where M' is a n-rowed minor of $T'T(A')$ and M'' the cofactor of the corresponding minor of $T'T(A'')$. For the minor

$$M_1=\begin{bmatrix}\frac{1}{2}\mathrm{I} & 0\\ 0 & \frac{1}{2}\mathrm{I}\end{bmatrix} \tag{A38}$$

of $T'T(A'')$, the cofactor of the corresponding minor in $T'T(A')$ is

$$M_2=\begin{bmatrix}-C & E\\ F & -D\end{bmatrix}. \tag{A39}$$

The determinant of $T'T(A)$ is equal to $\det(M_1)$ det (M_2), as all products of other corresponding pairs of minor determinants are zero. Thus we have from (A15), (A38), (A39), and (A36) that

$$\langle\sigma_{0,0}\sigma_{N,N}\rangle=2^{2N}\det(X_3), \tag{A40}$$

with X_3 given by (A11). The closed form for $\det(X_3)$ has been found in I to be

$$\det(X_3)=2^{2N(N-1)}\pi^{-N}[G(N)]^4[G(2N)]^{-1}, \tag{A41}$$

where

$$G(N)=1^{N-1}2^{N-2}3^{N-3}\cdots(N-1). \tag{A42}$$

From (A40) and (A41), we finally have

$$\langle\sigma_{0,0}\sigma_{N,N}\rangle=2^{N(2N-1)}\pi^{-N}[G(N)]^4[G(2N)]^{-1}. \tag{A43}$$

A few special cases will be given below

$$\langle\sigma_{0,0}\sigma_{0,0}\rangle=1, \tag{A44}$$

$$\langle\sigma_{0,0}\sigma_{1,1}\rangle=2/\pi, \tag{A45}$$

$$\langle\sigma_{0,0}\sigma_{2,2}\rangle=16/3\pi^2. \tag{A46}$$

And as $N\to\infty$, we have

$$\langle\sigma_{0,0}\sigma_{N,N}\rangle\sim e^{1/4}2^{1/12}A^{-3}N^{-1/4}(1-\tfrac{1}{64}N^{-2}+\cdots), \tag{A47}$$

where A is Glaisher's constant[7]

$$A\approx 1.282427130.$$

It is interesting to compare (A47) with

$$\langle\sigma_{0,0}\sigma_{0,\sqrt{2}N}\rangle\sim e^{1/4}2^{1/12}A^{-3}N^{-1/4}[1+(1/128)N^{-2}+\cdots] \tag{A48}$$

from I. On the basis of (A47) and (A48), it is conjectured that, for $E_1=E_2$ and $T=T_c$, the spin-spin correlation is asymptotically isotropic for large separations.

Most of the results of this Appendix, including the above conjecture, are known to Onsager.[8]

APPENDIX B

We present here the calculation of the asymptotic series of the spin correlation function below the critical temperature. After carrying out the integration over ϕ_2 and ϕ_4, (3.31) becomes

$$\langle\sigma_{0,0}\sigma_{M,N}\rangle\doteqdot S_\infty\left\{1-(8\pi^2)^{-1}\gamma_1\gamma_2^{-1}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}d\phi_1 d\phi_3\exp[-iM(\phi_1+\phi_3)-iN(\phi_2+\phi_4)]\right.$$
$$\left.\times\sin\tfrac{1}{2}(\phi_1-\phi_3)\sin\tfrac{1}{2}(\phi_2-\phi_4)[\sin\phi_2\sin\phi_4\sin\tfrac{1}{2}(\phi_1+\phi_3)\sin\tfrac{1}{2}(\phi_2+\phi_4)]^{-1}\right\}, \tag{B1}$$

where ϕ_2 and ϕ_4 are related to ϕ_1 and ϕ_3 by (3.29). To make the integral in (B1) more symmetrical, let us introduce the variables u and v

$$\gamma_1\cosh\psi_1=\tfrac{1}{2}a+u,\qquad \gamma_1\cosh\psi_3=\tfrac{1}{2}a+v, \tag{B2}$$

[7] J. W. L. Glaisher, Massenger Math. **24**, 1 (1894).
[8] See also J. Stephenson, J. Math. Phys. **5**, 1009 (1964).

where

$$\psi_n = i\phi_n, \qquad n=1,2,3,4.$$

Then (3.29) gives

$$\gamma_2 \cosh\psi_2 = \tfrac{1}{2}a - u, \qquad \gamma_2 \cosh\psi_4 = \tfrac{1}{2}a - v. \tag{B3}$$

With the variables u and v, (B1) is reduced to

$$\langle \sigma_{0,0}\sigma_{M,N}\rangle \doteqdot S_\infty \Big\{ 1-(8\pi^2)^{-1}\gamma_1^{-1}\gamma_2^{-1} \iint du\, dv \exp[-M(\psi_1+\psi_3)-N(\psi_2+\psi_4)]$$

$$\times \sinh\tfrac{1}{2}(\psi_1-\psi_3)\sinh\tfrac{1}{2}(\psi_2-\psi_4)[\sinh\tfrac{1}{2}(\psi_1+\psi_3)\sinh\tfrac{1}{2}(\psi_2+\psi_4)]^{-1}\cdot(\prod_{i=1}^{4}\sinh\psi_i)^{-1}\Big\}. \tag{B4}$$

The saddle point is easily found to be

$$\psi_1=\psi_3=\theta_1, \qquad \psi_2=\psi_4=\theta_2, \tag{B5}$$

where θ_1 and θ_2 are given by (5.3). We shall expand the integrand of (B4) in the neighborhood of the saddle point. From (B2) we have

$$d\psi_1/du = (\gamma_1 \sinh\psi_1)^{-1}, \tag{B6}$$

$$d^2\psi_1/du^2 = -\gamma_1^{-2}(\sinh\psi_1)^{-3}\cosh\psi_1, \tag{B7}$$

$$d^3\psi_1/du^3 = \gamma_1^{-3}(\sinh\psi_1)^{-5}(3+2\sinh^2\psi_1), \tag{B8}$$

$$d^4\psi_1/du^4 = -3\gamma_1^{-4}(\sinh\psi_1)^{-7}\cosh\psi_1(5+2\sinh^2\psi_1), \tag{B9}$$

and similarly for the derivations of ψ_3 with respect to v. The derivatives of ψ_2, for example, can be obtained from (B6)–(B9) by the replacement

$$u\to -u, \qquad \psi_1\to\psi_2, \qquad \gamma_1\to\gamma_2.$$

Writing

$$\xi = u-u_0, \qquad \eta = v-v_0, \tag{B10}$$

where u_0 and v_0 are the values of u and v at the saddle point, we have, in the neighborhood of the point

$$\sinh\tfrac{1}{2}(\psi_1-\psi_3)\sim(2\gamma_1\sinh\theta_1)^{-1}(\xi-\eta)$$

$$\times[1-(2\gamma_1\sinh^2\theta_1)^{-1}\cosh\theta_1(\xi+\eta)+(8\gamma_1^2\sinh^4\theta_1)^{-1}(4+3\sinh^2\theta_1)(\xi+\eta)^2-(2\gamma_1^2\sinh^4\theta_1)^{-1}\cosh^2\theta_1\xi\eta], \tag{B11}$$

$$\sinh\tfrac{1}{2}(\psi_2-\psi_4)\sim-(2\gamma_2\sinh\theta_2)^{-1}(\xi-\eta)$$

$$\times[1+(2\gamma_2\sinh^2\theta_2)^{-1}\cosh\theta_2(\xi+\eta)+(8\gamma_2^2\sinh^4\theta_2)^{-1}(4+3\sinh^2\theta_2)(\xi+\eta)^2-(2\gamma_2^2\sinh^4\theta_2)^{-1}\cosh^2\theta_2\xi\eta], \tag{B12}$$

$$[\sinh\tfrac{1}{2}(\psi_1+\psi_3)]^{-1}\sim(\sinh\theta_1)^{-1}$$

$$\times[1-(2\gamma_1\sinh^2\theta_1)^{-1}\cosh\theta_1(\xi+\eta)+(8\gamma_1^2\sinh^4\theta_1)^{-1}(3\sinh^2\theta_1+4)(\xi+\eta)^2-(2\gamma_1^2\sinh^4\theta_1)^{-1}\cosh^2\theta_1\xi\eta], \tag{B13}$$

$$[\sinh\tfrac{1}{2}(\psi_2+\psi_4)]^{-1}\sim(\sinh\theta_2)^{-1}$$

$$+[1+(2\gamma_2\sinh^2\theta_2)^{-1}\cosh\theta_2(\xi+\eta)+(8\gamma_2^2\sinh^4\theta_2)^{-1}(3\sinh^2\theta_2+4)(\xi+\eta)^2-(2\gamma_2^2\sinh^4\theta_2)^{-1}\cosh^2\theta_2\xi\eta], \tag{B14}$$

$$(\prod_{n=1}^{4}\sin\psi_n)^{-1}\sim(\sinh^2\theta_1\sinh^2\theta_2)^{-1}\{1+[(\gamma_2\sinh^2\theta_2)^{-1}\cosh\theta_2-(\gamma_1\sinh^2\theta_1)^{-1}\cosh\theta_1](\xi+\eta)$$

$$+[(2\gamma_1^2\sinh^4\theta_1)^{-1}(3+2\sinh^2\theta_1)+(2\gamma_2^2\sinh^4\theta_2)^{-1}(3+2\sinh^2\theta_2)-(\gamma_1\gamma_2\sinh^2\theta_1\sinh^2\theta_2)^{-1}\cosh\theta_1\cosh\theta_2]$$

$$\times(\xi+\eta)^2-[(\gamma_1^2\sinh^4\theta_1)^{-1}(2+\sinh^2\theta_1)+(\gamma_2^2\sinh^4\theta_2)(2+\sinh^2\theta_2)]\xi\eta\}, \tag{B15}$$

and

$$\exp[-M(\psi_1+\psi_3)-N(\psi_2+\psi_4)]\sim\exp(-2M\theta_1-2N\theta_2)$$

$$\times\exp\{[(2\gamma_1^2\sinh^3\theta_1)^{-1}M\cosh\theta_1+(2\gamma_2^2\sinh^3\theta_2)^{-1}N\cosh\theta_2](\xi^2+\eta^2)\}$$

$$\times\{1-\tfrac{1}{6}[(\gamma_1^3\sinh^5\theta_1)^{-1}(3+\sinh^2\theta_1)M-(\gamma_2^3\sinh^5\theta_2)^{-1}(3+2\sinh^2\theta_2)N](\xi^3+\eta^3)$$

$$+\tfrac{1}{8}[(\gamma_1^4\sinh^7\theta_1)^{-1}\cosh\theta_1(5+2\sinh^2\theta_1)M+(\gamma_2^4\sinh^7\theta_2)^{-1}\cosh\theta_2(5+2\sinh^2\theta_2)N](\xi^4+\eta^5)$$

$$+(1/72)[(\gamma_1^3\sinh^5\theta_1)^{-1}(3+2\sinh^2\theta_1)M-(\gamma_2^3\sinh^5\theta_2)^{-1}(3+2\sinh^2\theta_2)N]^2(\xi^3+\eta^3)^2\}. \tag{B16}$$

We also have

$$\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy\ (x-y)^2 \exp(-ax^2-ay^2)[1,\ (x+y)^2,\ xy,\ x^4+y^4,\ (x+y)(x^3+y^3),\ (x^3+y^3)^2]$$
$$=(\pi/a^2)(1,\ a^{-1},\ -\tfrac{1}{2}a^{-1},\ \tfrac{9}{2}a^{-2},\ 3a^{-2},\ \tfrac{51}{4}a^{-3}). \quad \text{(B17)}$$

Substituting (B.11)–(B16) into (B4), and putting

$$\xi = iy, \qquad \eta = ix,$$

we may obtain (5.1) by making use of (B17).

APPENDIX C

We shall derive the asymptotic form of $F_{M,N}$ defined in (4.22), when M^2+N^2 is large. This asymptotic form was first evaluated by Montroll, Potts, and Ward,[3] however, their calculation is not correct.[9]

We start with the alternative form for $F_{M,N}$,[3]

$$F_{M,N}=\int_0^{\infty} dx\ e^{-ax} I_M(x\gamma_1) I_N(x\gamma_2). \quad \text{(C1)}$$

Being content with obtaining the first two terms in the asymptotic series, we shall replace $I_n(Z)$ in (C1) by

$$I_n(Z) \sim \exp[(n^2+Z^2)^{1/2} - n \sinh^{-1}(n/Z)](2\pi)^{-1/2}(n^2+Z^2)^{-1/4}[1-(24n)^{-1}(1+Z^2/n^2)^{-3/2}(2-3Z^2/n^2)]. \quad \text{(C2)}$$

Substituting (C2) into (C1), we get

$$F_{M,N} \sim (2\pi)^{-1} \int_0^{\infty} dx \exp\left[-ax+(M^2+\gamma_1^2x^2)^{1/2} - M \sinh^{-1}\left(\frac{M}{\gamma_1 x}\right) + (N^2+\gamma_2^2x^2)^{1/2} - N \sinh^{-1}\left(\frac{N}{\gamma_2 x}\right)\right]$$
$$\times (M^2+\gamma_1^2x^2)^{-1/4}(N^2+\gamma_2^2x^2)^{-1/4}$$
$$\times[1-(24M)^{-1}(1+\gamma_1^2x^2/M^2)^{-3/2}(2-3\gamma_1^2x^2/M^2)-(24N)^{-1}(1+\gamma_2^2x^2/N^2)^{-3/2}(2-3\gamma_2^2x^2/N^2)]. \quad \text{(C3)}$$

We shall evaluate (C3) by the saddle-point method. Let us define

$$g(x)=ax-(M^2+\gamma_1^2x^2)^{1/2}-(N^2+\gamma_2^2x^2)^{1/2}+M\sinh^{-1}\left(\frac{M}{\gamma_1 x}\right)+N\sinh^{-1}\left(\frac{N}{\gamma_2 x}\right). \quad \text{(C4)}$$

The derivative of $g(x)$ is given by

$$g^{(1)}(x)=a-x^{-1}(M^2+\gamma_1^2x^2)^{1/2}-x^{-1}(N^2+\gamma_2^2x^2)^{1/2}. \quad \text{(C5)}$$

The saddle point x_0 at which $g^{(1)}(x)$ vanishes will now be determined. It is convenient to adopt the following notation

$$\theta_1=\sinh^{-1}(M/\gamma_1x_0), \qquad \theta_2=\sinh^{-1}(N/\gamma_2x_0). \quad \text{(C6)}$$

It follows from (C6) that

$$\gamma_1 N \sinh\theta_1 = \gamma_2 M \sinh\theta_2. \quad \text{(C7)}$$

From (C5) and (C6) we have

$$\gamma_1 \cosh\theta_1 + \gamma_2 \cosh\theta_2 = a. \quad \text{(C8)}$$

Solving (C7) and (C8), we may obtain θ_1 and θ_2 which are explicitly given in (5.3). The saddle point x_0 is given by

$$x_0=\{a^2(M^2+N^2)-(M^2-N^2)(\gamma_1^2-\gamma_2^2)+2a[M^2N^2a^2+(M^2-N^2)(M^2\gamma_2^2-N^2\gamma_1^2)]^{1/2}\}^{1/2}$$
$$\times[a^2-(\gamma_1+\gamma_2)^2]^{-1/2}[a^2-(\gamma_1-\gamma_2)^2]^{-1/2}. \quad \text{(C9)}$$

The higher derivatives of $g(x)$ at the saddle point are given by

$$g^{(2)}(x_0)=(M\tanh\theta_1+N\tanh\theta_2)x_0^{-2},$$

$$g^{(3)}(x_0)=x_0^{-3}[M\tanh^3\theta_2+N\tanh^3\theta_2-3(M\tanh\theta_1+N\tanh\theta_2)],$$

and

$$g^{(4)}(x_0)=3x_0^{-4}[M\tanh^5\theta_1+N\tanh^5\theta_2-3(M\tanh^3\theta_1+N\tanh^3\theta_2)+4(M\tanh\theta_1+N\tanh\theta_2)]. \quad \text{(C10)}$$

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[9] One of us (T.T.W.) wishes to thank Professor E. W. Montroll for a most helpful discussion on this point.

Expanding the integrand of (C3) about x_0, and denoting $\xi=x-x_0$, we get

$$F_{M,N}\sim(2\pi)^{-1}(M^2+\gamma_1^2x_0^2)^{-1/4}(N^2+\gamma_2^2x_0^2)^{1/4}\exp(-M\theta_1-N\theta_2)$$

$$\times\int_{-\infty}^{\infty}d\xi\,\exp[-\tfrac{1}{2}g^{(2)}(x_0)\xi^2]\{1-\tfrac{1}{6}g^{(3)}(x_0)\xi^3-(24)^{-1}g^{(4)}(x_0)\xi^4+(1/72)[g^{(3)}(x_0)]^2\xi^6\}$$

$$\times[1-\tfrac{1}{2}x_0^{-1}(2-\tanh^2\theta_1-\tanh^2\theta_2)\xi+\tfrac{1}{8}x_0^{-2}(8-10\tanh^2\theta_1-10\tanh^2\theta_2+2\tanh^2\theta_1\tanh^2\theta_2+5\tanh^4\theta_1+5\tanh^4\theta_2)\xi^2]$$

$$\times[1-(24M)^{-1}(1+\gamma_1^2x_0^2/M^2)^{-3/2}(2-3\gamma_1^2x_0^2/M^2)-(24N)^{-1}(1+\gamma_2^2x_0^2/N^2)^{-3/2}(2-3\gamma_2^2x_0^2/N^2)]. \quad \text{(C11)}$$

Now we have

$$\int_{-\infty}^{\infty}e^{-a\xi^2}(1,\xi^2,\xi^4,\xi^6)=(\pi/a)^{1/2}(1,\tfrac{1}{2}a^{-1},\tfrac{3}{4}a^{-2},\tfrac{15}{8}a^{-3}). \quad \text{(C12)}$$

From (C11) and (C12) we obtain

$$F_{M,N}\sim(2\pi\gamma_1\gamma_2)^{-1/2}(M\sinh\theta_1\cosh\theta_2+N\cosh\theta_1\sinh\theta_2)^{-1/2}\exp(-M\theta_1-N\theta_2)\{1-(24)^{-1}(M\tanh\theta_1+N\tanh\theta_2)^{-3}$$

$$\times[3M^2\tanh^2\theta_1(1+\tanh^2\theta_2)+3N^2\tanh^2\theta_2(1+\tanh^2\theta_1)-\tanh^3\theta_1\tanh\theta_2(3-5\tanh^2\theta_2)M^3/N$$

$$-\tanh^3\theta_2\tanh\theta_1(3-5\tanh^2\theta_1)N^3/M+2\tanh\theta_1\tanh\theta_2MN(3+3\tanh^2\theta_1+3\tanh^2\theta_2-5\tanh^2\theta_1\tanh^2\theta_2)]+\cdots\}. \quad \text{(C13)}$$

We may obtain (5.2) from (C13).

APPENDIX D

To determine the singularities of the function (5.9), let us carry out the integration over δ_1 and δ_2. We obtain

$$(\pi\gamma_1)^{-1}\int_0^{2\pi}d\delta_2\,[-a^2+2a\gamma_2\cos\tfrac{1}{2}\psi_2\cos\delta_2-\gamma_2^2\cos^2\tfrac{1}{2}\psi_2\cos^2\delta_2-a\gamma_2\sin\tfrac{1}{2}\psi_2\sin\delta_2$$

$$+\gamma_2^2\sin\tfrac{1}{2}\psi_2\cos\tfrac{1}{2}\psi_2\sin\delta_2\cos\delta_2+\gamma_1^2\cos^2\tfrac{1}{2}\psi_1](\sin\tfrac{1}{2}\psi_1\sin\delta_2)$$

$$\times[\gamma_2^2(\cos\psi_2-\cos\psi_1)\cos^2\delta_2-4a\gamma_2\cos\tfrac{1}{2}\psi_2\sin^2\tfrac{1}{2}\psi_1\cos\delta_2+2\sin^2\tfrac{1}{2}\psi_1a^2+2\cos^2\tfrac{1}{2}\psi_1(\gamma_2^2\sin^2\tfrac{1}{2}\psi_2-\gamma_1^2\sin^2\tfrac{1}{2}\psi_1)]^{-1}$$

$$\times\{[a-\gamma_2\cos(\tfrac{1}{2}\psi_2+\delta_2)]^2-\gamma_1^2\}^{-1/2}. \quad \text{(D1)}$$

The integrand in the integral above, considered as a function of $\exp(i\delta_2)$, has four simple poles. Let us denote

$$\cos\delta_{2\pm}=\{a\cos\tfrac{1}{2}\psi_2\sin^2\tfrac{1}{2}\psi_1\pm\cos\tfrac{1}{2}\psi_1[a^2\sin^2\tfrac{1}{2}\psi_1\sin^2\tfrac{1}{2}\psi_2+(\sin^2\tfrac{1}{2}\psi_2-\sin^2\tfrac{1}{2}\psi_1)(\gamma_2^2\sin^2\tfrac{1}{2}\psi_2-\gamma_1^2\sin^2\tfrac{1}{2}\psi_1)]^{1/2}\}$$

$$\times[\gamma_2(\cos^2\tfrac{1}{2}\psi_2-\cos^2\tfrac{1}{2}\psi_1)]^{-1}, \quad \text{(D2)}$$

then the four poles are located at

$$\exp(i\delta_{2+}),\qquad \exp(-i\delta_{2+}),\qquad \exp(i\delta_{2-}),\qquad \exp(-i\delta_{2-}). \quad \text{(D3)}$$

When two poles pinch the contour, the integral is singular. The pinching occurs only at

$$\exp(i\delta_{2+})=\exp(-i\delta_{2+}), \quad \text{(D4)}$$

or

$$\exp(i\delta_{2-})=\exp(-i\delta_{2-}). \quad \text{(D5)}$$

From these, we conclude that there are four singularity curves given by

$$a\pm\gamma_1\cos\tfrac{1}{2}\psi_1\pm\gamma_2\cos\tfrac{1}{2}\psi_2=0. \quad \text{(D6)}$$