Motion of the Vortex Structure in Type-II Superconductors in High Magnetic Field*

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We study microscopically the resistive state in Type-II superconductors in high field in two extreme limits (i.e., the dirty limit and the pure limit). We establish that in high magnetic field, a constant electric field induces (1) a uniform motion of the order parameter in the direction perpendicular to both the electric and the magnetic fields with velocity $-u=-E/H_{c2}$, and (2) a polarization along the electric field proportional to this velocity. This gives rise to additional contributions to the electric and thermoelectric transport coefficients. We calculate in both limits the changes of longitudinal resistivity and Ettinghausen coefficient due to the above effect. In the pure limit, we also obtain the variations of the Hall angle and the Peltier coefficient.

1. INTRODUCTION

THERE have been recently a number of experiments[1] associated with the motion of vortex lines in type-II superconductors. They involve measurements of the voltages and temperature differences (or heat currents) in the presence of a finite transport current. Many theories,[2] based on the analogy between this situation and the similar one in superfluid helium, have been proposed to account for these dissipative phenomena. Most of them treat the mixed state semiphenomenologically (essentially using a two-fluid model and hydrodynamic assumptions). This is probably the source of the discrepancies which have sometimes appeared between different theories. The above assumptions, which are good for He II are more or less arbitrary here, since they are both incompatible with the expression of the electromagnetic conductivity in a pure superconductor. The hydrodynamic approach is ruled out in this case by the presence of a logarithmic frequency-dependent term in the conductivity[3] (this corresponds to a non-Ohmic behavior of the "normal" current, i.e., to a logarithmic increase of this current at very long times[4,5]).

On the other hand the two-fluid model cannot describe consistently the responses to perturbations having different time-reversal symmetries (i.e., associated with different coherence factors), e.g., the electromagnetic and thermal conductivities.

Finally, all these theories deal with the low-field region ($H \gtrsim H_{c1}$) where the distance between vortex lines is much larger than the coherence distance.

Recently, Schmid[6] and Kulik[7] have discussed this problem with the help of time-dependent Ginzburg-Landau equations (Schmid derives a diffusionlike equation, while Kulik uses an inadequate phenomenological wavelike one). In particular, the former proves that, close to H_{c2} and in the presence of an electric field, the order parameter moves with a uniform velocity, and is able to account for the flux-flow resistivity close to the transition temperature T_{c0}.

In the present paper, we generalize this type of approach to both dirty and pure superconductors at arbitrary temperatures and in the vicinity of the upper critical field H_{c2}. Using the fact that close to H_{c2} the order parameter Δ is small, we can describe its motion microscopically.

In previous works[8] we have obtained equations which describe the time-dependent fluctuations of the order parameter in the high-field region. In the same sense as an electromagnetic wave is a coherent superposition of photons, the motion of the order parameter in the mixed state can be understood as a coherent emission of quanta of this fluctuation field (although in low

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[1] For a comprehensive review of experiments, see for instance, Y. B. Kim and M. J. Stephen, in *Treatise on Superconductivity*, edited by R. D. Parks (Marcel Dekker, Inc., New York, to be published); F. A. Otter, Jr. and P. R. Solomon, Phys. Rev. Letters **16,** 681 (1966); A. T. Fiory and B. Serin, *ibid.* **21,** 308 (1966); J. Lowell, J. S. Munoz, and J. Sousa, Phys. Letters **27,** 376 (1967).

[2] See for instance: M. J. Stephen and J. Bardeen, Phys. Rev. Letters **14,** 112 (1965); Phys. Rev. **140,** A1197 (1965); M. Tinkham, Phys. Rev. Letters **13,** 804 (1964); M. J. Stephen and H. Suhl, *ibid.* **13,** 797 (1964); P. W. Anderson, N. R. Werthamer, and J. M. Luttinger, Phys. Rev. **138,** A1157 (1965); P. Nozieres and W. F. Vinen, Phil. Mag. **14,** 667 (1966); M. P. Kemoklidze and L. P. Pitaevskii, Zh. Eksperim. i Teor. Fiz. **50,** 243 (1966) [English transl.: Soviet Phys.—JETP **23,** 160 (1966)]; P. G. de Gennes and P. Nozieres, Phys. Letters **15,** 216 (1965).

[3] A. A. Abrikosov, L. P. Gor'kov, and I. M. Khalatnikov, Zh. Eksperim. i Teor. Fiz. **35,** 265 (1958) [English transl.: Soviet Phys.—JETP **8,** 182 (1959)].

[4] This difficulty has been independently noticed recently by W. A. B. Evans (private communication).

[5] Precisely speaking, this objection does not apply to dirty gapless superconductors (e.g., dirty type-II materials in high field). In this case, due to the short collision lifetime, the low-frequency complex conductivity can be expanded in powers of the frequency [see K. Maki, Phys. Rev. **141,** 331 (1966)].

[6] A. Schmid, Physik. Kondensienten Materie **5,** 302 (1966).

[7] I. O. Kulik, Zh. Eksperim. i Teor. Fiz. **50,** 1617 (1966) [English transl.: Soviet Phys.—JETP **23,** 1077 (1966)].

[8] C. Caroli and K. Maki, Phys. Rev. **159,** 306 (1967); **159,** 316 (1967).

fields these fluctuations are coupled with other (e.g., electromagnetic) fluctuations of the superconductor). The spectrum of fluctuations of the order parameter is of a diffusion type at all temperatures, with a diffusion coefficient

$$D \cong v(1/l+1/\xi_0)^{-1}$$

(where l is the electronic mean free path, ξ_0 is the BCS coherence distance, and v the Fermi velocity). Therefore, the motion of the Abrikosov structure in an electric field can be visualized as the motion of a set of charged Brownian particles in a magnetic and an electric field at right angle; that is the Abrikosov structure drifts at right angle from both the magnetic field and the electric field with a velocity E/H_{c2}. This analogy with Brownian particles cannot be pursued too far. For example we have to calculate the currents by using precise expressions in terms of the moving order parameter $\Delta(\mathbf{r}, t)$.

In Sec. 2 we develop a formalism, which allows us to describe the time evolution of the order parameter, and to calculate various currents in terms of the time-dependent order parameter $\Delta(\mathbf{r}, t)$ in the presence of both a magnetic field and an electric field. Furthermore we establish that the order parameter (i.e., the Abrikosov structure) moves uniformly with a velocity $u=E/H_{c2}$ in the electric field independently of the impurity scattering and at all temperatures.

Restricting ourselves to the dirty limit, we calculate in Sec. 3 the longitudinal resistivity and the Ettingshausen effect for all temperatures (in the vicinity of H_{c2}). As has already been pointed out by Kulik,[7] the expression of the local current contains oscillating parts with frequencies nku, where n is an integer, $k \simeq [\xi(T)]^{-1}$ (i.e., the inverse coherence distance of a superconducting alloy) and $u=E/H_{c2}$, which is analogous to the ac Josephson effect.

In Sec. 4, we will discuss various currents in a pure type-II superconductor in the high-field region.

We do not calculate here the Hall current and the Peltier (i.e., longitudinal) heat current. However, we can show that the contribution to those quantities due to the motion of the order parameter is at most of order ω_c/T_{c0} (where $\omega_c=eH_{c2}/m$ and T_{c0} is the transition temperature in zero field) compared with the contributions to the longitudinal electric current and the transverse heat current, respectively. Therefore, in the pure limit we can neglect completely the contributions to the Hall and the Peltier currents due to the motion of the order parameter, which gives rise to a correction of the order of (ξ_0/l). In this case the variations of the Hall angle and the Peltier coefficient should reflect simply the variations of the longitudinal resistivity and of the Ettingshausen coefficient, respectively. In the dirty limit, on the other hand, the corrections due to the motion of the order parameter are of the same order in l/ξ_0 as those in the normal state. In order to obtain them, one should improve the accuracy of the calculations up to order $(\tau E_f)^{-1}$ (where τ is the collision lifetime of an electron and E_f is the Fermi energy).

Finally, we assume everywhere that one can completely discard the pinning effects on the motion of the order parameter, which is known from experiments to be a quite reasonable assumption in the high-field region ($H \simeq H_{c2}$). Furthermore, we assume in all the calculations that the Fermi surface is spherical, which may be adequate for relatively pure Nb (say, with a resistivity ratio ~10 000) but may not be valid for extremely pure Nb.

2. FORMULATION

The purpose of this section is to present a general formalism which allows us to calculate various physical quantities in a state where the order parameter varies in time. Since we are interested here in studying the resistive state of type-II superconductors in the high-field region to the lowest order in the order parameter, it is possible to treat the effect of the order parameter as a perturbation (i.e., we consider the normal state with $\Delta(\mathbf{r}, t)=0$ as the unperturbed ground state).

The time variations of the various physical quantities (i.e., observables) $A(\mathbf{r}, t)$ which are bilinear in the electron creation and annihilation operators $\psi_\sigma^\dagger(\mathbf{r})$ and $\psi_\sigma(\mathbf{r})$ are ruled by the interaction Hamiltonian

$$\mathcal{H}_{\mathrm{I}}=e\int n(\mathbf{r}, t)\phi(\mathbf{r}, t)d^3r + \int [\Delta(\mathbf{r}, t)\Psi^\dagger(\mathbf{r}, t)+\Delta^\dagger(\mathbf{r}, t)\Psi(\mathbf{r}, t)]d^3r, \quad (1)$$

where

$$n(\mathbf{r}, t)=\sum_\sigma \psi_\sigma^\dagger(\mathbf{r}, t)\psi_\sigma(\mathbf{r}, t),$$

$$\Psi(\mathbf{r}, t)=\psi_\uparrow(\mathbf{r}, t)\psi_\downarrow(\mathbf{r}, t),$$

$$\Psi^\dagger(\mathbf{r}, t)=\psi_\downarrow^\dagger(\mathbf{r}, t)\psi_\uparrow^\dagger(\mathbf{r}, t), \quad (2)$$

and

$$\Delta^\dagger(\mathbf{r}, t)=-|g|\langle\Psi^\dagger(\mathbf{r}, t)\rangle \quad \text{etc.} \quad (3)$$

Equations (1) and (3) involve only one assumption, namely that one can still make, as in the static case, a generalized time-dependent Hartree–Fock approximation. This is a very good approximation—at least in weak-coupling superconductors—since the electron–electron interaction can be considered as instantaneous compared with the scale of time variations of the order parameter (i.e., $\omega_D \gg T_{c0}$, where ω_D is the Debye frequency of the material). Here $\phi(\mathbf{r}, t)=-Ex$ is the scalar potential which describes a constant electric field in the x direction, while the time-dependent order parameter $\Delta(\mathbf{r}, t)$ has to be determined consistently from Eq. (3). In the absence of the electric field, H_{I} can be treated to lowest order in perturbation with respect to the state with $\Delta=0$ (i.e., normal state). We know that in this situation the self-consistency equation, Eq. (3), reduces to a generalized linear

Ginzburg–Landau equation, whose solution is the Abrikosov solution.

We will now treat the electric field as a perturbation acting on this equilibrium state. Since we will see that it induces a drastic change of the structure (i.e., motion of the order parameter), we will first write Eq. (3) to lowest order in Δ but to all orders in the electric field. This allows us to describe quite generally the motion of the order parameter. Then we can use this result to calculate the currents to first order in the electric field.

A. Equation for the Order Parameter

Let us consider the equation which describes the time variations of $\Delta(\mathbf{r},t)$ in an electric field E. Equation (3) is formally rewritten as

$$\Delta^\dagger(\mathbf{r},t)=-|g|\left\langle \exp\left[i\int_{-\infty}^{t}\mathcal{H}_{\mathrm{I}}(t')dt'\right]\Psi^\dagger(\mathbf{r},t)\times\exp\left[-i\int_{-\infty}^{t}\mathcal{H}_{\mathrm{I}}(t'')dt''\right]\right\rangle$$
$$=i|g|\int_{-\infty}^{t}dt'\int d^3r'\times\langle[\Psi^\dagger(\mathbf{r},t),\Psi(\mathbf{r}',t')]\rangle\Delta^\dagger(\mathbf{r}',t'),\quad(4)$$

where $i\langle[A(t),B(t')]\rangle\theta(t-t')$ is the retarded product and can be obtained by making use of the temperature Green's function technique.[9] We have kept here only the lowest order term in $\Delta(\mathbf{r},t)$, since we are interested in the behavior of the order parameter at H_{c2}, which is adequate for the following calculations. It is important to notice that we have to take into account the effect of the external field to all orders. For this purpose it is convenient to start with the case of no electric field (i.e., $E=0$). Here, the time dependence of $i\langle[\Psi^\dagger(\mathbf{r}',t'),\Psi(\mathbf{r},t)]\rangle$ is very simple (i.e., depends only on the relative time $(t'-t)$] and Eq. (4) reduces to[10]

$$\{1-|g|\langle[\Psi^\dagger,\Psi]\rangle_{\mathbf{q}\omega}\}\Delta^\dagger(\mathbf{q},\omega)=0.\quad(5)$$

This equation has been obtained previously in the study of the fluctuations of the order parameter.[8] Using these results, we have in the dirty limit:

$$\left\{\ln\frac{T}{T_{c0}}+\psi\left(\tfrac{1}{2}+\frac{\Lambda}{4\pi T}\right)-\psi(\tfrac{1}{2})\right\}\Delta^\dagger(\mathbf{r},t)=0,\quad(6)$$

where

$$\Lambda=-i\omega+DQ^2,\qquad D=\tfrac{1}{3}(l_{\mathrm{tr}}v),$$

and

$$\mathbf{Q}=(1/i)\nabla+2e\mathbf{A}.$$

Furthermore, in the dirty limit, the effect of the electric field can be incorporated simply by replacing $i\omega$ by $-(\partial/\partial t)+2ie\phi(x)$. Other terms which come from the noncommutative nature of operators Q_μ and $i\omega=-(\partial/\partial t)+2ie\phi(x)$ give rise to terms of higher order in l/ξ_0 (where l is the electron mean free path and ξ_0 the BCS coherence distance) and thus negligible. This is analogous to the treatment of the magnetic field in the dirty limit.[11] As in the case of the calculation of the upper critical field H_{c2}, we can solve Eq. (6) by considering the set of equations:

$$[-(\partial/\partial t)+2ie\phi(x)+DQ^2]\Delta^\dagger(\mathbf{r},t)=\epsilon_0\Delta^\dagger(\mathbf{r},t)\quad(7)$$

and

$$\ln(T/T_{c0})+\psi(\tfrac{1}{2}+\epsilon_0/4\pi T)-\psi(\tfrac{1}{2})=0.\quad(8)$$

We will defer discussion of the above equations until the next section.

In the pure limit, on the other hand, Eq. (5) is still an integral equation. In this case, by analogy with the treatment of the vector potential,[12] we may introduce the effect of the external electric field by means of a phase factor:

$$\Delta^\dagger(\mathbf{r},t)=-i|g|\int_{-\infty}^{t}dt'\int d^3r'\exp[iS(\mathbf{r}t;\mathbf{r}'t')]\times\langle[\Psi^\dagger(\mathbf{r},t),\Psi(\mathbf{r}',t')]\rangle_{E=0}\Delta^\dagger(\mathbf{r}',t'),\quad(9)$$

where

$$S(\mathbf{r}t;\mathbf{r}'t')=2e\int_{t'}^{t}\phi(x'')dt'',\quad(10)$$

and $\langle\ \rangle_{E=0}$ is the average in the normal state in the absence of the electric field. In fact, we can show that, as in the treatment of the vector potential, the above procedure gives a correct prescription as long as we neglect the curvature of the electron orbit due to the static magnetic field. Such an approximation is adequate for the calculations of the longitudinal electric current and the transverse heat current, but is not for the calculation of the transverse electric current (Hall effect) and of the longitudinal heat current (Peltier effect), since these effects arise from terms of the order of ω_c/T_{c0}. Fortunately, in the discussion of these quantities in the pure limit, we can completely neglect the corrections to those contributions associated with the order parameter $\Delta^\dagger(\mathbf{r},t)$, since they are smaller than the normal-state values by a factor of order $(\xi_0/l)^2$. Therefore, throughout the following discussions the electric field will always be treated by introducing an appropriate phase factor in the relevant kernel.

B. Current Operators

The currents in the system with a time-dependent order parameter (and in the electric field) are given

[9] See for example: A. A. Abrikosov, L. P. Gor'kov, and I. E. Dzialoshinski, *Methods of Quantum Field Theory in Statistical Physics* (Prentice-Hall, Englewood Cliffs, New Jersey, 1963), Chap. 7, Sec. 37-2.

[10] We use the standard notation, i.e., $\langle[A,B]\rangle_{q\omega}$ is the Fourier transform of the corresponding retarded product (see Ref. 9).

[11] K. Maki, Physics **1**, 21 (1964).

[12] L. P. Gor'kov, Zh. Eksperim. i Teor. Fiz. **37**, 835 (1959) [English transl.: Soviet Phys.—JETP **10**, 593 (1960)].

as

$$\tilde{A}(\mathbf{r},t)=\left\langle \exp\left[i\int_{-\infty}^{t}\mathcal{H}_{\mathrm{I}}(t_1)dt_1\right]A(\mathbf{r},t)\exp\left[-i\int_{-\infty}^{t}\mathcal{H}_{\mathrm{I}}(t_2)dt_2\right]\right\rangle$$
$$=A_1(\mathbf{r},t)+A_2(\mathbf{r},t), \tag{11}$$

where

$$A_1(\mathbf{r},t)=-ie\int_{-\infty}^{t}dt'\int d^3r'\,\langle[A(\mathbf{r},t),n(\mathbf{r}',t')]\rangle\phi(\mathbf{r}',t'), \tag{12}$$

and

$$A_2(\mathbf{r},t)=\int_{-\infty}^{t}dt_1\int_{-\infty}^{t}dt_2\int d^3l d^3m\{\langle[[A(\mathbf{r},t),\Psi(\mathbf{l},t_1)],\Psi^\dagger(\mathbf{m},t_2)]\rangle$$
$$+\langle[[A(\mathbf{r},t),\Psi^\dagger(\mathbf{m},t_2)],\Psi(\mathbf{l},t_1)]\rangle\}\Delta^\dagger(\mathbf{l},t_1)\Delta(\mathbf{m},t_2). \tag{13}$$

Here the terms linear in $\Delta(\mathbf{r},t)$ [or $\Delta^\dagger(\mathbf{r},t)$] do not contribute (since they do not conserve the number of particles). $A_1(\mathbf{r},t)$ is the current in the normal state in the presence of the electric field, while $A_2(\mathbf{r},t)$ is the lowest-order contribution due to the (moving) order parameter.

The effect of the electric field in the various retarded products involving $\Psi(\mathbf{r},t)$ operators is treated as in Subsec. A. That is, the scalar potential $\phi(x)$ can be considered as a simple shift of frequencies in the dirty limit, while in the pure limit, it appears through an appropriate phase factor (see Sec. 4).

3. DIRTY LIMIT

A. Uniform Motion of the Order Parameter

We will confine ourselves to the dirty limit, where the electronic mean free path l is short with respect to the BCS coherence distance ξ_0. We have seen that the time dependence of the order parameter in an electric field is described by the equation

$$\{(\partial/\partial t)-2ie\phi(\mathbf{r})+D(\nabla+2ie\mathbf{A})^2\}\Delta^\dagger(\mathbf{r},t)=\epsilon_0\Delta^\dagger(\mathbf{r},t), \tag{14}$$

where $\epsilon_0=2DeH_{c2}$ is determined by

$$-\ln(T/T_{c0})=\psi(\tfrac{1}{2}+\epsilon_0/4\pi T)-\psi(\tfrac{1}{2}) \tag{15}$$

and $\psi(z)$ is the di-gamma function.

Here we take $\phi(\mathbf{r})=-Ex$ and $\mathbf{A}=(0,Hx,0)$ (i.e., the magnetic field is along the z axis). As is well-known, in the absence of electric field, the equilibrium solution of Eq. (14) is a linear combination of Abrikosov's degenerate solutions ($H=H_{c2}$);

$$\Delta^\dagger(\mathbf{r},t)=\sum_n C_n^*\exp(ikny)\times\exp[-eH(x+nk/2eH)^2], \tag{16}$$

where the constants k and C_n are such that Δ corresponds to the usual triangular configuration.

In the presence of an electric field E, the solution of Eq. (14) becomes

$$\Delta^\dagger(\mathbf{r},t)=\sum_n C_n^*\exp[ikn(y+ut)]\times\exp\left[-eH\left(x+\frac{kn}{2eH}+\frac{iu}{4eHD}\right)^2\right], \tag{17}$$

where $u=E/H_{c2}$. This solution has been previously derived by Schmid[13] with the help of the time-dependent Ginzburg–Landau equation, which is valid for $T\lesssim T_{c0}$. We have neglected here the shift of the upper critical field in the electric field, since it is of second order in u—i.e., H_{c2} is given now by

$$H_{c2}(E)=H_{c2}[1-(1/8H_{c2})(E/H_{c2}D)^2].$$

The modifications of the vortex configuration (i.e., changes in the C_n's) due to the electric field are always negligible, since they would introduce only a correction of higher order in u. We can then take for k and the C_n's the value for $E=0$. Equation (17) indicates that the electric field induces a uniform motion of the Abrikosov structure in the y direction. The velocity of this motion is such that in a frame moving with the order parameter (i.e., with the velocity $-u=-E/H_{c2}$), there is no net Lorentz force. It is sometimes useful to note that $\Delta(\mathbf{r},t)$ satisfies the relation

$$(\partial/\partial t)\Delta^\dagger(\mathbf{r},t)=u(\partial/\partial y)\Delta^\dagger(r,t) \tag{18}$$

B. Transport Equations in the Resistive State

One of the most fascinating properties of the socalled resistive state is that a finite electric field can exist in this state inside the bulk of the type-II superconductor, giving rise to electric dissipation and thermoelectric effects. This feature is in strong contrast to the behavior of a type-I superconductor, where the electric field $E=0$ everywhere in the bulk of the specimen, so that there is no electric dissipation and thermoelectric effect.[14] Whatever description one gives of the resistive

[13] A similar solution has also been found by Kulik (see Ref. 7). However, his result is not valid, since he starts from a wavelike equation instead of the appropriate diffusionlike equation.

[14] This obviously does not hold in the intermediate state, where the normal regions and the $N-S$ boundaries allow a finite average electric field to exist.

state, the appearance of a finite voltage always has to be interpreted in terms of vortex-line motion.[2]

We are now going to show that the uniformly moving solution for the order parameter discussed in the preceding subsection gives rise to finite corrections to the average electric and heat currents with respect to their normal state values. Making use of the formalism developed in Sec. 2, we can express the electric current as

$$\mathbf{j}(\mathbf{r},t)=\mathbf{j}_1(\mathbf{r},t)+\mathbf{j}_2(\mathbf{r},t), \tag{19}$$

with

$$\mathbf{j}_1(\mathbf{r},t)=-ie\int_{-\infty}^{t}dt'\int d^3r'\,\langle[\mathbf{j}(\mathbf{r},t),n(\mathbf{r}',t')]\rangle\phi(\mathbf{r}',t'), \tag{20}$$

$$\mathbf{j}_2(\mathbf{r},t)=\frac{1}{2}\int_{-\infty}^{t}dt_1\int_{-\infty}^{t}dt_2\int d^3ld^3m\{\langle[[\mathbf{j}(\mathbf{r},t),\Psi(\mathbf{l},t_1)],\Psi^\dagger(\mathbf{m},t_2)]\rangle +\langle[[\mathbf{j}(\mathbf{r},t),\Psi^\dagger(\mathbf{m},t_2)],\Psi(\mathbf{l},t_1)]\rangle\}\Delta^\dagger(\mathbf{l},t_1)\Delta(\mathbf{m},t_2), \tag{21}$$

where

$$\mathbf{j}(\mathbf{r},t)=-\frac{ie}{2m}\sum_\sigma(\nabla-\nabla'-2ie\mathbf{A}(\mathbf{r},t))\psi_\sigma^\dagger(\mathbf{r}',t)\psi_\sigma(\mathbf{r},t)|_{\mathbf{r}'=\mathbf{r}}. \tag{22}$$

We also have a similar expression for the heat current, where the heat current operator is given by

$$\mathbf{j}^h(\mathbf{r},t)=-\frac{1}{2m}\sum_\sigma\left\{(\nabla-ie\mathbf{A})\left(\frac{\partial}{\partial t'}-ie\phi\right)+(\nabla'+ieA)\left(\frac{\partial}{\partial t}+ie\phi\right)\right\}\psi_\sigma^\dagger(\mathbf{r}',t')\psi_\sigma(\mathbf{r},t)|_{\mathbf{r}'=\mathbf{r},t'=t}, \tag{23}$$

$\mathbf{j}_1(\mathbf{r},t)$ and $\mathbf{j}_1^h(\mathbf{r},t)$ are the currents in the normal state, and we have

$$j_{1x}(\mathbf{r},t)=\sigma(1+\eta^2)^{-1}E,$$
$$j_{1y}(\mathbf{r},t)=\sigma[\eta/(1+\eta^2)]E, \tag{24}$$

and

$$\mathbf{j}_1^h(\mathbf{r},t)=ST\mathbf{j}_1(\mathbf{r},t), \tag{25}$$

where $\sigma=e^2\tau_{\mathrm{tr}}N/m$ is the conductivity of the normal metal in zero magnetic field, $\eta=\tau_{\mathrm{tr}}\omega_c$ (where $\omega_c=eH_{c2}/m$ is the cyclotron frequency) and $S(=\pi^2T/3e\mu$ for a spherical Fermi surface) the thermoelectric power coefficient. $\mathbf{j}_2(\mathbf{r},t)$ and $\mathbf{j}_2^{(h)}(\mathbf{r},t)$ are the contributions due to the motion of the vortex structure (i.e., that of the order parameter), in which we are interested here. Equation (21) and the corresponding one for the heat current are treated with the help of the standard-temperature Green's function technique (see Appendix A) and we have,

$$\mathbf{j}_2(\mathbf{r},t)=(e\tau_{\mathrm{tr}}N/4\pi mT)(\mathbf{q}_1-\mathbf{q}_2)\psi^{(1)}(\tfrac{1}{2}+\rho)\Delta(1)\Delta^\dagger(2)|_{1=2=(\mathbf{r},t)}, \tag{26}$$

$$\mathbf{j}_2^h(\mathbf{r},t)=(\tau_{\mathrm{tr}}N/8\pi mT)(\mathbf{q}_1-\mathbf{q}_2)(\omega_2-\omega_1)[\psi^{(1)}(\tfrac{1}{2}+\rho)+\tfrac{1}{2}\rho\psi^{(2)}(\tfrac{1}{2}+\rho)]\Delta(1)\Delta^\dagger(2)|_{1=2=(\mathbf{r},t)}, \tag{27}$$

where

$$\mathbf{q}_1=(1/i)\nabla_1-2e\mathbf{A}(1),\qquad \mathbf{q}_2=(1/i)\nabla_2+2e\mathbf{A}(2),$$
$$\omega_1=i(\partial/\partial t_1)-2e\phi(1),\qquad \omega_2=i(\partial/\partial t_2)+2e\phi(2). \tag{28}$$

Here we have made use of the relation

$$(-i\omega_1+Dq_1^2)\Delta(1)=\epsilon_0\Delta(1),$$
$$(-i\omega_2+Dq_2^2)\Delta^\dagger(2)=\epsilon_0\Delta^\dagger(2), \tag{29}$$

$-\epsilon_0=2DeH_{c2}(T)$, $\rho=\epsilon_0/4\pi T$, and $\psi^{(1)}(z)$ is the trigamma function. We note that both expressions (26) and (27) contain ac contributions oscillating with frequencies $\omega_0=nku$ where n is an integer and k is of the order of $\xi(t)^{-1}=[2eH_{c2}(t)]^{1/2}$. The basic harmonic is in the radio wave region (say 10^5 cps). This ac current is associated with the local variations of the amplitude of $\Delta(\mathbf{r},t)$ due to the uniform motion of the order parameter. An analysis of the low-frequency noise might reveal this ac effect.[7] Making a space average we finally have

$$\langle\mathbf{j}_{2x}\rangle=(e\tau_{\mathrm{tr}}N/4\pi mT)(u/D)\psi^{(1)}(\tfrac{1}{2}+\rho)\langle|\Delta|^2\rangle$$
$$=-Mu/D, \tag{30}$$

$$\langle\mathbf{j}_{2y}\rangle=0,$$
$$\langle\mathbf{j}_{2x}^h\rangle=0,$$
$$\langle j_{2y}^h\rangle=-(\tau_{\mathrm{tr}}N/2\pi mT)[\psi^{(1)}(\tfrac{1}{2}+\rho)+\tfrac{1}{2}\rho\psi^{(2)}(\tfrac{1}{2}+\rho)]\langle|\Delta|^2\rangle eE$$
$$=ME\{2+[\rho\psi^{(2)}(\tfrac{1}{2}+\rho)/\psi^{(1)}(\tfrac{1}{2}+\rho)]\}\equiv MEL_D(t). \tag{31}$$

$L_D(t)$ is plotted on Fig. 1.

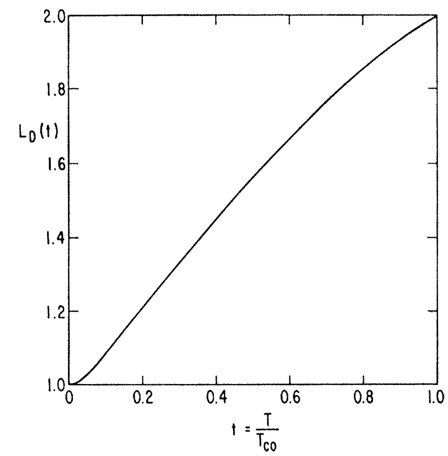

FIG. 1. $L_D(t)$, which appears in the expression of the entropy carried by the vortices in the dirty limit, is plotted against the reduced temperature.

Here we have made use of the expression of the magnetization:

$$-4\pi M=(e\tau_{\text{tr}}N/mT)\langle|\Delta|^2\rangle\psi^{(1)}(\tfrac{1}{2}+\rho). \quad (32)$$

Furthermore, in the above calculation we neglect systematically all terms of order of $l/\xi_0(=\tau_{\text{tr}}T_{c0})$. Therefore, because of this approximation, we do not find any contribution to the Hall current and to the Peltier effect due to the motion of the order parameter. In the evaluation of these quantities a more accurate treatment of the various terms would thus be necessary.

We note here that the ratio of the heat and electric currents is given by

$$\langle j_{2y}{}^h\rangle/\langle j_{2x}\rangle=-\tfrac{1}{2}[\epsilon_0(t)L_D(t)]=-DH_{c2}L_D(t); \quad (33)$$

$t=T/T_{c0}$ is the reduced temperature.

The above result suggests that a temperature gradient (perpendicular to the magnetic field) produces the reciprocal effect (i.e., a heat current parallel to the temperature gradient and an electric current perpendicular to it), so that the complete set of transport equations are

$$j_x=\sigma_sE_x+\beta(\nabla T)_y,$$
$$j_y{}^h=\alpha E_x-K_s(\nabla T)_y, \quad (34)$$

where

$$\sigma_s=\sigma\left[1+\frac{4\kappa_1{}^2(0)}{(2\kappa_2{}^2(t)-1)\beta_A}\left(1-\frac{H_0}{H_{c2}}\right)\right], \quad (35)$$

$$\alpha=-\frac{\sigma\epsilon_0(t)}{2e^2}\frac{4\kappa_1{}^2(0)}{(2\kappa_2{}^2(t)-1)\beta_A}L_D(t)\left(1-\frac{H_0}{H_{c2}}\right). \quad (36)$$

$\beta_A=1.16$, $\kappa_1(0)=(\pi^2/2[14\zeta(3)]^{1/2})\kappa\cong1.20\kappa$ and $\sigma=e^2\tau_{\text{tr}}N/m$ is the conductivity of the normal metal. Since we do not calculate the superconducting corrections to the Hall and Peltier currents, we have consistently kept in Eqs. (34) only the terms of zeroth order in $\tau\omega_c$. In particular, we have considered the normal value of α as negligible. Furthermore, we have made use of the expression of the magnetization

$$-4\pi M=(H_{c2}-H_0)/(2\kappa_2{}^2(t)-1)\beta_A. \quad (37)$$

K_s is the thermal conductivity in the mixed state obtained previously,[15]

$$K_s=K_n-\frac{1}{2e}\frac{H_{c2}-H_0}{(2\kappa_2{}^2(t)-1)\beta_A}\rho\left[1+\rho\frac{\psi^{(2)}(\frac{1}{2}+\rho)}{\psi^{(1)}(\frac{1}{2}+\rho)}\right]. \quad (38)$$

We prove in Appendix A that $\beta=\alpha/T$, as expected from the reciprocity principle. Using the above set of equations, we can discuss various transport properties.

C. Resistivity, Ettingshausen, and Nernst Effects

As an application of the above equations, we consider the usual situation met in the resistivity measurements in a type-II superconductor. In this case, the experiments are done with the condition

$$j_y{}^{(h)}=0. \quad (39)$$

The solution of Eq. (34) is then given by

$$j_x=\{\sigma_s+\alpha^2/TK_s\}E_x, \quad (40)$$

and

$$(\nabla T)_y=+(\alpha/K_s)E_x. \quad (41)$$

Since α is proportional to $\langle|\Delta|^2\rangle$, we can neglect the second term in Eq. (40) in the discussion of the resistivity. Substituting the expressions for σ_s, α and K_s given in Eqs. (35), (36), (38), respectively, we have

$$\frac{R_s}{R_n}=1-\frac{4\kappa_1{}^2(0)}{(2\kappa_2{}^2(t)-1)\beta_A}\left(1-\frac{H_0}{H_{c2}}\right), \quad (42)$$

and

$$\frac{(\nabla T)_\perp}{E}=+\frac{\sigma eDH_{c2}4\kappa_1{}^2(0)}{e^2K_s(2\kappa_2{}^2(t)-1)\beta_A}\left(1-\frac{H_0}{H_{c2}}\right)L_D(t). \quad (43)$$

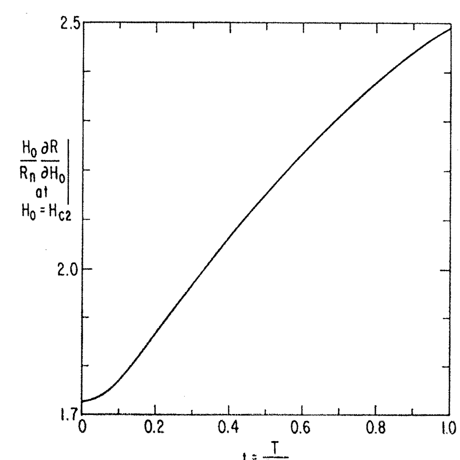

FIG. 2. $(H_0/R_n)(\partial R/\partial H_0)\,|_{H_0=H_{c2}}$ is plotted against the reduced temperature for a dirty material with $\kappa_2\gg1$.

[15] C. Caroli and M. Cyrot, Physik Kondensienten Materie 4, 285 (1965).

R_n is the resistivity in the normal state. Equation (42) has been obtained previously by Schmid[6] in the Ginzburg–Landau region (i.e., $T_{c0}-T\ll T_{c0}$), and he has shown that in this region of temperature, it is in good agreement with the existing experimental data. Equation (42) shows that $H_0(\partial R_s/\partial H_0)|_{H_0=H_{c2}}$ decreases monotonically[16] as temperature decreases. This behavior should be easily tested experimentally. We plot in Fig. 2 the temperature dependence of $(H_0/R_n)(\partial R_n/\partial H_0)|_{H_0=H_{c2}}$ for a material with a large Ginzburg–Landau parameter [i.e., $2\kappa_2^2(t)-1\cong 2\kappa_2^2(t)$] without Pauli paramagnetism effect.

Equation (43) might be interpreted by saying that each vortex line carries an amount of entropy

$$\mathcal{S}=\frac{\sigma DH_{cz}}{e^2T}\frac{4\pi\kappa_1^2(0)L_D}{(2\kappa_2^2(t)-1)\beta_A}\left(1-\frac{H_0}{H_{c2}}\right). \tag{44}$$

This entropy vanishes at $H_0=H_{c2}$ (as it should be at a second-order transition). We can also consider the reciprocal effect (i.e., Nernst effect). In the situation where a finite temperature gradient is applied along the y axis, a heat current along the y axis and a finite electric field in the x direction are induced. They are given by

$$j_y^{(h)}=\{K_s+\alpha^2/T\sigma_s\}(-\nabla T)_y, \tag{45}$$

and

$$E_x=(\alpha/T\sigma_s)(\nabla T)_y. \tag{46}$$

In Eq. (45) the second term is negligible, since it is of higher order in Δ^2, even though it gives rise to an additional anisotropic contribution to the heat conduction in lower fields. We rewrite Eq. (46) as

$$\frac{E_\perp}{(\nabla T)}\cong\frac{DH_{c2}(t)}{eT}L_D(t)\frac{4\kappa_1^2(0)}{(2\kappa_2^2(t)-1)\beta_A}\left(1-\frac{H_0}{H_{c2}}\right). \tag{47}$$

IV. PURE LIMIT

A. Motion of the Order Parameter

In the pure limit, Eq. (4), which rules the variations of the order parameter at H_{c2}, can no longer be reduced to a simple differential equation as in the dirty case, but is an intrinsically nonlocal integral equation. We therefore need here a more elaborate treatment of the electric field than in the dirty limit, where $\phi(x)$ was essentially treated as a c number shifting the frequencies. We can rewrite Eqs. (9) and (10) as

$$\Delta^\dagger(\mathbf{r},t)=|g|\int_{-\infty}^{\infty}dt'\int d^3r'\exp\left\{2ie\int_{t'}^{t}\phi[\mathbf{r}(t'')]dt''-2ie\int_{\mathbf{r}'}^{\mathbf{r}}\mathbf{A}(1)\cdot d\mathbf{l}\right\}K_0(\mathbf{r}t;\mathbf{r}'t')\Delta^\dagger(\mathbf{r}',t'), \tag{48}$$

where the kernel

$$K_0(\mathbf{r}t;\mathbf{r}'t')=i\langle[\Psi^\dagger(\mathbf{r},t),\Psi(\mathbf{r}',t')]\rangle^{A=E=0}\theta(t-t') \tag{49}$$

is an electron–electron or hole–hole propagator in the pure metal in the absence of any electric or magnetic field. Let us note again that Eq. (48) neglects the curvature of the electron orbit due to the magnetic field.

K_0 can be calculated easily (see Appendix B) with the help of the temperature-dependent Green's functions technique, and we obtain

$$K_0(\mathbf{r}t;\mathbf{r}'t')=\pi TN(0)\int\frac{d\Omega_v}{4\pi}\theta(t-t')\{\delta[\mathbf{R}-\mathbf{v}(t-t')]+\delta[\mathbf{R}+\mathbf{v}(t-t')]\}[\sinh(2\pi T|\mathbf{R}|/v)]^{-1}, \tag{50}$$

where Ω_v stands for the two angles θ and ϕ defining the direction of the velocity $\mathbf{v}$ of a quasiparticle, $|v|$ is always equal to the Fermi velocity v and $\mathbf{R}=\mathbf{r}-\mathbf{r}'$.

Substituting Eq. (50) into Eq. (48), we can eliminate the variable t' and obtain

$$\Delta^\dagger(\mathbf{r},t)=|g|\int d^3r'K(\mathbf{r},\mathbf{r}')\Delta^\dagger(r',t')|_{t'=t-|x-x'|/|v_x|}, \tag{51}$$

where

$$K(\mathbf{r},\mathbf{r}')=\int\frac{d\Omega_v}{4\pi}\frac{1}{|v_x|}\exp\left\{-ieE\frac{(x+x')|x-x'|}{|v_x|}-ieH(x^2-x'^2)\tan\phi\right\}$$
$$\times\int\frac{d^3q}{(2\pi)^3}\exp[i\mathbf{q}\cdot(\mathbf{r}-\mathbf{r}')]\left[\ln\left(\frac{2\gamma\omega_D}{\pi T}\right)-\frac{1}{2}\left\{\psi\left(\frac{1}{2}-\frac{i\mathbf{q}\cdot\mathbf{v}}{4\pi T}\right)+\psi\left(\frac{1}{2}+\frac{i\mathbf{q}\cdot\mathbf{v}}{4\pi T}\right)\right\}\right], \tag{52}$$

and

$$v_x=v\sin\theta\cos\phi.$$

[16] In the so-called "high field superconductors," which have a strong Pauli paramagnetic effect, this behavior can be different, due to the different temperature variation of κ_2 as well as to the presence of a paramagnetic contribution to the current.

In order to analyze Eq. (51), we will first briefly describe one of the methods which can be used for solving the static problem (i.e., in the absence of the electric field E). This method is then easily extended to the present case. From the study of the linearized Ginzburg–Landau equation, it is reasonable to expand $\Delta^\dagger(\mathbf{r})$ in series of eigenfunctions ϕ_n of the operator $\pi^2=(-i\nabla+2e\mathbf{A})^2$, which are the wave functions of the harmonic oscillator

$$\phi_n^*=(\Pi^+)^n \exp(iky)\exp[-eH(x+k/2eH)^2],$$

$$\Delta^\dagger(\mathbf{r})=\sum_{n=0}^{\infty} a_n\phi_n^*, \tag{53}$$

where $\Pi^+=\Pi_x+i\Pi_y$ and k and a_n are constants. Substituting this expression of $\Delta^\dagger(\mathbf{r})$ in Eq. (48) (in the absence of E), we can determine a_n from:

$$\sum_m\{\langle n\mid n\rangle\delta_{nm}-\langle n\mid K\mid m\rangle\}a_m=0. \tag{54}$$

Here we consider solutions with a given k, since K does not mix ϕ_n functions having different k values and $\langle nk\mid K\mid mk\rangle$ is independent of k. (This remains true for the moving solution.) As has been shown by Helfand and Werthamer,[17] $\langle n\mid K\mid m\rangle$ is diagonal and hence each ϕ_n is a solution of Eq. (26). The physical one, which describes $\Delta^\dagger(\mathbf{r})$ at $H=H_{c2}$, is ϕ_0. In the presence of a uniform electric field, since the equation ruling the fluctuations is, as mentioned before, essentially of diffusion type,[8] it is quite natural to expand $\Delta^\dagger(\mathbf{r},t)$ in series of eigenfunctions of the diffusion operator

$$((\partial/\partial t)-2ie\phi)+\mathfrak{D}(i^{-1}\nabla+2e\mathbf{A})^2,$$

$$\Pi_{unk}^*(\mathbf{r},t)=(\Pi^+)^n\exp[ik(y+ut)]$$
$$\times\exp\left[-eH\left(x+\frac{k}{2eH}+\frac{iu}{4e\mathfrak{D}H}\right)^2\right], \tag{55}$$

where $u=E/H_{c2}$. These functions describe the stationary state of the system as corresponding to a uniformly moving-order parameter as in the case of the dirty limit. The fact that $\Delta^\dagger(\mathbf{r},t)$ moves with the uniform velocity $-u=-E/H_{c2}$ follows from gauge invariance and ensures that there is no net electric field in the frame moving with the order parameter.

At this point, $\mathfrak{D}$ is an arbitrary diffusion constant which we are now going to determine. We expand $\Delta^\dagger(\mathbf{r},t)$ as[18]

$$\Delta^\dagger(\mathbf{r},t)=\sum_n a_n\phi_{un}^*(\mathbf{r},t). \tag{56}$$

The a_n's are constants which have to be determined with the help of Eq. (54). Keeping in mind that in the limit $u=0$ (i.e., $E=0$) $\langle n\mid K\mid m\rangle$ is diagonal and $\Delta^\dagger(\mathbf{r},t)\equiv\phi_0^*(\mathbf{r})$, we can solve Eq. (54) by iteration. (The first iteration is sufficient in order to calculate $\Delta^\dagger(\mathbf{r},t)$ up to first order in u)

$$\Delta^\dagger(\mathbf{r},t)=\phi_{u0}^*(\mathbf{r},t)+\sum_{n\neq0}\frac{\langle n_u\mid K\mid 0_u\rangle}{\langle n_u\mid n_u\rangle}\phi_{un}^*(\mathbf{r},t). \tag{57}$$

We notice that

$$\langle(2p+1)_u\mid K\mid 0_u\rangle=O(u)$$

while

$$\langle(2p)_u\mid K\mid 0_u\rangle=O(u^2),$$

where p is an integer. In the evaluation of the various linear responses we can thus neglect the contribution of $n=2p$ (with $p\geq1$)

$$\Delta^\dagger(\mathbf{r},t)=\phi_{u0}^*(\mathbf{r},t)+\frac{\langle 1_u\mid K\mid 0_u\rangle}{\langle 1_u\mid 1_u\rangle}\phi_{u1}^*(\mathbf{r},t)$$
$$+\frac{\langle 3_u\mid K\mid 0_u\rangle}{\langle 3_u\mid 3_u\rangle}\phi_{u3}^*(\mathbf{r},t)+\cdots. \tag{58}$$

Up to this point we are free to choose any value of $\mathfrak{D}$. We can therefore choose it so as to eliminate the contribution of ϕ_{u1}^* in Eq. (58). This procedure is analogous to the usual treatment of the Stark effect; that is the true ground state in the presence of the electric field has a finite dipole moment (which is here proportional to $u/\mathfrak{D}$). This choice of $\mathfrak{D}$ is convenient in the calculation of the various currents, since for this purpose it is then sufficient to write $\Delta^\dagger(\mathbf{r},t)=C\phi_{0u}^*(\mathbf{r},t)$, neglecting all the terms ϕ_{pu}^* with $p\geq1$. This follows from the fact that the current operators connect ϕ_{u0}^* only with ϕ_{u1}^* (as far as we are concerned with terms of first order in u only). Because K is diagonal in the limit $u=0$, $\langle 1_u\mid K\mid 0_u\rangle$ does not contain any terms of zeroth order in u. Keeping only the first-order terms and using the expression of K given in Eq. (52), we easily reduce the condition

$$\langle 1_u\mid K\mid 0_u\rangle=0 \tag{59}$$

to

$$0=T\int\frac{d\Omega_v}{v\sin\theta\mid\cos\phi\mid}\int_{-\infty}^{\infty}d\zeta\frac{\exp(-eH\zeta^2/2\cos^2\phi)}{\sinh[2\pi T\mid\zeta\mid/v\sin\theta\mid\cos\phi\mid]}\left\{-\frac{\zeta^2}{2\mathfrak{D}}+\frac{\mid\zeta\mid(1-eH\zeta^2\tan^2\phi)}{v\sin\theta\mid\cos\phi\mid}\right\}. \tag{60}$$

[17] E. Helfand and N. R. Werthamer, Phys. Rev. Letters **13**, 686 (1964); Phys. Rev. **147**, 288 (1966).

[18] We emphasize here the fact that the choice of functions made in Eq. (55) is completely general. For example, instead of Eq. (56), we could consider the most general form

$$\Delta^+(\mathbf{r},t)=\Sigma a_n(t)\Phi_{un}^*(\mathbf{r},t),$$

where the $a_n(t)$ are arbitrary time-dependent functions. We can show that this leads to

$$a_n(t)=e^{i\lambda t}a_n,$$

where the a_n's are time-independent constants and λ is independent of n, because space and time are essentially separable in Eq. (45). The stationary solution can further be proved to correspond to $\lambda=0$, whereas the other solutions are all decaying in time (i.e., $\mathrm{Im}\lambda>0$).

Finally, this gives

$$\mathfrak{D}=\frac{v^2}{8\pi T}\left(\int_0^1 \frac{dz}{(1-z^2)^{1/2}}\int_0^\infty d\zeta\,\zeta^2 \exp(-\rho^2\zeta^2)\left\{\sinh\left[\frac{\zeta}{(1-z^2)^{1/2}}\right]\right\}^{-1}\right)$$
$$\times\left(\int_0^1 \frac{dz}{1-z^2}\int_0^\infty d\zeta\,\zeta(1-\rho^2\zeta^2)\exp(-\rho^2\zeta^2)\left\{\sinh\left[\frac{\zeta}{(1-z^2)^{1/2}}\right]\right\}^{-1}\right)^{-1}, \quad (61)$$

where

$$\rho=\epsilon/2\pi T, \qquad \epsilon=v[\tfrac{1}{2}eH_{c2}(T)]^{1/2}.$$

From expression (61), the diffusion coefficient is seen to be positive (as is physically required) and real (which ensures that the motion of the pairs is a true diffusion process, and not a wave-like propagation). Equation (61) shows that $\mathfrak{D}$ is roughly of the order of $v\xi_0$. This means that the density or velocity correlations of the pairs decay on a distance of the order of ξ_0. This is to be contrasted to the dirty-limit situation, where the density and velocity correlations of the pairs decay on distances of order $(\xi_0 l)^{1/2}$ and l, respectively. The following asymptotic behaviors can be useful:

$$\mathfrak{D}=\frac{v^2}{2\pi T}\left[\frac{7\zeta(3)}{3\pi^2}-\left(\frac{124\zeta(5)}{5\pi^2}-\frac{14\zeta(3)}{9}\right)\rho^2\right]$$
$$=\frac{v^2}{2\pi T}\{0.284-0.734\rho^2\}, \qquad T\lesssim T_{c0}$$
$$\mathfrak{D}=\frac{v^2}{\pi^{3/2}\epsilon}\left\{1-(12\rho^2)^{-1}\left[\ln(\pi^2\gamma\rho^2)^{-1}+1+\frac{2\zeta'(2)}{\zeta(2)}\right]\right.$$
$$\left.+\frac{2}{\pi^2\rho^3}\zeta(\tfrac{3}{2})(1-2^{-1/2})\right\}, \qquad T\ll T_{c0}. \quad (62)$$

Since there are well-known discrepancies between the experimental and theoretical temperature variations of H_{c2}, it is probably better to insert in the formula for $\mathfrak{D}$ the measured value of $\epsilon=v[\frac{1}{2}eH_{c2}(T)]^{1/2}$. Using such a procedure, we find for Nb,

$$\mathfrak{D}_{T_{c0}}=33\ \text{cm}^2\ \text{sec}^{-1},$$
$$\mathfrak{D}_{T=0}=30.8\ \text{cm}^2\ \text{sec}^{-1}.$$

We will see in the following calculations that this diffusion coefficient is related to a physical property of the Abrikosov structure (i.e., its polarizability), which will appear in the expression of the electric current.

B. Transport Equations in the Resistive State

As we have pointed out in the preceding subsection, it is sufficient, in the calculation of the currents, to take for the order parameter the expression

$$\Delta^\dagger(\mathbf{r},t)=\sum_n C_n^*\exp[ikn(y+ut)]\psi_u(x+kn/2eH) \quad (63)$$

with

$$\psi_u(x)=\exp[-eH(x+iu/4e\mathfrak{D}H)^2],$$

where $\mathfrak{D}$ is given by Eq. (61).

The electric current $\mathbf{j}(\mathbf{r},t)=\mathbf{j}_1(\mathbf{r},t)+\mathbf{j}_2(\mathbf{r},t)$ is again obtained from the general expressions Eqs. (20) and (21). The normal state response $\mathbf{j}_1$ is given by Eqs. (24) and (25) as in the dirty limit. In order to calculate the superconducting correction $\mathbf{j}_2$, as in the case of the calculation of the order parameter, it is necessary to study Eq. (21) more carefully in the pure than in the dirty limit. This can be done by a method analogous to the one used in the study of the order parameter (see Appendix B), and we obtain

$$\langle j_{2x}\rangle=eN(0)\frac{u\langle|\Delta|^2\rangle}{2\mathfrak{D}}\left(\frac{v}{2\pi T}\right)^2\int_0^1\frac{dz}{(1-z^2)^{1/2}}$$
$$\times\int_0^\infty\frac{\zeta^2\exp(-\rho^2\zeta^2)}{\sinh[\zeta(1-z^2)^{-1/2}]}d\zeta$$
$$=-Mu/\mathfrak{D}, \quad (64)$$
$$\langle j_{2y}\rangle=0, \quad (65)$$

where we have made use of the expression of the magnetization

$$-4\pi M=[6\pi eN/m(2\pi T)^2]\langle|\Delta(\mathbf{r})|^2\rangle g(\rho),$$
$$g(\rho)=\frac{1}{2}\int_0^1\frac{dz}{(1-z^2)^{1/2}}\int_0^\infty\frac{\zeta^2\exp(-\rho^2\zeta^2)}{\sinh[\zeta(1-z^2)^{-1/2}]}d\zeta. \quad (66)$$

Therefore, if one measures the magnetization, the temperature dependence of $\langle j_{2x}(\mathbf{r},t)\rangle$ is completely determined by that of the diffusion constant $\mathfrak{D}$. We notice that the expression Eq. (64) of $\langle j_{2x}\rangle$ is formally equivalent to Eq. (30) in the dirty limit. This suggests that the relation

$$\langle j_{2x}\rangle=-Mu/\mathfrak{D}$$
$$=\mathfrak{D}^{-1}\frac{(H_{c2}-H_0)}{4\pi(2\kappa_2^2(t)-1)\beta_A}\frac{E}{H_{c2}} \quad (67)$$

should hold for a type-II superconductor with an arbitrary electronic mean free path, if one defines appropriately the diffusion constant $\mathfrak{D}$. We give the general proof of this in Appendix A. Equation (67) can be interpreted simply: The Abrikosov order parameter moves in a nonuniform magnetic field, the amplitude of the inhomogeneity being characterized by the magnetization. Since the moving-order parameter has a finite polarizability (proportional to $u/\mathfrak{D}$), the motion involves a dissipation corresponding to the absorption of energy from the dipolar part of the inhomogeneous field.

Exactly as in the case of the dirty limit, besides the average current $\langle j\rangle$, one should be able to observe an ac effect containing the various harmonics of the basic frequency $\omega_0=ku$ where k is of the order of $[\xi(T)]^{-1}$. In the case of pure Nb at low temperature, this frequency is of the order of 10^4 cps.

Similarly, we can calculate the heat current

$$\langle j_{2x}^{(h)}\rangle=0,$$

$$\langle j_{2y}^{(h)}\rangle=-\frac{3eN}{4m}\frac{\langle|\Delta|^2\rangle}{(2\pi T)^2}E\int_0^1\frac{dz}{(1-z^2)^{1/2}}\int_0^\infty\frac{d\zeta(3\zeta^2-2\rho^2\zeta^4)}{\sinh[\zeta(1-z^2)^{-1/2}]}\exp(-\rho^2\zeta^2) \tag{68}$$

$$=-\frac{H_{c2}-H_0}{4\pi(2\kappa_2^2(t)-1)\beta_A}L_P(t)E, \tag{69}$$

and

$$L_P(t)=\left(\int_0^1\frac{dz}{(1-z^2)^{1/2}}\int_0^\infty d\zeta\frac{(3\zeta^2-2\rho^2\zeta^4)}{\sinh[\zeta(1-z^2)^{-1/2}]}\exp(-\rho^2\zeta^2)\bigg/\int_0^1\frac{dz}{(1-z^2)^{1/2}}\int_0^\infty d\zeta\frac{\zeta^2}{\sinh[\zeta(1-z^2)^{-1/2}]}\exp(-\rho^2\zeta^2)\right). \tag{70}$$

$L_P(t)$ has the following asymptotic forms

$$L_P(t)=3\left(1-\frac{268}{35}\frac{\zeta(5)}{\zeta(3)}\rho^2+\cdots\right),\qquad T\lesssim T_{c0}$$

$$L_P(t)=1+\frac{1}{3\rho^2}\left[\ln(\pi\rho\sqrt{\gamma})-1-\frac{6\zeta'(2)}{\pi^2}\right]+\cdots,\qquad T\ll T_{c0}, \tag{71}$$

where $\gamma=1.78$ and $\zeta'(2)=-0.941$. The temperature dependence of $L_P(t)$ is quite different from that of $\mathfrak{D}^{-1}(T)$, which appears in the expression of the current. [That is, $L_P(t)$ is much more strongly temperature dependent, and decreases monotonically as T decreases.] One can notice on Eq. (69) that, as in the dirty case, the entropy carried by each line vanishes at $H=H_{c2}$, as it should. The ratio of the heat and electric currents is given by

$$\langle j_{2y}^{(h)}\rangle/\langle j_{2x}\rangle=-H_{c2}\mathfrak{D}L_P(t), \tag{72}$$

where $L_P(t)$ varies with temperature from $L_P(1)=3$ to $L_P(0)=1$, while in the dirty limit the analogous factor varies from $L_D(1)=2$ to $L_D(0)=1$. On the other hand, if one would make a phenomenological extension of the Ginzburg–Landau equation in the spirit of the two-fluid model, one would get[6]

$$\mathbf{j}_0=\frac{e^*}{2m^*}[(1/i)(\nabla-\nabla')-2e^*A]\psi_0^\dagger(\mathbf{r}'t')\psi_0(\mathbf{r}t)|_{\mathbf{r}=\mathbf{r}'},$$

$$\mathbf{j}_0^{(h)}=\frac{1}{2m^*}\left[(\nabla-ie^*\mathbf{A})\left(\frac{\partial}{\partial t'}-ie^*\phi\right)+(\nabla'+ie^*\mathbf{A})\left(\frac{\partial}{\partial t}+ie^*\phi\right)\right]\psi_0^\dagger(\mathbf{r}',t')\psi_0(\mathbf{r},t)|_{\mathbf{r}=\mathbf{r}',t=t'}, \tag{73}$$

where ψ_0 is the suitably normalized Ginzburg–Landau wave function, m^* and e^* are the effective mass and charge, respectively. From the above expression one would find

$$\langle j_{0y}^{(h)}\rangle/\langle j_{0x}\rangle=-2H_{c2}\mathfrak{D} \tag{74}$$

which describes neither the pure nor the dirty-limit behavior. We also show in Appendix C that in the Ginzburg–Landau region the "two-fluid behavior," Eq. (74) is obeyed only in the dirty limit and for $T\cong T_{c0}$. Therefore, the two-fluid approach is inconsistent with the microscopic calculations except at most in the dirty limit and for $T\cong T_{c0}$. This inconsistency seems to originate from the existence, in a superconductor, of the two different coherence factors. A two-fluid model cannot describe this internal structure of the condensate in a superconductor. This is in contrast to the case of liquid helium II where the condensate, being formed by real bosons, has no internal structure, which allows for many purposes a two-fluid description.

Making use of the same reciprocity arguments as in the dirty-limit calculation, we can now write the transport equations for the resistive state of a pure type-II material in high field.

$$\langle j_x\rangle=\sigma_s E_x-\sigma' E_y+P(\nabla T)_x+(\alpha/T)(\nabla T)_y, \quad (75a)$$

$$\langle j_y\rangle=\sigma' E_x-\sigma_s E_y+(\alpha/T)(\nabla T)_x+P(\nabla T)_y, \quad (75b)$$

$$\langle j_x^{(h)}\rangle=-TPE_x-\alpha E_y+K_s(-\nabla T)_x, \quad (75c)$$

$$\langle j_y^{(h)}\rangle=\alpha E_x-TPE_y+K_s(-\nabla T)_y, \quad (75d)$$

where

$$\sigma_s=\sigma\frac{1}{1+\eta^2}+\frac{1}{4\pi\mathfrak{D}}\frac{(1-H_0/H_{c2})}{[2\kappa_2^2(t)-1]\beta_A},$$
$$\sigma'=\sigma[\eta/(1+\eta^2)],$$
$$\alpha=\alpha_1-\frac{(H_{c2}-H_0)}{4\pi[2\kappa_2^2(t)-1]\beta_A}L_p(t), \quad (76)$$

and $\sigma=e^2\tau_{\text{tr}}N/m$, $\eta=\tau\omega_c$, P is the coefficient which appears in the expression of the thermoelectric power in the normal state, α_1 is the Ettingshausen coefficient in the normal state, and K_s is the thermal conductivity in the mixed state and has been obtained previously.[19]

It is important to point out that σ' is the same as that in the normal state. This follows from the fact that the superconducting corrections σ_2 and σ_2' to the longitudinal and Hall conductivities, respectively, are such that, at most, $\sigma_2'=0(\xi_0/r_c)\sigma_2$, where $r_c=v/\omega_c$ is the cyclotron radius for a quasiparticle with the Fermi velocity. This order of magnitude can be obtained by dimensional considerations from a more accurate expression of the current (including the cyclotron motion of the quasiparticles). This order of magnitude can also be understood on the basis of the following qualitative argument: we have seen from the expression Eq. (58) of the diffusion constant that, in the pure limit, ξ_0 is the characteristic distance for the decay of the velocity correlation of the pairs. On this distance, the angular deviation of a pair is of the order of ξ_0/r_c, this implies $\sigma_2'\simeq\xi_0\sigma_2/r_c\simeq\sigma'(\xi_0/l)^2$ (where we made use of $\sigma_2\simeq\sigma\xi_0/l$). Therefore, this correction to the Hall conductivity is negligible in all usual experimental situations related to measurements of the Hall angle, the variation of which is mainly due to that of the longitudinal conductivity.

C. Electric and Thermoelectric Effects

Equations (75) describe the general transport properties of the resistive state in a pure type-II superconductor in the high-field region. Let us concentrate now, for example, on the situation where a given electric current density j_x is sent in the x direction, with the conditions

$$j_y^{(h)}=0, \quad j_y=0, \quad \text{and} \quad (\nabla T)_x=0.$$

[19] K. Maki, Phys. Rev. **158**, 397 (1967).

In this case, the solution is given by

$$E_x=(TP^2-\sigma_s K_s)(\text{Det})^{-1}j_x, \quad (77a)$$

$$E_y=(\sigma' K_s+\alpha P)(\text{Det})^{-1}j_x, \quad (77b)$$

$$(\nabla T)_y=-(\alpha\sigma_s+TP\sigma')(\text{Det})^{-1}j_x, \quad (77c)$$

$$j_x^{(h)}=-(\alpha\sigma' K_s-K_s\sigma_s TP+\alpha^2 P+T^2P^3)(\text{Det})^{-1}j_x, \quad (77d)$$

where

$$\text{Det}=-[(\sigma_s^2+\sigma'^2)K_s+\alpha^2\sigma_s/T-\sigma_s TP^2+2\sigma'\alpha P]. \quad (77e)$$

Looking into the relative orders of magnitude of the different terms involved (and noticing for instance that P, α and σ' are small in the normal state) and neglecting terms of higher order in $\langle|\Delta|^2\rangle$, we can simplify Eqs. (77) into

$$R_s\left(\equiv\frac{E_x}{j_x}\right)\cong\left(\sigma_s+\frac{\alpha^2}{TK_s}\right)^{-1}\cong\sigma_s^{-1}$$
$$=R_n\left\{1-\frac{(1-H_0/H_{c2})}{4\pi\sigma\mathfrak{D}(2\kappa_2^2(t)-1)\beta_A}\right\}, \quad (78)$$

$$\Phi(\equiv E_y/E_x)=-(\sigma'/\sigma_s)-(\alpha P/K_s\sigma_s). \quad (79)$$

The variation of the Hall angle Φ with the magnetic field in the superconducting region comes mainly from the first term (i.e., is essentially the same as that of the resistivity).

$$K_s(\nabla T)_y/j_x=TP(\sigma'/\sigma_s^2)+(\alpha/\sigma_s). \quad (80)$$

The main variation with the magnetic field in the superconducting region is due to the second term;

$$\left(\frac{K_s(\nabla T)_y}{j_x}\right)_s-\left(\frac{K_s(\nabla T)_y}{j_x}\right)_n=\frac{(H_{c2}-H_0)L_p(t)}{4\pi\sigma[2\kappa_2^2(t)-1]\beta_A}. \quad (81)$$

Finally the Peltier coefficient is given by:

$$\Pi(\equiv j_x^{(h)}/j_x)=(1/\sigma_s)[-TP+\alpha\sigma'/\sigma_s]. \quad (82)$$

Here in the superconducting region the main variation comes from that of the second term (i.e., it is the same as for the Ettingshausen effect).

Everywhere in the above calculations we have assumed that the Fermi surface is spherical. This should not be too serious a limitation to the applicability of our theory, provided that the measurements are done in not too extremely pure samples ($l\lesssim10^{-3}$ cm).

5. CONCLUSION

The microscopic study of the resistive state of type-II superconductors in high field shows that, independently of the mean free path and temperature:

(1) The order parameter in the presence of an electric field E moves with the uniform velocity $u=E/H$ in the direction perpendicular to both the magnetic and electric fields.

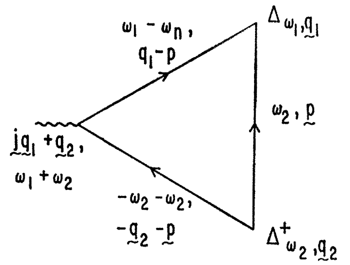

FIG. 3. Diagram corresponding to the superconducting contribution to the electric current (to lowest order in Δ).

(2) The motion of the pairs is controlled by a diffusion equation. The diffusion coefficient $\mathfrak{D}$ is independent of the temperature in the dirty limit and slightly dependent of it in the pure limit.

(3) $\mathfrak{D}^{-1}$ characterizes the polarizability of the Abrikosov structure. The corresponding correction to the longitudinal electric current thus obeys the relation

$$\langle j_{2x}\rangle = -Mu/\mathfrak{D},$$

where M is the magnetization of the sample.

(4) The electric field also gives rise to a transverse heat current. This can be interpreted as an entropy current carried by the vortex lines, of the order of $T^{-1}uH_{c2}M$. However, it should be noted that this energy current has so unusual temperature dependence that the conventional interpretation in terms of entropy flow will be unadequate.

(5) Besides these average currents, the inhomogeneity of the Abrikosov structure gives rise to ac currents which should be observable in the radio-wave range. These currents can be understood as an intrinsic ac Josephson effect.

In the pure limit the superconducting corrections to the Hall conductivity and to the longitudinal heat current are negligible. Therefore, the changes of the Hall angle and Peltier coefficient are essentially due to the changes of longitudinal conductivity and Ettingshausen coefficient.

This type of situation with a moving order parameter is not restricted to type-II superconductors in high field. It is known that it exists in type-II materials in low field, where it cannot be described by the kind of approach used in this paper. However, situations very closely analogous to the one studied here should also occur in gapless superconductors having a spatially varying order parameter along the direction of the electric field—such as in the surface sheath regime close to H_{c3} or the intermediate state.

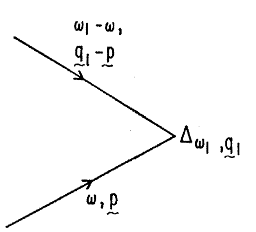

FIG. 4. S-wave diagram.

For instance, in the case of a dirty superconducting film much thicker than the coherence distance $\xi(T)$ with an electric field perpendicular to the film surface, we can show that the order parameter is given by

$$\Delta^\dagger(\mathbf{r},t) = Cf(y+ut,\, x+iu/4e\mathfrak{D}H), \tag{83}$$

where $u=E/H_{c3}$ and $f(y,x)$ is the solution in the absence of the electric field, which is expressed in terms of Weber functions.[20] [For a thinner film ($d \lesssim 2\xi(T)$), Eq. (83) should be slightly modified in order to satisfy

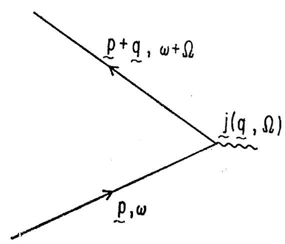

FIG. 5. P-wave diagram.

the boundary condition.] Using this solution we find

$$\langle j_{2x}\rangle = \frac{e\tau_{\rm tr}N}{4\pi mT}\psi^{(1)}(\tfrac{1}{2}+\rho)\langle|\Delta|^2\rangle\frac{0.59u}{\mathfrak{D}}$$

$$= \frac{0.59}{4\pi}\frac{u}{\mathfrak{D}}\left(\frac{\pi}{2}\right)^{1/2}\frac{2\xi(T)}{d}\frac{H_{c3}-H_0}{2\kappa_2^2(t)-0.334}, \tag{84}$$

$$\langle j_{2y}^{(h)}\rangle = -\frac{\tau_{\rm tr}N}{2\pi mT}\left[\psi^{(1)}(\tfrac{1}{2}+\rho)+\frac{\rho}{2}\psi^{(2)}(\tfrac{1}{2}+\rho)\right]$$

$$\times\langle|\Delta|^2\rangle 0.59eE$$

$$= \frac{0.59}{4\pi}\left(\frac{\pi}{2}\right)^{1/2}\frac{2\xi(T)}{d}L_D(t)\frac{H_{c3}-H_0}{2\kappa_2^2(t)-0.334}E, \tag{85}$$

where $\xi(T)=(1.18eH_{c3}(T))^{-1/2}$ and d is the thickness of the film. We have here considered the case where the surface sheath appears symmetrically on both surfaces of the (thick) film. Equation (84) predicts that in the surface sheath region the resistivity decreases linearly in the field as

$$\frac{R_s(H)}{R_n} = \left\{1-0.59\left(\frac{\pi}{2}\right)^{1/2}\frac{2\xi(T)}{d}\right.$$

$$\left.\times\frac{5.76\kappa^2}{2\kappa_2^2(t)-0.334}\left(1-\frac{H_0}{H_{c3}}\right)\right\}. \tag{86}$$

Furthermore, we have an ac current oscillating with a frequency $(2u/(\pi-2)^{1/2})\xi(T)^{-1}$, which is in the radio-wave region. The amplitude of this ac current decreases exponentially with $d/2\xi(T)$ [for $d \gg 2\xi(T)$]. It could

[20] D. Saint-James and P. G. de Gennes, Phys. Letters **7**, 306 (1963).

be of interest to detect this intrinsic Josephson effect in thinner films [$d\simeq 2$ or $3\xi(T)$]. In the thin film limit [i.e., $d\lesssim\xi(T)$], where the order parameter is almost constant in space, we expect similar but very small effects. In this limit, due to the absence of zeros of the order parameter, there is no ac effect [this happens when $d<1.8\xi(T)$].

ACKNOWLEDGMENTS

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APPENDIX A

Electric and Heat Currents in the Dirty Limit

Since in the dirty limit the effect of the electric and magnetic fields can be incorporated by the simple transformation

$$\omega\to\omega\pm 2ie\phi,\qquad \mathbf{q}\to\mathbf{q}\mp 2ie\mathbf{A},\tag{A1}$$

depending on whether ω and $\mathbf{q}$ act on Δ or $\Delta^\dagger$, it is sufficient to calculate formally $\mathbf{j}(\mathbf{q},\omega)$ and $\mathbf{j}^{(h)}(\mathbf{q},\omega)$ in the absence of any field. We then use transformation (A1) and finally let the external frequency and momentum go to zero (as far as we are concerned only with the average currents).

In order to calculate the retarded product in Eq. (21), we first evaluate the corresponding thermal product

$$\mathbf{j}_2(\mathbf{r},\omega_1+\omega_2)=-\left[\frac{ie}{m}(\boldsymbol{\nabla}'-\boldsymbol{\nabla})-\frac{2e^2\mathbf{A}(\mathbf{r})}{m}\right]T\sum_{n=-\infty}^{+\infty}\int d^3l d^3m$$
$$\times\langle G_{\omega_1-\omega_n}(\mathbf{r},1)G_{\omega_n}(\mathbf{m},1)G_{-\omega_2-\omega_n}(\mathbf{m},\mathbf{r}')\rangle_i\Delta_{\omega_1}(1)\Delta_{\omega_2}{}^\dagger(\mathbf{m})|_{\mathbf{r}=\mathbf{r}'},\tag{A2}$$

$\langle\ \rangle_i$ stands for the average over random impurity configurations,

$$\omega_n=(2n+1)\pi T,\qquad \omega_1=2n_1\pi T,\qquad \omega_2=2n_2\pi T,$$

where n is any integer and n_1 and n_2 are positive integers [since the real time-retarded product Eq. (21) is the analytic continuation of (A2), where frequencies ω_1 and ω_2 lie in the upper half-plane]. In the momentum representation expression (A2) corresponds to the diagram of Fig. 3.

As usual the effect of impurity scattering is taken into account by means of the following renormalizations:

(1) In each Green's function ω has to be replaced by $\tilde\omega=\omega[1+(1/2\tau|\omega|)]$ where τ is the collision lifetime of the electron.

(2) Each vertex (Fig. 4) corresponding to $\Delta(\mathbf{q},\Omega)$ or $\Delta^\dagger(\mathbf{q},\Omega)$ (s-wave vertex) introduces a factor

$$\begin{aligned}\eta_{\mathbf{q},\Omega}&=\{1-(1/\tau\,|2\tilde\omega-\tilde\Omega|)(1-\tfrac{1}{3}(\tau\tau_{\mathrm{tr}})v^2q^2)\}&&\text{if }(\omega-\Omega)\omega>0\\&=1&&\text{if }(\omega-\Omega)\omega<0.\end{aligned}$$

τ_{tr} is the transport lifetime of the electrons.

(3) In the vertex (Fig. 5) corresponding to the current operator (P-wave vertex) we have to replace $(2\mathbf{p}+\mathbf{q})$ by $(2\mathbf{p}+\alpha\mathbf{q})$, where

$$\begin{aligned}\alpha&=(2p_0v/3i)(\omega/|\omega|)(|\Omega|+Dq^2)^{-1}&&\text{if }\omega(\omega+\Omega)>0\\&=0&&\text{if }\omega(\omega+\Omega)<0,\end{aligned}$$

where p_0 is the Fermi momentum.

Furthermore, confining ourselves to the case ω_1, $\omega_2>0$, since we are interested in the corresponding retarded product, we have

$$\mathbf{j}_2(\mathbf{q},\omega_1+\omega_2)=\frac{ep_0v\tau N(0)}{3m}(\mathbf{q}_1-\mathbf{q}_2)\left\{(\omega_1+\omega_2+Dq_2^2-Dq_1^2)^{-1}\left[\psi\left(\tfrac{1}{2}+\frac{\omega_2+2\omega_1+Dq_2^2}{4\pi T}\right)-\psi\left(\tfrac{1}{2}+\frac{\omega_1+Dq_1^2}{4\pi T}\right)\right]\right.$$
$$\left.-(\omega_1+\omega_2+Dq_1^2-Dq_2^2)^{-1}\left[\psi\left(\tfrac{1}{2}+\frac{\omega_2+Dq_2^2}{4\pi T}\right)-\psi\left(\tfrac{1}{2}+\frac{2\omega_2+\omega_1+Dq_1^2}{4\pi T}\right)\right]\right\}\Delta(\mathbf{q}_1\omega_1)\Delta^\dagger(\mathbf{q}_2\omega_2),\tag{A3}$$

where $\mathbf{q}=\mathbf{q}_1+\mathbf{q}_2$. Furthermore, for low frequency $\omega_1+\omega_2\ll\pi T_{c0}$, the above expression reduces to

$$\mathbf{j}_2(\mathbf{q},\omega_1+\omega_2)=\frac{e\tau N}{2m}(\mathbf{q}_1-\mathbf{q}_2)\left\{\frac{1}{2\omega_2}\left[\psi\left(\tfrac{1}{2}+\frac{2\omega_2}{4\pi T}+\rho\right)-\psi(\tfrac{1}{2}+\rho)\right]+\frac{1}{2\omega_1}\left[\psi\left(\tfrac{1}{2}+\frac{2\omega_1}{4\pi T}+\rho\right)-\psi(\tfrac{1}{2}+\rho)\right]\right\}\Delta(\mathbf{q}_1\omega_1)\Delta^\dagger(\mathbf{q}_2\omega_2),$$

$$\mathbf{j}_2(\mathbf{q},\omega_1+\omega_2)\cong\frac{e\tau N}{4\pi mT}(\mathbf{q}_1-\mathbf{q}_2)\left[\psi^{(1)}(\tfrac{1}{2}+\rho)+\frac{(\omega_1+\omega_2)}{8\pi T}\psi^{(2)}(\tfrac{1}{2}+\rho)\right]\Delta(\mathbf{q}_1\omega_1)\Delta^\dagger(\mathbf{q}_2\omega_2),\tag{A4}$$

where we have made use of the relation

$$(\omega_1+Dq_1^2)\Delta(1)=\epsilon_0\Delta(1),$$
$$(\omega_2+Dq_2^2)\Delta^\dagger(2)=\epsilon_0\Delta^\dagger(2)\tag{A5}$$

with $\epsilon_0=2DeH_{c2}$ and $\rho=\epsilon_0/4\pi T$.

The calculation of the heat current follows exactly the same step. We give here the relevant calculation of the thermal product. The heat current is given by

$$\mathbf{j}_2^{(h)}(\mathbf{r},\omega_1+\omega_2)=\frac{1}{m}\left[\frac{\partial}{\partial t'}\nabla+\frac{\partial}{\partial t}\nabla'\right]T\sum_{n=-\infty}^{+\infty}\int d^3l d^3m\langle G_{\omega_1-\omega_n}(\mathbf{r},1)G_{\omega_n}(\mathbf{m},1)G_{-\omega_2-\omega_n}(\mathbf{m},\mathbf{r}')\rangle_i\Delta_{\omega_1}(1)\Delta_{\omega_2}^\dagger(\mathbf{m})|_{\mathbf{r}=\mathbf{r}',t=t'}.\tag{A6}$$

The impurity average can be carried out as before and we have

$$\mathbf{j}_2^{(h)}(\mathbf{q},\omega_1+\omega_2)=\frac{N\tau}{4m}(\mathbf{q}_1-\mathbf{q}_2)\left\{\psi\left(\tfrac{1}{2}+\frac{\omega_2+Dq_2^2}{4\pi T}\right)-\psi\left(\tfrac{1}{2}+\frac{2\omega_1+\omega_2+Dq_2^2}{4\pi T}\right)+\frac{\omega_2-Dq_1^2}{\omega_1+\omega_2-D(q_1^2-q_2^2)}\right.$$
$$\times\left[\psi\left(\tfrac{1}{2}+\frac{2\omega_1+\omega_2+Dq_2^2}{4\pi T}\right)-\psi\left(\tfrac{1}{2}+\frac{\omega_1+Dq_1^2}{4\pi T}\right)\right]+\frac{\omega_2+Dq_1^2}{\omega_1+\omega_2+D(q_1^2-q_2^2)}$$
$$\left.\times\left[\psi\left(\tfrac{1}{2}+\frac{2\omega_2+\omega_1+Dq_1^2}{4\pi T}\right)-\psi\left(\tfrac{1}{2}+\frac{\omega_2+Dq_2^2}{4\pi T}\right)\right]\right\}\Delta(\mathbf{q}_1\omega_1)\Delta^\dagger(\mathbf{q}_2\omega_2)\tag{A7}$$

which we can reduce to

$$\mathbf{j}_2^{(h)}(\mathbf{q},\omega_1+\omega_2)=(N\tau/8\pi mT)(\mathbf{q}_1-\mathbf{q}_2)(\omega_2-\omega_1)\{\psi^{(1)}(\tfrac{1}{2}+\rho)+\tfrac{1}{2}\rho\psi^{(2)}(\tfrac{1}{2}+\rho)\}\Delta(\mathbf{q}_1\omega_1)\Delta^\dagger(\mathbf{q}_2\omega_2)+0(\omega_1+\omega_2).\tag{A8}$$

The expressions for $\mathbf{j}_2$ and $\mathbf{j}_2^{(h)}$ should be continued analytically for ω_1 and ω_2 to $-i\omega_1$ and $-i\omega_2$ then the limit $\omega_1+\omega_2=0$ should be taken.

If we are interested in the dc term (i.e., $\omega_1+\omega_2\to0$) of the above expressions, we can calculate the corresponding quantities in expanding in powers of $\phi(r)$ (the scalar potential). In this case, we have

$$\mathbf{j}(\mathbf{q})=\langle[\mathbf{j},n]\rangle_{\mathbf{q},0}\phi_{\mathbf{q}}\tag{A9}$$

and

$$\mathbf{j}^{(h)}(\mathbf{q})=\langle[\mathbf{j}^{(h)},n]\rangle_{\mathbf{q},0}\phi_{\mathbf{q}}.\tag{A10}$$

The second-order terms in $\Delta(\mathbf{r})$ are given in terms of square diagrams. Similarly in the presence of a temperature gradient, we have

$$\mathbf{j}(\mathbf{q})=\langle[\mathbf{j},h]\rangle_{\mathbf{q},0}(\delta T/T)_{\mathbf{q}},\tag{A11}$$

$$\mathbf{j}^{(h)}(\mathbf{q})=\langle[\mathbf{j}^{(h)},h]\rangle_{\mathbf{q},0}(\delta T/T)_{\mathbf{q}},\tag{A12}$$

where δT is the shift of local temperature and

$$h(\mathbf{r},t)=(1/2i)\sum_\sigma\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial t'}\right)\psi_\sigma^\dagger(\mathbf{r},t')\psi_\sigma(\mathbf{r},t)|_{t'=t}\tag{A13}$$

is the local energy operator. From the explicit form of the square diagrams of Eqs. (A10) and (A11), we can immediately see that the reciprocal relation follows.

APPENDIX B

1. Calculation of $K_0(rt;\ r't')=i\langle[\Psi^\dagger(r,\ t),\ \Psi(r',\ t')]\rangle^{A=E=0}\theta(t-t')$ in the Pure Limit

First let us consider the corresponding thermal product, which is given by

$$K_0(\mathbf{q},\omega_\nu)=T\sum_{n=-\infty}^{+\infty}\int\frac{d^3p}{(2\pi)^3}G_{\omega_n+\omega_2}(\mathbf{p}+\mathbf{q})G_{-\omega_n}(\mathbf{p})$$

$$=2\pi TN(0)\sum_n\int\frac{d\Omega_v}{4\pi}\frac{\theta[\omega_n(\omega_n+\omega_\nu)]}{|2\omega_n+\omega_\nu|+i\mathbf{v}\cdot\mathbf{q}}$$

$$=N(0)\int\frac{d\Omega_v}{4\pi}\left\{\ln\frac{\omega_D}{2\pi T}-\tfrac{1}{2}\left[\psi\left(\tfrac{1}{2}+\frac{\omega_\nu+i\mathbf{v}\cdot\mathbf{q}}{4\pi T}\right)+\psi\left(\tfrac{1}{2}+\frac{\omega_\nu-i\mathbf{v}\cdot\mathbf{q}}{4\pi T}\right)\right]\right\}. \tag{B1}$$

The retarded product is then obtained by analytical continuation and we find

$$K_0(\mathbf{q},\omega)=N(0)\int\frac{d\Omega_v}{4\pi}\left\{\ln\frac{\omega_D}{2\pi T}-\tfrac{1}{2}\left[\psi\left(\tfrac{1}{2}+\frac{i\mathbf{v}\cdot\mathbf{q}-i\omega}{4\pi T}\right)+\psi\left(\tfrac{1}{2}-\frac{i\omega+i\mathbf{v}\cdot\mathbf{q}}{4\pi T}\right)\right]\right\}. \tag{B2}$$

Doing the Fourier transform, we finally obtain

$$K_0(\mathbf{r}t;\ \mathbf{r}'t')=\pi TN(0)\theta(t-t')\frac{1}{\sinh(2\pi TR/v)}\int\frac{d\Omega_v}{4\pi}\{\delta[\mathbf{R}-\mathbf{v}(t-t')]+\delta[\mathbf{R}+\mathbf{v}(t-t')]\}, \tag{B3}$$

where $\mathbf{R}=\mathbf{r}-\mathbf{r}'$. If one integrates over t, one obtains a well-known result;

$$K_0(R)=\int_{-\infty}^{\infty}K_0(\mathbf{r}t;\ \mathbf{r}'t')$$

$$=\frac{m^2T}{(2\pi R)^2}\frac{1}{\sinh(2\pi TR/v)}. \tag{B4}$$

In the presence of both magnetic and electric fields, we have

$$K(\mathbf{r}t;\ \mathbf{r}'t')=\exp[iS(\mathbf{r}t;\ \mathbf{r}'t')]K_0(\mathbf{r}t;\ \mathbf{r}'t'), \tag{B5}$$

where

$$S(\mathbf{r}t;\ \mathbf{r}'t')=2e\int_{t'}^{t}\phi[\mathbf{r}(t'')]dt''-2e\int_{\mathbf{r}'}^{\mathbf{r}}\mathbf{A}(\mathbf{l})\cdot d\mathbf{l}. \tag{B6}$$

Using expression (B3) and integrating over t', we reduce Eq. (48) to the form

$$\Delta^\dagger(\mathbf{r},t)=|g|\int d^3r'\int\frac{d\Omega_v}{4\pi}\exp\left\{-2ieE(x+x')\frac{|x-x'|}{|v_x|}-ieH(x+x')(y-y')\right\}K_0(|\mathbf{r}-\mathbf{r}'|)\Delta^\dagger\left(\mathbf{r}',t-\frac{|x-x'|}{|v_x|}\right). \tag{B7}$$

It is easy to calculate the matrix element which appeared in Eq. (54).

2. Calculations of $\mathbf{j}_2(\mathbf{r},t)$ and $\mathbf{j}_2^{(h)}(\mathbf{r},t)$

Since the procedures used in evaluation of the relevant integral kernels follows exactly the same line, we sketch them here briefly. We rewrite Eq. (21) as

$$\mathbf{j}_2(\mathbf{r},t)=-\int_{-\infty}^{t}dt_1\int_{-\infty}^{t}dt_2\int d^3l d^3m\exp[iS(\mathbf{m}t_2;\ \mathbf{l}t_1)]K_1(\mathbf{r}t;\mathbf{l}t_1;\mathbf{m}t_2)\Delta^\dagger(\mathbf{m},t_2)\Delta(\mathbf{l},t_1), \tag{B8}$$

where $K_1(\mathbf{r}_0,t_0;\ \mathbf{r}_1,t_1;\ \mathbf{r}_2,t_2)$ is obtained as

$$K_1(\mathbf{r}_0t_0;\ \mathbf{r}_1t_1;\ \mathbf{r}_2t_2)=2\pi eTN(0)\int\frac{d\Omega_v}{4\pi}\mathbf{v}[\delta(\mathbf{R}_{12}+\mathbf{v}t_{12})+\delta(\mathbf{R}_{12}-\mathbf{v}t_{12})]\{\theta(-t_{10})\theta(t_{12})\delta(\mathbf{R}_{10}+\mathbf{v}t_{10})[\sinh(2\pi Tt_{12})]^{-1}$$
$$-\theta(-t_{20})\theta(t_{21})\delta(\mathbf{R}_{20}+\mathbf{v}t_{20})[\sinh(2\pi Tt_{21})]^{-1}\}, \tag{B9}$$

where $\mathbf{R}_{ij}=\mathbf{r}_i-\mathbf{r}_j$ and $t_{ij}=t_i-t_j$ $(i,j=0,1,2)$. After integration over t_1, t_2 and $\mathbf{r}_0$ we find

$$\langle \mathbf{j}_2\rangle = ieN(0)2\pi T\langle|\Delta|^2\rangle\int\frac{d\Omega_v}{4\pi}\frac{\mathbf{v}}{v_x|v_x|}\int_{-\infty}^{\infty}dx\int_{-\infty}^{\infty}dx'\exp[-\Lambda(x,x',\mathbf{v})]\cos\left(\frac{eE(x^2-x'^2)}{v_x}\right)\left[\sinh\left(\frac{2\pi T|x-x'|}{|v_x|}\right)\right]^{-1}, \tag{B10}$$

and

$$\Lambda = eH(x^2+x'^2)-(iu/2\mathfrak{D})(x-x')+ieH\tan\phi(x^2-x'^2). \tag{B11}$$

A similar calculation gives the expression for the heat current

$$\langle \mathbf{j}_2^{(h)}\rangle = -i\pi TN(0)\langle|\Delta|^2\rangle\int\frac{d\Omega_v}{4\pi}\frac{\mathbf{v}}{v_x^2}\int_{-\infty}^{+\infty}dx\int_{-\infty}^{\infty}dx'\exp[-\Lambda(x,x',\mathbf{v})]$$
$$\times\sin\left(eE\frac{x^2-x'^2}{v_x}\right)\frac{|x-x'|\cosh(2\pi T|x-x'|/|v_x|)}{\sinh^2(2\pi T|x-x'|/|v_x|)}. \tag{B12}$$

APPENDIX C

Generalization to the Case of an Arbitrary Electronic Mean Free Path *l*.

(a) Determination of the Diffusion Constant

It is not difficult to generalize the discussion given in the text in Sec. 4A. For this purpose we note that the effect of the impurity scattering can be taken into account by renormalization of the frequencies and of the proper vertices.

$$\omega\rightarrow\tilde{\omega}=\omega(1+1/2\tau|\omega|),\qquad \phi_0\rightarrow\hat{\phi}_{0u}=\eta_{\omega 0}\phi_{0u},\qquad \phi_1\rightarrow\hat{\phi}_{1u}=\eta_{\omega 1}\phi_{1u}\ \text{etc}\cdots,$$

where Φ_n has been defined already in Eq. (55). Since we are interested in the corrections of first order in u we can consider the $\eta_{\omega i}$'s as independent of u (they do not contain any term of order u). They are given (following the procedure of Helfand and Werthamer[17]) by,

$$\eta_{\omega_n 0}=\left[1-(\tau\epsilon)^{-1}\int_0^{\infty}dpe^{-p^2}\arctan\left(\frac{\epsilon p}{|\tilde{\omega}_n|}\right)\right]^{-1},$$
$$\eta_{\omega_n 1}=\left[1-(\tau\epsilon)^{-1}\int_0^{\infty}dp(2p^2-1)e^{-p^2}\arctan\left(\frac{\epsilon p}{|\tilde{\omega}_n|}\right)\right]^{-1}, \tag{C1}$$

where

$$\epsilon=v(\tfrac{1}{2}eH_{c2})^{1/2},\qquad \tilde{\omega}_n=\omega_n(1+1/2\tau|\omega_n|).$$

Including these modifications, we obtain the diffusion coefficient as

$$\mathfrak{D}=\frac{v^2}{8\pi T}\left(\sum_{n\geq 0}\eta_{\omega_n 0}\eta_{\omega_n 1}\int_0^1 dz(1-z^2)^{-1/2}\int_0^{\infty}d\zeta\zeta^2\exp[-\rho^2\zeta^2-b_n\zeta(1-z^2)^{-1/2}]\right)$$
$$\times\left(\sum_{n\geq 0}\eta_{\omega_n 0}\eta_{\omega_n 1}\int_0^1 dz(1-z^2)^{-1}\int_0^{\infty}d\zeta\zeta(1-\rho^2\zeta^2)\exp[-\rho^2\zeta^2-b_n\zeta(1-z^2)^{-1/2}]\right)^{-1}, \tag{C2}$$

where

$$\omega_n=(2n+1)\pi T,\qquad b_n=2n+1+1/2n\tau T,\qquad \rho=\epsilon/2\pi T.$$

In particular in the vicinity of the transition temperature, the above expression reduces to

$$\mathfrak{D}=\frac{v^2}{6\pi T}\left(\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2(2n+1+y)}\bigg/\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}\right)$$
$$=(7\zeta(3)v^2/6\pi^3T_{c0})X(y), \tag{C3}$$

where

$$X(y)=\frac{8}{7\zeta(3)}\sum_{n>0}^{\infty}\frac{1}{(2n+1)^2(2n+1+y)}$$

was first introduced by Gor'kov in the microscopic derivation of the Ginzburg–Landau equations, and $y=(2\pi\tau T)^{-1}$.

(b) Expressions of the Currents

The electric current in this general situation is obtained as

$$\langle j_{2x}\rangle = eN(0)\,\frac{u\langle|\Delta|^2\rangle}{\mathfrak{D}}\left(\frac{v}{2\pi T}\right)^2\sum_{n\geq 0}(\eta_{\omega_n 0})^2\int_0^1\frac{dz}{(1-z^2)^{1/2}}\int_0^\infty d\zeta\zeta^2\exp\left[-\rho^2\zeta^2-b_n\frac{\zeta}{(1-z^2)^{1/2}}\right]$$

$$=-\frac{Mu}{\mathfrak{D}}=\frac{1}{4\pi\mathfrak{D}}\,\frac{H_{c2}-H_0}{(2\kappa_2^2(t)-1)\beta_A}\,u, \tag{C4}$$

$$\langle j_{2y}\rangle = 0, \tag{C5}$$

where we have made use of the expression of the magnetization

$$-4\pi M=\frac{3eN}{2\pi mT^2}\langle|\Delta|^2\rangle\sum_{n\geq 0}(\eta_{\omega_n 0})^2\int_0^1\frac{dz}{(1-z^2)^{1/2}}\int_0^\infty d\zeta\zeta^2\exp\left[-\rho^2\zeta^2-b_n\frac{\zeta}{(1-z^2)^{1/2}}\right]. \tag{C6}$$

The heat current is given similarly by

$$\langle j_{2y}^{(h)}\rangle=-\frac{3eN\langle|\Delta|^2\rangle}{m(2\pi T)^2}E\sum_{n\geq 0}(n+\tfrac{1}{2})\int_0^1\frac{dz}{(1-z^2)^{1/2}}\int_0^\infty d\zeta\zeta^2\left(\frac{\zeta\eta_{\omega_n 0}{}^2}{(1-z^2)^{1/2}}-2\pi T\eta_{\omega_n 0}\frac{\partial\eta_{\omega_n 0}}{\partial\omega_n}\right)\exp\left[-\rho^2\zeta^2-b_n\frac{\zeta}{(1-z^2)^{1/2}}\right]$$

$$=-\frac{H_{c2}-H_0}{4\pi(2\kappa_2^2(t)-1)\beta_A}L(t)E, \tag{C7}$$

$$\langle j_{2x}^{(h)}\rangle=0, \tag{C8}$$

where

$$L(t)=\frac{48}{\pi}\left(\sum_{n\geq 0}(n+\tfrac{1}{2})\int_0^1 dz(1-z^2)^{-1/2}\int_0^\infty d\zeta\,\zeta^2\left(\zeta\eta_{\omega_n 0}{}^2(1-z^2)^{1/2}-2\pi T\eta_{\omega_n 0}\frac{\partial\eta_{\omega_n 0}}{\partial\omega_n}\right)\exp[-\rho^2\zeta^2-b_n\zeta(1-z^2)^{-1/2}]\right)$$

$$\times\left(\sum_{n\geq 0}(\eta_{\omega_n 0})^2\int_0^1 dz(1-z^2)^{-1/2}\int_0^\infty d\zeta\zeta^2\exp[-\rho^2\zeta^2-b_n\zeta(1-z^2)^{-1/2}]\right)^{-1}. \tag{C9}$$

In the dirty and the pure limits, the above expression reduces to Eq. (31) and Eq. (70), respectively.

In particular in the Ginzburg–Landau region (i.e., $T\lesssim T_{c0}$), we obtain from (C9)

$$L(t)=\frac{\sum_{n\geq 0}[1/(2n+1)(2n+1+y)][(2/2n+1)+(1/2n+1+y)]}{\sum_{n\geq 0}[1/(2n+1)^2(2n+1+y)]}, \tag{C10}$$

where $y=(2\pi\tau T)^{-1}$. In this limit, $L(t)$ changes smoothly from 3 to 2 as the electronic mean free path decreases.

It may be instructive to express $\mathbf{j}$ and $\mathbf{j}^{(h)}$ in the Ginzburg–Landau region as

$$\mathbf{j}(\mathbf{r},t)=[eN/2m(2\pi T)^2][7\zeta(3)/8]X(y)i^{-1}(\nabla'-\nabla)\Delta^\dagger(\mathbf{r},t)\Delta(\mathbf{r}',t)|_{\mathbf{r}'=\mathbf{r}}, \tag{C11}$$

and

$$\mathbf{j}^{(h)}(\mathbf{r},t)=\frac{eN}{2m(2\pi T)^2}\,\frac{7\zeta(3)}{8}X_1(y)\left[\nabla'\frac{\partial}{\partial t}+\nabla\frac{\partial}{\partial t'}\right]\Delta(\mathbf{r}'\,t')\Delta^\dagger(\mathbf{r}\,t)|_{\mathbf{r}'=\mathbf{r},t'=t}, \tag{C12}$$

where $X(y)$ is given in (C3) and

$$X_1(y)=\frac{8}{7\zeta(3)}\sum_{n=0}^{\infty}\frac{1}{(2n+1)(2n+1+y)}\left[\frac{2}{2n+1}+\frac{1}{2n+1+y}\right]. \tag{C13}$$

In the presence of both magnetic field and electric field $\partial/\partial t$ and ∇ have to be replaced by the appropriate gauge-invariant generalized operators. The fact that $X(y)$ and $X_1(y)$ depend differently on y, shows clearly the inadequacy of the two-fluid approach.