Thermodynamic Fluctuations of the Order Parameter in Type-II Superconductors near the Upper Critical Field H_{c2}†

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In this paper we investigate the mean-square deviation of the order parameter from its equilibrium for type-II superconductors, assuming that the Gibbs free energy is given by a Ginzburg-Landau-type functional. It turns out that the equilibrium function, which is already varying with position because of the penetrating magnetic flux, resists distortion much less than the constant order parameter in type-I superconductors. Therefore, the mean-square deviation becomes equal in magnitude to the square of the equilibrium order parameter in the magnetic field range $(H_{c2}-B)/H_{c2}\lesssim 10^{-4}$ to 10^{-5}. This means that the range of magnetic fields in which Ginzburg-Landau-type descriptions should fail and a singular behavior might show up seems not entirely out of reach of present experimental techniques, in contrast to the situation in type-I superconductors. As a necessary byproduct of our investigation, we prove that in the limit as B goes to H_{c2}, the equilibrium function corresponding to a triangular fluxoid lattice gives a local minimum of the free energy in function space; and we obtain an orthogonal set of fluctuations of the equilibrium function which are normal modes in the sense that the second-order increment of the free energy does not mix their amplitudes. The special properties of the most important fluctuations suggest that the vortex crystal melts slightly below H_{c2}, but that the identity of the vortices is maintained.

I. BASIC FORMULATION

THE thermodynamic potential of a type-II superconductor in the external magnetic field H_e is given by

$$\phi=\int d^3r\{b_2\,|\,\mathbf{\Pi}\Delta(\mathbf{r})\,|^2-a_2\,|\,\Delta(\mathbf{r})\,|^2 +\tfrac{1}{2}a_4\,|\,\Delta(\mathbf{r})\,|^4+(8\pi)^{-1}[\mathbf{B}(\mathbf{r})-\mathbf{H}_e]^2\},\quad(1.1)$$

where the constants derived from the microscopic theory of superconductivity are[1]

$$a_2=N(0)\,\ln(T_c/T),\quad(1.2)$$

$$a_4=N(0)\tfrac{7}{8}\zeta(3)\,(\beta/\pi)^2,\quad(1.3)$$

$$b_2=N(0)\tfrac{7}{8}\zeta(3)\tfrac{1}{6}\chi(\beta/\pi\hbar v_F)^2,\quad(1.4)$$

$$\mathbf{\Pi}=(\nabla/i)-(2e/\hbar c)\,\mathbf{A}(\mathbf{r}).\quad(1.5)$$

Here, $N(0)=k_F^2/2\pi^2\hbar v_F$ and χ is the Gorkov function[1] depending on the electronic mean free path.

Besides being based on the concept of a mean field $\Delta(\mathbf{r})$, the expression (1.1) contains technical approximations. It is obtained by expanding the true functional $\phi(\{\Delta(\mathbf{r}),\,\mathbf{A}(\mathbf{r})\})$ simultaneously into powers of $\Delta(\mathbf{r})$ and $\mathbf{\Pi}$, neglecting all powers higher than fourth order. The neglect of higher powers of $|\,\Delta(\mathbf{r})\,|$ is perfectly sound, as we are only interested in the limit $H_e\to H_{c2}$ where $|\,\Delta(\mathbf{r})\,|$ goes to zero. The neglect of higher powers of $\mathbf{\Pi}$, however, is not justified at temperatures well below T_c, but it can be seen from the complete Δ^2 and Δ^4 contribution to ϕ that the features of (1.1) which are essential for our subsequent conclusions remain unchanged.[2] We shall, therefore, not burden our derivations with the unessential and complicated details of the true ϕ, but rather take (1.1) as a convenient model.

It is then practical to introduce new units, namely the coherence length $(\hbar c/2e)\,(a_4/8\pi b_2 a_2)^{1/2}$ and the *upper* critical field $\hbar c a^2/2eb_2=H_{c2}$ for lengths and fields, respectively, and also to use a normalized order parameter $\psi(\mathbf{r})=(a_4/a_2)^{1/2}\Delta(\mathbf{r})$. Thus (1.1) becomes

$$\phi=C\int d^3r\{|\,(\nabla/i-\kappa^2\mathbf{A}(\mathbf{r}))\psi(\mathbf{r})\,|^2 +\kappa^2(\tfrac{1}{2}\,|\,\psi(\mathbf{r})\,|^4-|\,\psi(\mathbf{r})\,|^2)+\kappa^4(\mathbf{B}(\mathbf{r})-\mathbf{H}_e)^2\},\quad(1.6)$$

where κ is the well-known Ginzburg-Landau parameter[3]

$$\kappa=(\hbar c/2e)\,(a_4/8\pi b_2^2)^{1/2},\quad(1.7)$$

and the other constant C is given by

$$\frac{C}{\kappa}=\frac{b_2(a_2b_2)^{1/2}}{a_4} =(\tfrac{7}{8}\zeta(3))^{1/2}\chi^{3/2}\left(\frac{\hbar k_F v_F}{k_BT}\right)^2\frac{k_BT(\ln T_c/T)^{1/2}}{12(6)^{1/2}\pi^3}.\quad(1.8)$$

The equilibrium function $\psi_0(\mathbf{r})$ is determined by minimizing $\phi(\{\psi(\mathbf{r})\})$ with respect to $\psi(\mathbf{r})$ while calculating $\mathbf{A}(\mathbf{r})$ from the supercurrents:

$$\nabla\times\mathbf{B}(\mathbf{r})=(1/2\kappa^2)[\psi^*(\mathbf{r})\,(\nabla/i-\kappa^2\mathbf{A}(\mathbf{r}))\psi(\mathbf{r})+\text{c.c.}].\quad(1.9)$$

According to our policy of keeping only terms up to order ψ^4 in ϕ, we can replace[4] $\mathbf{A}(\mathbf{r})$ in (1.9) by the vector potential $\mathbf{A}_i(\mathbf{r})=-\hat{x}\bar{B}y$ produced by the averaged magnetic flux density $\mathbf{B}_i=\bar{B}\cdot\hat{z}$ inside the super-

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[1] See, for instance, N. R. Werthamer, in Superconductivity, edited by R. D. Parks (to be published).
[2] Gert Eilenberger, Phys. Rev. **153**, 584 (1957).
[3] V. L. Ginzburg and L. D. Landau, Zh. Eksperim. i Teor. Fiz. **20**, 1064 (1950) [German transl.: Phys. Abhandlungen Sovjetunion **1**, 1 (1958)].
[4] Gert Eilenberger, Z. Physik **180**, 33 (1964).

conductor. The deviation $\mathbf{B}_0(\mathbf{r})=\mathbf{B}(\mathbf{r})-\mathbf{B}_i$ from the mean-flux density contributes a fourth-order term to $|(\nabla/i-\kappa^2\mathbf{A})\psi|^2$, which equals $-2\kappa^4\mathbf{B}_0^2(\mathbf{r})$, so that we have

$$\phi=C\int d^3r\{|(\nabla/i-\kappa^2\mathbf{A}_i(\mathbf{r}))\psi|^2 +\kappa^2(\tfrac{1}{2}|\psi|^4-|\psi|^2)-\kappa^4\mathbf{B}_0^2+\kappa^4(\bar{B}-H_e)^2\}. \quad (1.10)$$

To our knowledge, no recipe has yet been derived from first principles for calculating properly the contribution of fluctuations in a mean-field theory like the one we have to deal with. We shall, therefore, make two assumptions the validity of which remains to be proved from the microscopic theory.

The first assumption is that we shall maintain the relation (1.9) even if $\psi(\mathbf{r})$ is not the equilibrium function $\psi_0(\mathbf{r})$. The reason is that a fluctuation of $\psi(\mathbf{r})$ causes a fluctuation of the supercurrent and this in turn leads to a fluctuating $\mathbf{B}_0(\mathbf{r})$. We shall maintain a constant $\bar{B}$, however, assuming that the surface currents which produce the difference $\bar{B}-H_e$ react to fluctuations inside the specimen so as to maintain the total enclosed magnetic flux.

The second assumption is that the probability of finding $\psi(\mathbf{r})$ in any volume element $\mathfrak{D}\psi(\mathbf{r})$ of the space spanned by a complete orthonormal set of functions $\psi(\mathbf{r})$ is given by $\exp[-\beta\mathfrak{F}(\{\psi(\mathbf{r})\})]\mathfrak{D}\psi(\mathbf{r})$ with $\mathfrak{F}=\phi-V\kappa^4(\bar{B}-H_e)^2$. This assumption has previously been employed by Rice[5] for type-I superconductors, neglecting the $\mathbf{B}_0^2$ term.

A peculiar difficulty is connected with our first assumption, which cannot be resolved within our present phenomenological approach. It will turn out in the following that for nonequilibrium functions $\psi(\mathbf{r})$ the divergence of the current in (1.9) will not be zero in general, and therefore (1.9) cannot be solved rigorously. This fact is *not* to be interpreted as indicating temporary accumulation of charge by the fluctuations of the supercurrent; instead, as shown below, it is inherent in the particle-nonconserving mean field formulation of the effective Hamiltonian which leads to the Gorkov equations.[1]

These equations follow as exact consequences, if one uses the Hamiltonian (including summation over repeated spin variables α):

$$\mathfrak{IC}=\int d^3r\{[\psi_\alpha^+(\nabla/i-(e/c)\mathbf{A})^2\psi_\alpha-\mu\psi_\alpha^+\psi_\alpha] -[\Delta(r)\psi_\uparrow^+\psi_\downarrow^+ + \text{c.c.}]\}. \quad (1.11)$$

From this we get

$$\begin{aligned}0&=Z^{-1}\,\mathrm{Tr}(\exp(-\beta\mathfrak{IC})[\mathfrak{IC},\psi_\alpha^+(\mathbf{r})\psi_\alpha(\mathbf{r})])\\&=(\partial/i\partial t)\langle\psi_\alpha^+(\mathbf{r})\psi_\alpha(\mathbf{r})\rangle\\&=-\langle\psi_\alpha^+(r)(\nabla/i-(e/c)\mathbf{A}(\mathbf{r}))^2\psi_\alpha(\mathbf{r})-\text{c.c.}\rangle\\&\quad-2\langle\Delta(\mathbf{r})\psi_\uparrow^+(\mathbf{r})\psi_\downarrow^+(\mathbf{r})-\text{c.c.}\rangle\end{aligned}$$

or

$$\begin{aligned}&(4\pi c/e)\,\mathrm{div}\mathbf{j}\\&=\nabla\langle\psi_\alpha^+(r)(\nabla/i-(e/c)\mathbf{A}(\mathbf{r}))\psi_\alpha(\mathbf{r})+\text{c.c.}\rangle\\&=4\,\mathrm{Im}(\Delta(\mathbf{r})\langle\psi_\uparrow^+(\mathbf{r})\psi_\downarrow^+(\mathbf{r})\rangle). \quad (1.12)\end{aligned}$$

The Hamiltonian (1.11) has to be supplemented by the prescription

$$\Delta^*(\mathbf{r})=\lambda\langle\psi_\uparrow^+(\mathbf{r})\psi_\downarrow^+(\mathbf{r})\rangle \quad (1.13)$$

(λ being the interaction parameter), which condition is exactly equivalent to requiring that $\Delta(\mathbf{r})$ minimizes the (true) functional $\phi(\{\Delta(\mathbf{r})\})$, and which through (1.12) automatically ensures $\mathrm{div}\mathbf{j}(\mathbf{r})\equiv 0$ for the equilibrium function $\Delta_0(\mathbf{r})$. In general, however, (1.12) leads to $\mathrm{div}\mathbf{j}(\mathbf{r})\neq 0$ for functions $\Delta(\mathbf{r})$ deviating from $\Delta_0(\mathbf{r})$. In our case we shall see that $\mathrm{div}\mathbf{j}$ is of an order of magnitude which is systematically neglected anyway and thus we can rid ourselves of the difficulty by ignoring it.

II. "NORMAL" FLUCTUATIONS OF THE SYSTEM

To construct an orthonormal set of functions $\psi(\mathbf{r})$ suitable for treating the fluctuations of ψ_0, we consider the eigenfunctions of the operator

$$(\nabla/i-\kappa^2\mathbf{A}_i(\mathbf{r}))^2-\kappa^2=(\partial/i\partial z)^2+\kappa^2(\bar{B}-1)+F_+F_-, \quad (2.1)$$

with

$$F_\pm=((\partial/i\partial x)+\kappa^2\bar{B}y\mp(\partial/\partial y)). \quad (2.2)$$

A complete orthogonal set of eigenfunctions of (2.1) is given by[6]

$$\psi_{n,k,r_0}(\mathbf{r})=\exp(ikz)[(2\kappa^2\bar{B})^n n!]^{-1/2}F^n{}_+\varphi(r\mid r_0) \quad (2.3)$$

with corresponding eigenvalues

$$E_{n,k}=2n\kappa^2\bar{B}+k^2+\kappa^2(\bar{B}-1), \quad (2.4)$$

where the $\varphi(r\mid r_0)$ are any orthogonal set of functions (labeled by r_0) which spans the complete function subspace S_0 that is annihilated by F_-. We shall normalize the functions φ so that

$$V^{-1}\int|\varphi(r\mid r_0)|^2d^3r=1. \quad (2.5)$$

It is well known[7] that for magnetic fields close to H_{c2}, i.e., for $1-\bar{B}\ll 1$, a very good approximation for the function $\psi_0(\mathbf{r})$ that minimizes $\mathfrak{F}$ is obtained if one restricts $\psi_0(\mathbf{r})$ to belong S_0. The reason is that the above inequality is identical with $E_{0,k=0}\ll E_{1,k=0}$, so that any contribution from the $\psi_{n\neq 0}$ greatly increases

[5] T. M. Rice, Phys. Rev. **140**, A1889 (1965).

[6] We shall denote three-dimensional vectors (x, y, z) by $\mathbf{r}$ and two-dimensional vectors (x, y) by r or by $z=x+iy$ depending upon convenience. We write $r\doteq z$ to denote equivalence of r and z.

[7] A. A. Abrikosov, Zh. Eksperim. i Teor. Fiz. **32**, 1442 (1957) [English transl.: Soviet Phys.—JETP **5**, 1174 (1957)].

the ψ^2 contribution to $\mathfrak{F}$. For the same reason, fluctuations with $n\neq 0$ are not important and we shall only consider those with $n=0$. We now denote by $\varphi(r \mid 0)$ the function which describes the triangular fluxoid lattice and which is supposed[4,8] to minimize (with a proper amplitude α) the functional $\mathfrak{F}(\{\psi(\mathbf{r})\})$ within S_0. $\mid \varphi(r \mid 0) \mid$ is then periodic with periods

$$r_{\mathrm{I}}=(x_{\mathrm{I}}, y_{\mathrm{I}})=(l, 0), \qquad r_{\mathrm{II}}=(x_{\mathrm{II}}, y_{\mathrm{II}})=(\tfrac{1}{2}l, \tfrac{1}{2}l\sqrt{3}),$$

with an l such that the area Q of the fundamental cell is given by the flux quantization condition

$$Q=\tfrac{1}{2}l^2\sqrt{3}=2\pi/\kappa^2\bar{B}. \tag{2.6}$$

If we shift this function by $r_0=(x_0, y_0)$ in the x-y plane and multiply it by $\exp(i\kappa^2\bar{B}y_0x)$ to take care of the vector potential, we get another function of S_0 which we call $\varphi(r \mid r_0)$; it has essentially the same properties as $\varphi(r \mid 0)$. Permitting r_0 to take on any value within one lattice cell, we get a set of functions $\varphi(r \mid r_0)$ which are mutually orthogonal and span S_0 completely. We introduced this set recently in Ref. 2, and we describe it more extensively in Appendix A.

To count modes, we consider from now on a finite volume $V=LA$ with an extension L in the z direction and a cross section A containing A/Q lattice cells. By means of the periodicity properties (A7) and (A8) it can be shown that r_0 then can take on A/Q different values and

$$dN=(V/Q)\,(dk/2\pi)\,(d^2r_0/Q) \tag{2.7}$$

is therefore the number of functions, whose indices k, r_0 lie within any given volume dkd^2r_0.

To calculate the coefficients which determine the increment of $\mathfrak{F}$ to second order in the amplitudes of the fluctuations we shall now evaluate $\mathfrak{F}$ for

$$\psi(\mathbf{r})=\alpha\varphi(r \mid 0)+V^{-1/2}(a_1\varphi(r \mid r_1)+a_2\varphi(r \mid r_2)), \qquad 0\neq r_1\neq r_2\neq 0. \tag{2.8}$$

Using $F_{-}\psi(\mathbf{r})=0$, (1.9) is solved by[9]

$$\mathbf{B}_0(\mathbf{r})=(-1/2\kappa^2)\,(\mid\psi(\mathbf{r})\mid^2-\langle\mid\psi(\mathbf{r})\mid^2\rangle_s) \tag{2.9}$$

and we get from (1.10) and (2.4) to second order in the α_ν:

$$\mathfrak{F}(\{\psi(r)\})=C\left\{\kappa^2(\bar{B}-1)\,(V\alpha^2+\mid a_1\mid^2+\mid a_2\mid^2)+\tfrac{1}{4}\alpha^2(V\alpha^2+2\mid a_1\mid^2+2\mid a_2\mid^2)+\tfrac{1}{4}(2\kappa^2-1)\int\mid\psi(\mathbf{r})\mid^4 d^3r\right\}. \tag{2.10}$$

The second term on the right-hand side of (2.10) stems from the term $\langle\mid\psi(\mathbf{r})\mid^2\rangle_s=\mid\alpha\mid^2+V^{-1}(\mid a_1\mid^2+\mid a_2\mid^2)$ in (2.9).

Evaluating the integral over $\mid\psi(\mathbf{r})\mid^4$ we shall drop the terms linear in a_ν immediately; by the same method of integration we use in the following they can be shown to vanish identically, as they must.

We are then left with

$$\begin{aligned}\int\mid\psi(\mathbf{r})\mid^4 d^3r=V\alpha^4\int\varphi^2(r \mid 0)\varphi^{*2}(r \mid 0)\,d^2r/A\\ +\alpha^2\left(\sum_{\nu=1,2;\mu=1,2}a_\nu a_\mu\int\varphi(r \mid r_\nu)\varphi(r \mid r_\mu)\varphi^{*2}(r \mid 0)\,d^2r/A+\text{c.c.}\right)\\ +4\alpha^2\sum_{\nu=1,2;\mu=1,2}a_\nu a_\mu^*\int\varphi(r \mid r_\nu)\varphi(r \mid 0)\varphi^*(r \mid r_\mu)\varphi^*(r \mid 0)\,d^2r/A.\end{aligned} \tag{2.11}$$

The integrations are easily done if we employ the addition theorem (A19):

$$\begin{aligned}&\varphi(r \mid r_1)\varphi(r \mid r_2)\\ &\quad=\sqrt{2}^{-1}\{\tilde{\varphi}(r \mid \tfrac{1}{2}(r_1+r_2))\tilde{\varphi}(\tfrac{1}{2}(r_1-r_2) \mid 0)\\ &\quad+\tilde{\varphi}(r \mid \tfrac{1}{2}(r_1+r_2)+\tfrac{1}{2}r_1)\tilde{\varphi}(\tfrac{1}{2}(r_1-r_2) \mid \tfrac{1}{2}r_1)\},\end{aligned} \tag{2.12}$$

together with the orthogonality of the functions $\tilde{\varphi}(r \mid r_0)$, which are the same type of functions as $\varphi(r \mid r_0)$ with the difference of having a fundamental cell with $\tilde{r}_{\mathrm{II}}=\frac{1}{2}r_{\mathrm{II}}$.

One then gets from (2.11)

$$\begin{aligned}\int\mid\psi(\mathbf{r})\mid^4 d^3r=V\alpha^4\xi_0+2\alpha^2(a_1a_2\xi_n(r_1)\delta(r_1,-r_2)+\text{c.c.})\\ +4\alpha^2(\xi_d(r_1)\mid a_1\mid^2+\xi_d(r_2)\mid a_2\mid^2),\end{aligned} \tag{2.13}$$

where $\delta(r_1, -r_2)$ is Kronecker's delta and the other new quantities are[10]

$$\begin{aligned}&\xi_0=\tfrac{1}{2}(\mid\tilde{\varphi}(0 \mid 0)\mid^2+\mid\tilde{\varphi}(0 \mid \tfrac{1}{2}r_{\mathrm{I}})\mid^2)=1.1596,\\ &\xi_n(r_0)=\tfrac{1}{2}(\tilde{\varphi}(r_0 \mid 0)\tilde{\varphi}^*(0 \mid 0)+\tilde{\varphi}(r_0 \mid \tfrac{1}{2}r_{\mathrm{I}})\tilde{\varphi}^*(0 \mid \tfrac{1}{2}r_{\mathrm{I}})),\\ &\xi_d(r_0)=\tfrac{1}{2}(\mid\tilde{\varphi}(\tfrac{1}{2}r_0 \mid 0)\mid^2+\mid\tilde{\varphi}(\tfrac{1}{2}r_0 \mid \tfrac{1}{2}r_{\mathrm{I}})\mid^2).\end{aligned} \tag{2.14}$$

[8] W. H. Kleiner, L. M. Roth, and S. H. Autler, Phys. Rev. 133, A1226 (1964).

[9] We use $\langle\ \rangle_s$ for the spatial and $\langle\ \rangle_{th}$ for the thermodynamic average.

[10] ξ_0 is usually called β in the literature (Refs. 7 and 8); we reserve this letter for $(k_BT)^{-1}$.

We see that the only mixtures occur between the amplitudes of $\varphi(r \mid r_0)$ and $\varphi(r \mid -r_0)$, and the normal modes of fluctuations can be constructed by proper combinations of both, namely

$$u_\pm(r \mid r_0) = (\xi_n(r_0)/2 \mid \xi_n(r_0) \mid)^{1/2}(\varphi(r \mid r_0) \pm \varphi(r \mid -r_0)),$$
$$v_\pm(r \mid r_0) = (\xi_n(r_0)/2 \mid \xi_n(r_0) \mid)^{1/2} \times (i\varphi(r \mid r_0) \mp i\varphi(r \mid -r_0)). \quad (2.15)$$

These are now modes of fluctuations, whose amplitudes (which have to be *real*) do not mix in the second order increment of $\mathfrak{F}$; the set (2.15) is complete in S_0 if r_0 takes on all permitted values in *half* of the original fundamental cell [since we have combined $\varphi(r \mid r_0)$ with $\varphi(r \mid -r_0)$].

So far we have not permitted variation in the z direction. To include this we use the fact that u_+ and u_- are energetically degenerate with v_+ and v_-, respectively, and the result of Appendix B that (2.9) is still the proper solution for B_0 even if we include modes with $k \neq 0$ in ψ. Then, with

$$u_\pm(r \mid r_0, k) = (\xi_n(r_0)/2 \mid \xi_n(r_0) \mid)^{1/2}(\exp(ikz)\varphi(r \mid r_0) \pm \exp(-ikz)\varphi(r \mid -r_0)),$$
$$v_\pm(r \mid r_0, k) = (\xi_n(r_0)/2 \mid \xi_n(r_0) \mid)^{1/2}(i\exp(ikz)\varphi(r \mid r_0) \mp i\exp(-ikz)\varphi(r \mid -r_0)), \quad (2.16)$$

the general function ψ we consider is

$$\psi(\mathbf{r}) = (2\kappa^2(1-\bar{B})/(1+[2\kappa^2-1]\xi_0))^{1/2}\varphi(r \mid r_0) + V^{-1/2}\sum_{r_0,k;\nu=\pm}(a_\nu(r_0, k)u_\nu(r \mid r_0, k) + b_\nu(r_0, k)v_\nu(r \mid r_0, k)), \quad (2.17)$$

where we inserted the equilibrium amplitude $\alpha^2 = 2\kappa^2(1-\bar{B})/(1+(2\kappa^2-1)\xi_0)$. The calculation of the corresponding $\mathfrak{F}$ is straightforward using (2.10), (2.13), and (2.14) and yields

$$\mathfrak{F}(\{\psi(\mathbf{r})\}) = \kappa^2 CV\left\{-\frac{\kappa^2(1-\bar{B})^2}{1+(2\kappa^2-1)\xi_0} + V^{-1}\sum_{k,r_0;\nu=\pm}\left(\frac{k^2}{\kappa^2} + \frac{(2\kappa^2-1)}{1+(2\kappa^2-1)\xi_0}(1-\bar{B})\xi_\nu(r_0)\right)(a_\nu^2(r_0, k) + b_\nu^2(r_0, k))\right\}, \quad (2.18)$$

with

$$\xi_\pm(r_0) = 2\xi_n(r_0) - \xi_0 \pm \mid \xi_n(r_0) \mid$$
$$= \mid \tilde{\varphi}(\tfrac{1}{2}r_0 \mid 0) \mid^2 + \mid \tilde{\varphi}(\tfrac{1}{2}r_0 \mid \tfrac{1}{2}r_1) \mid^2 - \xi_0 \pm \tfrac{1}{2} \mid \tilde{\varphi}(r_0 \mid 0)\tilde{\varphi}^*(0 \mid 0) + \tilde{\varphi}(r_0 \mid \tfrac{1}{2}r_1)\tilde{\varphi}^*(0 \mid \tfrac{1}{2}r_1) \mid. \quad (2.19)$$

Two interesting conclusions can be drawn from these results. First of all, numerical computation shows that the coefficients $\xi_\pm(r_0)$ are positive for all $r_0 \neq 0$, thus proving that for $1-\bar{B} \ll 1$ the triangular fluxoid lattice is indeed stable against infinitesimal distortions. More precisely, for all $r_0 \neq 0$ the inequalities

$$0 < \xi_-(r_0) \leq 0.6811 \leq \xi_+(r_0) < 2.3192 = 2\xi_0 \quad (2.20)$$

hold with $\xi_-(r_0) \to 0$, $\xi_+(r_0) \to 2\xi_0$ as $r_0 \to 0$ and $\xi_-(r_M) = \xi_+(r_M) = 0.6811$ at $r_M = (\frac{1}{2}l, (2\sqrt{3})^{-1}l)$. Tables of the functions $\xi_+(r_0)$ and $\xi_-(r_0)$ are given in Appendix C.

The second conclusion is that the increase in the number of modes of fluctuations with small energy as $\bar{B} \to 1$ is much faster here then it is in a type-I superconductor when $T/T_c \to 1$. This is due to the fact that *all* fluctuations in the x-y directions contribute only a free-energy proportional to $1-\bar{B}$. We expect, therefore, that the effect of fluctuations is much closer to being observable in type-II than in type-I superconductors, and shall demonstrate this in the following section.

III. MAGNITUDE OF FLUCTUATIONS IN TYPE-I AND TYPE-II SUPERCONDUCTORS

To estimate the range of magnetic fields $\bar{B}$ where the fluctuations could be of influence, we calculate $\langle\langle \mid \psi(\mathbf{r}) \mid^2\rangle_s\rangle_{\text{th}}$ and compare it with $\langle \mid \psi_0(r) \mid^2\rangle_s$. Using the relative probability $\exp[-\beta\mathfrak{F}(\{\psi(\mathbf{r})\})]$ for any $\psi(\mathbf{r})$ we get immediately

$$\langle\langle \mid \psi(\mathbf{r}) \mid^2\rangle_s\rangle_{\text{th}} - \langle \mid \psi_0(\mathbf{r}) \mid^2\rangle_s$$
$$= V^{-1}\sum_{k,r_0,\nu}\left\{2\beta C\left(k^2 + \frac{(2\kappa^2-1)}{1+(2\kappa^2-1)\xi_0}\kappa^2(1-\bar{B})\xi_\nu(r_0)\right)\right\}^{-1}$$
$$= \frac{\kappa\bar{B}}{8\pi\beta C}\left(\frac{(2\kappa^2-1)(1-\bar{B})}{1+(2\kappa^2-1)\xi_0}\right)^{-1/2}(Z_+ + Z_-), \quad (3.1)$$

where we used (2.6) and (2.7) for the last step and introduced

$$Z_+ = \int_{\text{cell}}(\xi_+(r_0))^{-1/2}d^2r_0/Q \approx 1,$$
$$Z_- = \int_{\text{cell}}(\xi_-(r_0))^{-1/2}d^2r_0/Q \approx 10. \quad (3.2)$$

The simple Ginzburg-Landau approach is supposed to break down when (3.1) becomes equal in magnitude to

$$\langle \mid \psi_0(\mathbf{r}) \mid^2\rangle_s = \alpha^2 = [2\kappa^2(1-\bar{B})/(1+(2\kappa^2-1)\xi_0)]. \quad (3.3)$$

The interesting range of magnetic fields, where this

happens, is given by

$$(1-\bar{B})^{3/2} \lesssim (10/8\pi)(\kappa/\beta C) \approx \frac{30\pi^2}{\chi^{3/2}(\ln(T_c/T))^{1/2}}\left(\frac{k_B T}{\hbar k_F v_F}\right)^2,$$

i.e., for temperatures T well between 0 and T_c when

$$(1-\bar{B}) \lesssim 10^{-4}\text{–}10^{-5}$$

or

$$(H_{c2}-H_e)/H_{c2} \lesssim 10^{-4}\text{–}10^{-5}, \tag{3.4}$$

which is at least close to being observable with present experimental techniques.

To compare this with the situation in type-I superconductors one gets a rough estimate by replacing the formulas following (2.16) by

$$u_\pm(\mathbf{r} \mid \mathbf{k}) = \sqrt{2}^{-1}[\exp(i\mathbf{k}\cdot\mathbf{r}) \pm \exp(-i\mathbf{k}\cdot\mathbf{r})],$$
$$v_\pm(\mathbf{r} \mid \mathbf{k}) = \sqrt{2}^{-1}[i\exp(i\mathbf{k}\cdot\mathbf{r}) \mp i\exp(-i\mathbf{k}\cdot\mathbf{r})], \tag{3.5}$$
$$\psi(\mathbf{r}) = 1 + V^{-1/2}\sum_{\mathbf{k},\nu=\pm}[a_\nu(\mathbf{k})u_\nu(\mathbf{r} \mid \mathbf{k}) + b_\nu(\mathbf{k})v_\nu(\mathbf{r} \mid \mathbf{k})], \tag{3.6}$$

and consequently

$$\mathfrak{F}(\{\psi(\mathbf{r})\}) = \kappa^2 CV\{-\tfrac{1}{2} + V^{-1}\sum_{\mathbf{k},\nu=\pm}(k^2/(\kappa^2+1+\nu)) \times (a_\nu^2(\mathbf{k}) + b_\nu^2(\mathbf{k}))\}, \tag{3.7}$$

where we have now neglected the B_0^2 term. Instead of (3.1) we get

$$\langle\langle|\psi(\mathbf{r})|^2\rangle_s\rangle_{\text{th}} - 1 = (2\beta C)^{-1}\int \frac{d^3k}{(2\pi)^3}[(k^2+2\kappa^2)^{-1} + (1/k^2)] = \frac{\kappa}{2\pi^2\beta C}\int_0^{k\max/\kappa}[1-(q^2+2)^{-1}]dq. \tag{3.8}$$

The results of this integration clearly depends on the cutoff we choose. Following Ginzburg,[11] we take the second, convergent term in the integrand (3.8) as an estimate of the relevant contribution, which amounts to an effective cutoff $k_{\max}$ of order κ. This finally yields for the interesting temperature range in type-I superconductors the well-known result[11]

$$1 \lesssim \kappa/8\pi\beta C \approx \frac{3\pi^2}{\chi^{3/2}(\ln(T_c/T))^{1/2}}\left(\frac{k_B T}{\hbar k_F v_F}\right)^2$$

or

$$(T_c-T)/T_c \lesssim 9\pi^4/\chi^3(k_B T_c/\hbar k_B v_F)^4 \approx 10^{-15}\text{–}10^{-17}. \tag{3.9}$$

The confrontation of the two situations clearly exhibits as the source of their profound difference the high degeneracy of the kinetic energy operator in magnetic fields.

One would like to ask, what happens in the critical region (3.4) of field magnitudes. Because the fluctuations which contribute are of the very special type such that the fluctuating function remains a ground-state function of the linearized Ginzburg-Landau equation, we suspect that the latter remains meaningful in this range, but that the vortex lattice melts at a certain external field strength below H_{c2}. This would be rather interesting if it were observable experimentally.

APPENDIX A: THE FUNCTIONS $\varphi(r \mid r_0)$

We shall describe here the most important and useful properties of the quasiperiodic[12] eigenfunctions of the operator

$$[\partial/i\partial x + 2\pi(y/\eta)]^2 + (\partial/i\partial y)^2 = 2\pi/\eta + [\partial/i\partial x + 2\pi(y/\eta) - \partial/\partial y] \times [\partial/i\partial x + 2\pi(y/\eta) + \partial/\partial y] = 2\pi/\eta + F_+F_-. \tag{A1}$$

It is easily shown[4] that any such eigenfunction, if it exists at all, must obey the "flux quantization condition"

$$(2\pi/\eta)Q = 2\pi n, \tag{A2}$$

where Q is the area of the fundamental cell and n is an integer. Given such a cell for $n=1$ it is convenient to scale its base to unit length, so that the cell is spanned by the periodicity vectors[6]:

$$2s_{\text{I}} = r_{\text{I}} = (1, 0) \doteq 1, \qquad 2s_{\text{II}} = r_{\text{II}} = (\zeta, \eta) \doteq \tau.$$

We then define the function

$$\varphi(r \mid 0 \mid t) = (2\eta)^{1/4}\sum_{p=-\infty}^{+\infty}\exp 2\pi/\eta \times \{-\tfrac{1}{2}(y-2t+p\eta)^2 + t^2 + ip\eta(x+\tfrac{1}{2}p\zeta)\}, \tag{A3}$$

where t is a generating parameter, the use of which will be convenient later. For $t=0$ this equals

$$\varphi(r \mid 0) = (2\eta)^{1/4}\exp(-\pi y^2/\eta)\vartheta_3(\pi z \mid \tau), \tag{A4}$$

where ϑ_3 is Jacobi's ϑ function.[13] Provided its magnitude is bounded, this function is a ground-state eigenfunction of (A1) since any $\varphi(r) = f(z)\exp\{-\pi y^2/\eta\}$ with an analytic $f(z)$ [and *only* such a $\varphi(r)$] satisfies

$$F_-\varphi = [\partial/i\partial x + 2\pi(y/\eta) + \partial/\partial y]\varphi(r) = 0. \tag{A5}$$

If we shift $\varphi(r \mid 0)$ by $r_0 = (x_0, y_0)$ we get a very similar function which can also be made a ground-state function of (A1) if we multiply it by $\exp\{2\pi i(y_0/\eta)x\}$ to provide a shift in the vector potential when F_- acts on

[11] V. L. Ginzburg, Fiz. Tverd. Tela **2**, 2031 (1960) [English transl.: Soviet Phys.—Solid State **2**, 1824 (1961)].

[12] We use the term quasiperiodic for the behavior exhibited by the formulas (A7).

[13] E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis* (Cambridge University Press, New York, 1952).

the shifted function. We denote these functions

$$\varphi(r \mid r_0) = \exp[2\pi i(y_0/\eta)x]\varphi(r+r_0 \mid 0). \quad \text{(A6)}$$

It is now easy to check the following properties of $\varphi(r \mid r_0)$:

(a) Periodicity:

$$\varphi(r+r_{\mathrm{I}} \mid r_0) = \exp[2\pi i(y_0/\eta)]\varphi(r \mid r_0),$$

$$\varphi(r+r_{\mathrm{II}} \mid r_0) = \exp(2\pi i)[(y_0/\eta)\zeta - x_0] \times \exp[-2\pi i(x+\tfrac{1}{2}\zeta)]\varphi(r \mid r_0), \quad \text{(A7)}$$

$$\varphi(r \mid r_0+r_{\mathrm{I}}) = \varphi(r \mid r_0),$$

$$\varphi(r \mid r_0+r_{\mathrm{II}}) = \exp[-2\pi i(x_0+\tfrac{1}{2}\zeta)]\varphi(r \mid r_0). \quad \text{(A8)}$$

(b) Normalization:

$$\int_{\text{cell}} |\varphi(r \mid r_0)|^2 (d^2r/\eta) = 1. \quad \text{(A9)}$$

(c) Symmetry and zeros:

$$\varphi(r+ns_{\mathrm{I}}+ms_{\mathrm{II}} \mid 0) = (-1)^{nm}\varphi(-r+ns_{\mathrm{I}}+ms_{\mathrm{II}} \mid 0) \quad n, m \text{ integers.} \quad \text{(A10)}$$

These symmetry relations follow for $n=m=0$ directly from the definition (A3), the other ones can then easily be derived by means of (A7). From (A10) also follows

$$\varphi(s_{\mathrm{I}}+s_{\mathrm{II}} \mid 0) = 0, \quad \text{(A11)}$$

and this is the only zero φ has in the fundamental cell, since ϑ_3 has no other zeros.[13]

(d) Except for a constant phase factor, $\varphi(r \mid 0)$ is uniquely determined by the conditions (A5), (A7) with $r_0=0$, and (A9). If there is another function $X(r)$ satisfying (A5) and (A7) with $r_0=0$, the ratio

$$X(r)/\varphi(r \mid 0)$$

is a doubly periodic analytic function which has at most one single pole in the fundamental cell. Hence,[13] this ratio is constant and of magnitude 1 because of (A9). We shall now proceed to say something about the completeness of the functions we have acquired.

For this purpose, we consider an operator $\mathfrak{K}(r_0)$ that operates in the Hilbert space of functions which are defined and square-integrable over one fundamental cell. For functions that are at least two times differentiable and obey the boundary conditions (A7) (with an r_0 kept fixed), let $\mathfrak{K}(r_0)$ be represented by (A1). It is obvious, that $F_-\varphi$ as well as $F_+\varphi$ and, consequently, $F_+F_-\varphi$ obey the relation (A7) if φ itself does so. Since $\varphi(r \mid r_0)$ is the only function which satisfies (A7) and $F_+\varphi(r)=0$, it seems reasonable to assume,[14] that the set of eigenfunctions of $\mathfrak{K}(r_0)$ for the given r_0,

$$\varphi_n(r \mid r_0) = \{(4\pi/\eta)^n n!\}^{-1/2} F_+^n \varphi(r \mid r_0) = \{(4\pi/\eta)^n n!\}^{-1/2} (\partial^n/\partial t^n)\varphi(r \mid r_0 \mid t=0) \quad \text{(A12)}$$

is a complete set of functions over one fundamental cell and determines $\mathfrak{K}(r_0)$ uniquely.

We observe now from (A7) that functions $\varphi_n(r \mid r_0)$ with different r_0 acquire different phase factors if we go from one fundamental cell in r space to another. In this respect, they resemble closely the well-known Bloch functions for electrons in periodic potentials; in fact, we can convert the index r_0 into a wave number k by setting

$$k_x = 2\pi(y_0/\eta), \qquad k_y = -2\pi(x_0/\eta). \quad \text{(A13)}$$

If r_0 runs through all positions within one fundamental cell, k runs through all positions in the Brillouin zone of the two-dimensional lattice with lattice vectors r_{I} and r_{II}. Invoking the well-known orthogonality relations of the Bloch functions, it follows from the completeness of the set (A12) for *each* r_0 in *one* fundamental cell the completeness in the entire x-y plane of the set

$$\varphi_n(r \mid r_0), \qquad n=0, 1, 2, \cdots,$$

r_0 in the fundamental cell. There follows also the orthogonality relation

$$\int d^2r \varphi_n^*(r \mid r_1)\varphi_m(r \mid r_2) = \eta^2\delta^2(r_1-r_2)\delta_{n,m}. \quad \text{(A14)}$$

We shall now consider the simplest of the addition theorems, which relates products of φ's to functions $\tilde{\varphi}$, which are defined like the φ but with fundamental cells of smaller size. These theorems are essentially the addition theorems of the ϑ functions, but they arise in a quite natural way viewed from the results gained above. To see this, we consider the product

$$P(r, r') = P(r', r) = \varphi(r+r' \mid 0)\varphi(r-r' \mid 0). \quad \text{(A15)}$$

From (A5) we get immediately

$$\tilde{F}_-P(r, r') = (\partial/i\partial x + 4\pi/\eta + \partial/\partial y)P(r, r') = 0. \quad \text{(A16)}$$

The property (A16), as well as the periodicity of P as function of r, which follows from (A7), is shared only by the functions

$$\tilde{\varphi}(r \mid 0), \qquad \tilde{\varphi}(r \mid s_{\mathrm{I}}), \quad \text{(A17)}$$

which are defined with $\tilde{\tau}=\tau/2$. Using the completeness theorem we see, then, that $P(r, r')$ must be a linear combination of the functions (A17) as far as its r dependence is concerned. Moreover, because $P(r, r') = P(r', r)$ an equality

$$P(r, r') = \sum_{\mu,\nu=0,1} A_{\nu\mu}\tilde{\varphi}(r \mid \nu s_{\mathrm{I}})\tilde{\varphi}(r' \mid \mu s_{\mathrm{I}})$$

must hold with constants $A_{\nu\mu}$. Directly inserting the series (A3) one finds then

$$\varphi(r+r' \mid 0)\varphi(r-r' \mid 0) = (\sqrt{2})^{-1}[\tilde{\varphi}(r \mid 0)\tilde{\varphi}(r' \mid 0) + \tilde{\varphi}(r \mid s_{\mathrm{I}})\tilde{\varphi}(r' \mid s_{\mathrm{I}})]. \quad \text{(A18)}$$

[14] This assumption is usually made tacitly, whenever one deals with unbounded operators in quantum mechanics.

This can be cast into the form

$$\varphi(r \mid r_1 \mid t_1)\varphi(r \mid r_2 \mid t_2) = \sqrt{2}^{-1}\{\tilde{\varphi}(r \mid \tfrac{1}{2}(r_1+r_2) \mid \tfrac{1}{2}(t_1+t_2))\tilde{\varphi}(\tfrac{1}{2}(r_1-r_2) \mid 0 \mid \tfrac{1}{2}(t_1-t_2)) + \tilde{\varphi}(r \mid \tfrac{1}{2}(r_1+r_2)+s_I \mid \tfrac{1}{2}(t_1+t_2))\tilde{\varphi}(\tfrac{1}{2}(r_1-r_2) \mid s_I \mid \tfrac{1}{2}(t_1-t_2))\} \quad \text{(A19)}$$

by observing the identity

$$\varphi(r \mid r_0 \mid t) = \exp[(2\pi/\eta)t^2] \times \exp[2\pi i(y_0/\eta)x]\varphi(x+x_0, y+y_0-2t \mid 0). \quad \text{(A20)}$$

The form (A19) is particularly useful in connection with the right-hand side of (A12); we used it in the main part of the paper only with $t_1=t_2=0$.

Finally, we shall comment on the numerical computation of the φ. The series (A3) generally converges quite fast, but it is obvious, that it converges the faster, the bigger η is. Therefore, if one has a long flat fundamental cell, as we had for the $\tilde{\varphi}$ in the main part of the paper, one is led to the idea of turning the cell around by turning the coordinate system and regauging to get the vector potential back into the x direction.

The goal of this procedure is most easily achieved by employing Jacobi's transformation[13]

$$\vartheta_3(\pi z \mid \tau) = (i/\tau)^{1/2}\exp[\pi i(z^2/-\tau)]\vartheta_3(-\pi z/\tau \mid -\tau^{-1}); \quad \text{(A21)}$$

together with (A4) it leads to

$$\varphi(r \mid r_0) = \varphi_T(r_T \mid 0) \times \exp\pi i[\tfrac{1}{2}\arg\tau_T - \tfrac{1}{4} + \mathrm{Re}(z_T^2/\tau_T)], \quad \text{(A22)}$$

where $r_T \doteqdot z_T = z\tau$ and $-\tau^{-1} = \tau_T \doteqdot r_{II,T}$ describes the transformed fundamental cell, i.e., the fundamental cell of the transformed function φ_T.

APPENDIX B: CALCULATION OF B

We consider here a function

$$\psi(\mathbf{r}) = \alpha\varphi_0(r) + \sum_k a(k)\exp(ikz)\varphi_k(r), \quad \text{(B1)}$$

where all the $\varphi(r)$ are supposed to satisfy $F_-\varphi=0$. Then we try to solve

$$2\kappa^2(\nabla \times \mathbf{B}_0(\mathbf{r})) = \psi^*(\mathbf{r})(\nabla/i - \kappa^2\mathbf{A}_i)\psi(\mathbf{r}) + \text{c.c.}, \quad \text{(B2)}$$

keeping in mind that we are interested in contributions to $\mathbf{B}_0^2$ of at most second order in a. Writing

$$2\kappa^2\mathbf{B}_0(\mathbf{r}) = -(|\psi(\mathbf{r})|^2 - \langle|\psi(\mathbf{r})|^2\rangle_s)\hat{z} + \mathbf{B}'(\mathbf{r}), \quad \text{(B3)}$$

we take care of the main contribution to $\mathbf{B}_0$ and we are then left with the equations for $\mathbf{B}'$:

$$(\partial/\partial x)B'_y - (\partial/\partial y)B'_x = \psi^*(\mathbf{r})\sum_k a_k k\exp(ikz)\varphi_k(r) + \text{c.c.}, \quad \text{(B4)}$$

$$(\partial/\partial x)B'_z - (\partial/\partial z)B'_x = 0,$$
$$(\partial/\partial y)B'_z - (\partial/\partial z)B'_y = 0, \quad \text{(B5)}$$

and

$$(\partial/\partial x)B'_x + (\partial/\partial y)B'_y + (\partial/\partial z)B'_z = -i[\psi^*(\mathbf{r})\sum_k a_k k\exp(ikz)\varphi_k(r) + \text{c.c.}]. \quad \text{(B6)}$$

The components B'_x and B'_y contribute to $\mathbf{B}_0^2$ the term $2\kappa^2(B'^2_x + B'^2_y)$; being interested in the a^2 contributions

TABLE I. The function $\xi_+(r)$.

M \ N	−6	−5	−4	−3	−2	−1	0	1	2	3	4	5	6
0	1.4035	1.4688	1.6443	1.8772	2.1021	2.2618	2.3192	2.2618	2.1021	1.8772	1.6443	1.4688	1.4035
1	1.3926	1.4908	1.6888	1.9269	2.1400	2.2759	2.3046	2.2201	2.0403	1.8071	1.5807	1.4263	1.3926
2	1.3619	1.4965	1.7171	1.9565	2.1527	2.2618	2.2618	2.1527	1.9565	1.7171	1.4965	1.3619	1.3619
3	1.3173	1.4923	1.7323	1.9662	2.1400	2.2201	2.1930	2.0632	1.8539	1.6089	1.3919	1.2772	1.3173
4	1.2690	1.4857	1.7370	1.9565	2.1021	2.1527	2.1021	1.9565	1.7370	1.4857	1.2690	1.1775	1.2690
5	1.2306	1.4827	1.7323	1.9269	2.0403	2.0632	1.9946	1.8386	1.6118	1.3524	1.1313	1.0731	1.2306
6	1.2158	1.4857	1.7171	1.8772	1.9565	1.9565	1.8772	1.7171	1.4857	1.2158	0.9837	0.9837	1.2158
7	1.2306	1.4923	1.6888	1.8071	1.8539	1.8386	1.7576	1.6001	1.3678	1.0866	0.8317	0.9452	1.2306
8	1.2690	1.4965	1.6443	1.7171	1.7370	1.7171	1.6443	1.4965	1.2690	0.9837	**0.6811**	0.9837	1.2690
9	1.3173	1.4908	1.5807	1.6089	1.6118	1.6001	1.5455	1.4146	1.2017	0.9411	0.8317	1.0731	1.3173
10	1.3619	1.4688	1.4965	1.4857	1.4857	1.4965	1.4688	1.3619	1.1775	0.9837	0.9837	1.1775	1.3619
11	1.3926	1.4263	1.3919	1.3524	1.3678	1.4146	1.4202	1.3437	1.2017	1.0866	1.1313	1.2772	1.3926
12	1.4035	1.3619	1.2690	1.2158	1.2690	1.3619	1.4035	1.3619	1.2690	1.2158	1.2690	1.3619	1.4035

TABLE II. The function $\xi_-(r)$.

N \ M	−6	−5	−4	−3	−2	−1	0	1	2	3	4	5	6
0	0.0645	0.0562	0.0363	0.0161	0.0040	0.0003	0	0.0003	0.0040	0.0161	0.0363	0.0562	0.0645
1	0.0754	0.0607	0.0358	0.0142	0.0030	0.0002	0.0000	0.0009	0.0073	0.0239	0.0486	0.0702	0.0754
2	0.1062	0.0779	0.0414	0.0147	0.0029	0.0003	0.0003	0.0029	0.0147	0.0414	0.0779	0.1062	0.1062
3	0.1509	0.0995	0.0475	0.0152	0.0030	0.0009	0.0014	0.0072	0.0285	0.0717	0.1275	0.1645	0.1509
4	0.1993	0.1172	0.0500	0.0147	0.0040	0.0029	0.0040	0.0147	0.0500	0.1172	0.1993	0.2424	0.1993
5	0.2377	0.1239	0.0475	0.0142	0.0073	0.0072	0.0088	0.0262	0.0799	0.1781	0.2931	0.3285	0.2377
6	0.2525	0.1172	0.0414	0.0161	0.0147	0.0147	0.0161	0.0414	0.1172	0.2525	0.4070	0.4070	0.2525
7	0.2377	0.0995	0.0358	0.0239	0.0285	0.0262	0.0255	0.0592	0.1589	0.3340	0.5378	0.4417	0.2377
8	0.1993	0.0779	0.0363	0.0414	0.0500	0.0414	0.0363	0.0779	0.1993	0.4070	**0.6811**	0.4070	0.1993
9	0.1509	0.0607	0.0486	0.0717	0.0799	0.0592	0.0470	0.0945	0.2299	0.4394	0.5378	0.3285	0.1509
10	0.1062	0.0562	0.0779	0.1172	0.1172	0.0779	0.0562	0.1062	0.2414	0.4070	0.4070	0.2414	0.1062
11	0.0754	0.0702	0.1275	0.1781	0.1589	0.0945	0.0623	0.1105	0.2299	0.3340	0.2931	0.1645	0.0754
12	0.0645	0.1062	0.1993	0.2525	0.1993	0.1062	0.0645	0.1062	0.1993	0.2525	0.1993	0.1062	0.0645

to $\mathbf{B}_0^2$ only we can then replace $\psi^*(\mathbf{r})$ by $\varphi_0^*(r)$ in (B4), and solve (B4) and (B6) simultaneously with $B'_z=0$. The resulting contributions to $\mathbf{B}_0^2$ would then have factors $\alpha^2k^2\sim(1-\bar{B})k^2$ which have to be neglected besides the term k^2 in the $\mathfrak{F}$ of (2.18), according to our policy of keeping only terms of lowest order in $(1-\bar{B})$.

The proposed solution for $\mathbf{B}'$ does not solve the equations (B5), however; (B4) and (B5) cannot be solved simultaneously since the divergence of the current, i.e., $\partial/\partial z$ of the right-hand side of (B4), does not vanish. The "misfit" of the equations (B5), i.e., the quantities $(\partial/\partial z)B'_x$ and $(\partial/\partial z)B'_y$, are of magnitude $\alpha\sum_k k^2a_k\varphi_k(r)$. Since the important contribution to the fluctuations comes from values $k^2\approx 1-\bar{B}$, this misfit is again smaller by a factor $(1-\bar{B})$ than the quantities we have already included into (B3). It seems therefore consequent within our framework of approximations to use

$$\mathbf{B}_0(\mathbf{r})=-(2\kappa^2)^{-1}(|\psi(\mathbf{r})|^2-\langle|\psi(\mathbf{r})|^2\rangle_s) \quad \text{(B7)}$$

as the "solution" of (B2).

APPENDIX C: NUMERICAL VALUES OF THE FUNCTIONS $\xi_+(r)$ AND $\xi_-(r)$

We calculated the functions $\xi_+(r)$ and $\xi_-(r)$ numerically, using (2.19), (A3), and (A22). In Tables I and II the values of the functions are listed for the arguments

$$r(M,N)\doteq z(M,N)=(M+\tfrac{1}{2}(\exp\tfrac{1}{3}\pi i)N)\tfrac{1}{12}l,$$
$$-6\leq M\leq +6, \qquad 0\leq N\leq 12,$$

i.e., for 169 points in the upper half of one original fundamental cell of basic length l, centered at $r=0$.