"normal" phases, i.e., polymorphic transitions, etc. This could be done by varying the density of impurities in doubtful cases and studying changes in the transition temperature. In the same way, a careful study of antiferromagnetism in chromium alloys could help to decide on the applicability of the Lomer two-band model to chromium.[9,10] Due to the crudeness of the model in contrast to the real band structure in the latter case, quantitative estimates might be very difficult, however.

Though we have investigated the effect of impurities on the excitonic phase only in the semimetallic limit, it is clear that qualitatively the results of this paper should hold true also in the semiconductor region (positive or zero band gap in the underlying two-band model). The quantitative description would be different, however. There are two important differences from the former situation. First, the modifications of the single-particle energies due to impurities could not be neglected; in the semimetallic case we only have a small negligible shift of the Fermi energies. Secondly, the collision times $\tau_{a,b}$ will become energy- (or temperature-) dependent. It remains to be seen whether and how the Abrikosov-Gorkov theory has to be modified in order to deal with this situation.

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Dielectric Constant of a Dense Electron Gas Containing a Fixed Point Charge

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An expression for the dielectric constant of a dense electron gas containing a positive point charge Ze with a neutralizing positive background is obtained by employing the diagram technique of quantum field theory. The present derivation leads to some more terms in addition to those obtained in the self-consistent-field approximation. Besides, our derivation rigorously takes into account the Pauli exclusion principle. The simplest evaluation of the dielectric constant is made in the region where collective effect dominates, and the results are compared with those obtained in the self-consistent-field approximation.

1. INTRODUCTION

SINGLE-PARTICLE Green's functions $G(x, x')$ for a homogeneous system consisting of an interacting electron gas are widely used to obtain information regarding its ground-state properties and the nature of its elementary excitations. But most systems which one finds in nature are inhomogeneous. An inhomogeneity in a system arises from any external field acting on it. The type of inhomogeneity considered here is that due to a point charge Ze fixed inside an electron gas. This is of great physical interest for the study of the discrete single-particle excitation spectrum. Recently, Layzer[1] has investigated the quasiparticle excitation of such a system. He has shown that for a positive point charge, there exists a discrete spectrum of bound holes, finite in number, which disappears beyond a certain limiting value of the electron density. A similar investigation has been made by Sziklas[2] on the collective oscillations of a dense electron gas containing a fixed point charge. He finds two distinct types of collective excitations of this system. The first one, called a free plasmon, has the same excitation spectrum as found for the homogeneous system; and the other, called a bound plasmon, belongs to a discrete type of spectrum, and has no counterpart in the homogeneous gas. It exists only if the impurity charge is negative. Layzer's investigation of the quasiparticle excitations is based on the one-particle Green's function $G^{(w)}(x, x')$ for the nonuniform many-fermion systems. Besides Layzer, Sham and Kohn[3] have recently studied the inhomogeneous system consisting of an interacting electron gas using its one-particle Green's function. In this paper, we shall, however, use the one-particle Green's function $G^{(w)}(x, x')$ for the inhomogeneous system to derive an expression for its dielectric constant.

In Sec. 2, a brief review of the Green's-function approach to a many-fermion system is presented. A perturbation expansion is obtained for the one-particle Green's function of the inhomogeneous system in terms of the corresponding Green's function of the homogeneous case.

In Sec. 3, an expression for the dielectric constant of the inhomogeneous system is obtained by using the perturbation expansion of its one-particle Green's function. It is found that the expression for the dielectric

[1] A. J. Layzer, Phys. Rev. **129**, 897 (1963); **129**, 908 (1963).
[2] E. A. Sziklas, Phys. Rev. **138**, A1070 (1965).
[3] L. J. Sham and W. Kohn, Phys. Rev. **145**, 561 (1966).

constant contains some more terms in addition to those obtained by Sziklas[2] using the self-consistent-field (SCF) approximation of Ehrenreich and Cohen.[4] Besides, the momentum restrictions present in the various terms show strict use of the Pauli exclusion principle, which is clearly absent in the expression for the dielectric constant obtained by the SCF method. This shows the superiority of the Green's-function approach over the SCF method. We have evaluated the expression for the dielectric constant in the simplest case, that is, in the region where collective effects on an electron dominate. The results of our calculation are compared with those of the SCF method.

In Sec. 4, we briefly discuss the form of the dielectric constant obtained in this paper, and this is then compared with the result obtained by the SCF method. The details of the mathematical steps leading to the evaluation of the dielectric constant are given in the Appendix.

2. GENERAL FORMULATION

The Hamiltonian of the system consisting of an interacting electron gas at zero temperature, with a neutralizing positive background and a fixed positive point charge Ze, may be written (in units such that $\hbar=1$)

$$H=\int d\mathbf{x}\,\psi^\dagger(\mathbf{x},t)\left(-\frac{\nabla^2}{2m}\right)\psi(\mathbf{x},t)+\tfrac{1}{2}\int d\mathbf{x}\,d\mathbf{x}'\psi^\dagger(\mathbf{x},t)\psi^\dagger(\mathbf{x}',t)\times v(\mathbf{x}-\mathbf{x}')\psi(\mathbf{x}',t)\psi(\mathbf{x},t)+\int d\mathbf{x}\,\psi^\dagger(\mathbf{x},t)w(\mathbf{x})\psi(\mathbf{x},t),\quad(1)$$

where $\psi(x)$ [x stands for $\mathbf{x}$, t] is the second-quantized Heisenberg field operator for the electrons, whose equal-time anticommutator is

$$[\psi(\mathbf{x},t),\psi^\dagger(\mathbf{x}',t)]_+=\delta(\mathbf{x}-\mathbf{x}').\quad(2)$$

$v(\mathbf{x}-\mathbf{x}')$ is the interparticle potential, and $w(\mathbf{x})$ is the static external potential due to the fixed point charge. In this case, v and w have the following forms:

$$v(\mathbf{x}-\mathbf{x}')=e^2/|\mathbf{x}-\mathbf{x}'|,\quad(3)$$

$$w(\mathbf{x})=-Ze^2/|\mathbf{x}|.\quad(4)$$

The field operator $\psi(x)$ obeys the following equation of motion:

$$i\partial\psi(x)/\partial t=[\psi(x),H]=[-\nabla^2/2m+w(\mathbf{x})]\psi(\mathbf{x},t)+\int d\mathbf{x}'\psi^\dagger(\mathbf{x}',t)v(\mathbf{x}-\mathbf{x}')\psi(\mathbf{x}',t)\psi(\mathbf{x},t).\quad(5)$$

[4] H. Ehrenreich and M. H. Cohen, Phys. Rev. **115**, 786 (1959).

Define the one-particle Green's function for the inhomogeneous system by

$$G^{(w)}(x,x')=-i\langle\Psi\,|\,T\{\psi(x)\psi^\dagger(x')\}\,|\,\Psi\rangle,\quad(6)$$

where T denotes the Wick time-ordering operator, and $|\Psi\rangle$ is the ground state of the many-fermion system in the presence of the static external field. The superscript w on the Green's function refers to the external field.

To make a perturbation expansion of the Green's function $G^{(w)}(x,x')$ in terms of the Green's function $G(x,x')$ of the homogeneous system, we use the interaction representation defined by arbitrarily breaking up the Hamiltonian H into unperturbed and interacting parts H_0 and H_1;

$$H=H_0+H_1,\quad(7)$$

where H_0 corresponds to the Hamiltonian of the system in the absence of the external field and $H_1(t)$ is the interaction Hamiltonian, which is given by

$$H_1(t)=\int d\mathbf{x}\,\psi^\dagger(\mathbf{x},t)w(\mathbf{x})\psi(\mathbf{x},t).\quad(8)$$

In the interaction representation,[5] we obtain

$$G^{(w)}(x,x')=-i\langle\Psi_0\,|\,T\{\psi_I(x)\psi_I^\dagger(x')S\}\,|\,\Psi_0\rangle/\langle\Psi_0\,|\,S\,|\,\Psi_0\rangle,\quad(9)$$

where

$$\psi_I(x)=\exp(iH_0t)\psi(\mathbf{x})\exp(-iH_0t)$$

is the field operator in the interaction representation, and $|\Psi_0\rangle$ is the ground state of the system in the absence of the external field. The symbol S in (9) refers to the S matrix

$$S=T\exp\left[-i\int_{-\infty}^{\infty}dtH_1(t)\right].\quad(10)$$

If we disregard all the disconnected diagrams, the denominator in Eq. (9) will be suppressed.[6]

Assuming the external field to be a weak one, we can ignore terms beyond first order in the external field. To this approximation, we have from Eq. (9)

$$G^{(w)}(x,x')=-i\langle\Psi_0\,|\,T\{\psi_I(x)\psi_I^\dagger(x')\}\,|\,\Psi_0\rangle+(-i)^2\int d^4x_1w(\mathbf{x}_1)\times\langle\Psi_0\,|\,T\{\psi_I(x)\psi_I^\dagger(x')\psi_I^\dagger(x_1)\psi_I(x_1)\}\,|\,\Psi_0\rangle,\quad(11)$$

where the first term of the expansion in Eq. (11) denotes the one-particle Green's function for the homogeneous case. Since the homogeneous system is a

[5] M. Gell-Mann and F. Low, Phys. Rev. **84**, 350 (1951).
[6] A. A. Abrikosov, L. P. Gorkov, and I. E. Dzyaloshinski, *Method of Quantum Field Theory in Statistical Physics* (Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1963).

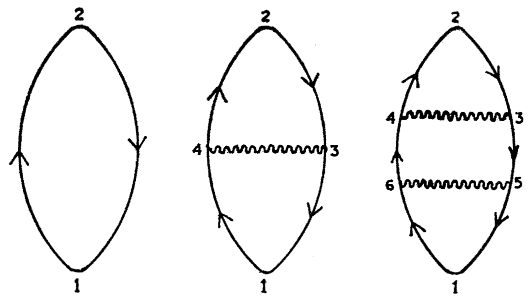

FIG. 1. Diagram representing the expansion of $Q(1, 2)$.

translationally invariant one, its one-particle Green's function is written as a function of the difference $x-x'$. Following the definition of the two-particle Green's function, we obtain from Eq. (11)

$$G^{(w)}(x, x') = G(x, x') - \int d^4x_1 w(\mathbf{x}_1) G_2(x, x_1; x', x_1), \quad (12)$$

where, in general,[7]

$$G_2(x_1, x_2; x'_1, x'_2) = (-i)^2 \langle | T\{\psi(x_1)\psi(x_2)\psi^\dagger(x'_2)\psi^\dagger(x'_1)\} | \rangle. \quad (13)$$

As has been shown by Gellmann and Low[5] and by Schwinger,[8] the following integral equation is satisfied by the two-particle Green's function:

$$G_2(1, 2; 1', 2') = G(1, 1')G(2, 2') - G(1, 2')G(2, 1') + \int d^4x_3 d^4x_4 d^4x_5 d^4x_6 G(1, 3)G(2, 4) \times W(3, 4; 5, 6) G_2(5, 6; 1', 2'), \quad (14)$$

where $1 \equiv \mathbf{x}_1 t_1$, etc. The symbol W in (14) represents the irreducible interaction operator. In the high-electron-density limit[9] and in the "shielded-interaction approximation,"[10] the two-particle Green's function G_2 can be written as

$$G_2(1, 2; 1', 2') = G(1, 1')G(2, 2') - G(1, 2')G(2, 1') + i\int d^4x_3 d^4x_4 G(1, 3)G(2, 4) \times V_S(3, 4) G(3, 1')G(4, 2'), \quad (15)$$

where V_S is the effective interaction between any two particles of the medium which takes into account all possible polarization processes of the medium. The integral equation[10] for V_S can be written as

$$V_S(1, 2) = U(1, 2) - \int d^4x_3 d^4x_4 V_S(1, 3) Q(3, 4) U(4, 2), \quad (16)$$

where

$$U(1, 2) = v(\mathbf{x}_1 - \mathbf{x}_2)\delta(t_1 - t_2). \quad (17)$$

Q in the above equation represents the polarization operator which includes the effect of all proper polarization diagrams; one can write the series expansion[7] for Q as

$$Q(1, 2) = iG(1, 2)G(2, 1) - \int d^4x_3 d^4x_4 G(1, 3)G(2, 4) \times V_S(3, 4) G(3, 2) G(4, 1) + \cdots. \quad (18)$$

The expansion for Q is represented by diagrams in Fig. 1. From Eq. (18) it follows that the lowest-order contribution to Q is

$$Q_0(1, 2) = iG(1, 2)G(2, 1). \quad (19)$$

In the presence of the external field one can write a similar type of expansion for $Q^{(w)}(1, 2)$, which in lowest order can be written as

$$Q_0^{(w)}(1, 2) = iG^{(w)}(1, 2) G^{(w)}(2, 1). \quad (20)$$

If we now substitute Eq. (19) in Eq. (16), we get the lowest-order contribution to V_S as

$$V_S(1, 2) = U(1, 2) - i\int d^4x_3 d^4x_4 V_S(1, 3) G(3, 4) G(4, 3) U(4, 2). \quad (21)$$

By using (15), we have from Eq. (12)

$$G^{(w)}(x, x') = G(x, x') + \int d^4x_1 w(\mathbf{x}_1) G(x, x_1) G(x_1, x') - i\int d^4x_1 d^4x_2 d^4x_3 w(\mathbf{x}_1) G(x, x_2) G(x_1, x_3) \times V_S(x_2, x_3) G(x_2, x') G(x_3, x_1), \quad (22)$$

where we have discarded the term corresponding to the disconnected diagram. If we represent the one-particle Green's function $G^{(w)}(x, x')$ by means of a

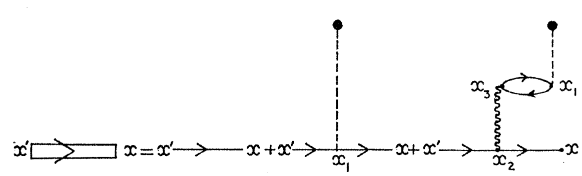

FIG. 2. Graphical representation of the perturbation expansion of $G^{(w)}(x, x')$.

[7] T. Kato, T. Kobayashi, and M. Namiki, Progr. Theoret. Phys. (Kyoto) Suppl. **15**, 3 (1960).
[8] J. Schwinger, Proc. Natl. Acad. Sci. (U.S.) **37**, 452 (1951).
[9] H. Kanazawa and M. Watabe, Progr. Theoret. Phys. (Kyoto) **23**, 408 (1960).
[10] G. Baym and L. P. Kadanoff, Phys. Rev. **124**, 287 (1961).

double line going from x' to x, the terms in Eq. (22) can be graphically represented by the diagrams in Fig. 2, where the single line represents the one-particle Green's function G for the homogeneous system, and the dashed line ending with shaded circle represents the external source of the potential. The third term in Eq. (22) corresponds to the screening of the external potential due to the static point charge, which has been represented by means of the third diagram in Fig. 2. By using Eq. (19), one gets from Eq. (22)

$$G^{(w)}(x,x')=G(x,x')+\int d^4x_1 w(\mathbf{x}_1)G(x,x_1)G(x_1,x')$$
$$-\int d^4x_1d^4x_2d^4x_3 w(\mathbf{x}_1)G(x,x_2)G(x_2,x')\times Q_0(x_3,x_1)V_S(x_2,x_3). \qquad (23)$$

3. DERIVATION OF THE DIELECTRIC CONSTANT

To work in the momentum representation, we define the Fourier transform of the Green's function $G^{(w)}(x,x')$ as

$$G^{(w)}(x,x')=\frac{1}{(2\pi)^7}\int d\mathbf{k}_1d\mathbf{k}_2\int_{-\infty}^{\infty}d\omega_0 G^{(w)}(\mathbf{k}_1,\mathbf{k}_2;\omega_0)\times\exp[i\mathbf{k}_1\cdot\mathbf{x}-i\mathbf{k}_2\cdot\mathbf{x}'-i\omega_0(t-t')]. \qquad (24)$$

In writing Eq. (24), one should note that in the case of a time-independent external field, $G^{(w)}(x,x')$ is a function only of the time difference $t-t'$ and is therefore diagonal in ω space, where ω is the energy variable congugate to time. This means one can write $G^{(w)}(x,x')$ as

$$G^{(w)}(x,x')=G^{(w)}(\mathbf{x},\mathbf{x}';t-t'). \qquad (25)$$

Since the system of electron gas is translationally invariant, one can write the Fourier transform of $G(x,x')$ as

$$G(x,x')=\frac{1}{(2\pi)^4}\int d\mathbf{k}\int_{-\infty}^{\infty}d\omega_0 G(\mathbf{k},\omega_0)\times\exp[i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')-i\omega_0(t-t')]. \qquad (26)$$

Defining the Fourier transforms of each of the functions appearing in Eq. (23) in a similar manner, we obtain

$$G^{(w)}(\mathbf{k},\mathbf{k}';\omega_0)=(2\pi)^3G(\mathbf{k},\omega_0)\delta(\mathbf{k}-\mathbf{k}')+G(\mathbf{k},\omega_0)w(\mathbf{k}-\mathbf{k}')G(\mathbf{k}',\omega_0)-G(\mathbf{k},\omega_0)w(\mathbf{k}-\mathbf{k}')G(\mathbf{k}',\omega_0)\times Q_0(\mathbf{k}-\mathbf{k}',0)V_S(\mathbf{k}-\mathbf{k}',0), \qquad (27)$$

where

$$w(\mathbf{k}-\mathbf{k}')=-4\pi Ze^2/(\mathbf{k}-\mathbf{k}')^2=-Zv(\mathbf{k}-\mathbf{k}'), \qquad (28)$$

and

$$V_S(\mathbf{k}-\mathbf{k}',0)=v(\mathbf{k}-\mathbf{k}')/[1+v(\mathbf{k}-\mathbf{k}')Q_0(\mathbf{k}-\mathbf{k}',0)]. \qquad (29)$$

Following the definition of the dielectric constant, which for a homogeneous system of electron gas is given by

$$\mathcal{E}(\mathbf{k},\omega)=v(\mathbf{k})/V_S(\mathbf{k},\omega), \qquad (30)$$

we obtain from Eq. (27)

$$G^{(w)}(\mathbf{k},\mathbf{k}';\omega_0)=(2\pi)^3G(\mathbf{k},\omega_0)\delta(\mathbf{k}-\mathbf{k}')+[G(\mathbf{k},\omega_0)G(\mathbf{k}',\omega_0)w(\mathbf{k}-\mathbf{k}')]/\mathcal{E}(\mathbf{k}-\mathbf{k}',0). \qquad (31)$$

To derive the expression for the dielectric constant for the inhomogeneous system, we take the Fourier transforms of the terms appearing in Eq. (20), and obtain

$$Q_0^{(w)}(\mathbf{q},\mathbf{q}';\omega_0)=\frac{2i}{(2\pi)^7}\int d\mathbf{k}d\mathbf{k}_1\int_{-\infty}^{\infty}d\epsilon G^{(w)}(\mathbf{k}+\mathbf{q},\mathbf{k}_1+\mathbf{q}';\epsilon+\omega_0)\times G^{(w)}(\mathbf{k}_1,\mathbf{k};\epsilon), \qquad (32)$$

where the factor of 2 in $Q_0^{(w)}$ comes from the spin summation. Using the expression for $G^{(w)}$ from Eq. (31), one obtains by retaining terms up to the first order in the external field:

$$Q_0^{(w)}(\mathbf{q},\mathbf{q}';\omega_0)=(2\pi)^3Q_0(\mathbf{q},\omega_0)\delta(\mathbf{q}-\mathbf{q}')-\frac{2Zi}{(2\pi)^4}\frac{v(\mathbf{q}-\mathbf{q}')}{\mathcal{E}(\mathbf{q}-\mathbf{q}',0)}\times\int d\mathbf{k}\int_{-\infty}^{\infty}d\epsilon[G(\mathbf{k}+\mathbf{q},\epsilon+\omega_0)G(\mathbf{k}+\mathbf{q}-\mathbf{q}',\epsilon)G(\mathbf{k},\epsilon)+G(\mathbf{k}+\mathbf{q},\epsilon+\omega_0)G(\mathbf{k}+\mathbf{q}',\epsilon+\omega_0)G(\mathbf{k},\epsilon)], \qquad (33)$$

where

$$Q_0(\mathbf{q},\omega_0)=\frac{2i}{(2\pi)^4}\int d\mathbf{k}\int_{-\infty}^{\infty}d\epsilon G(\mathbf{k}+\mathbf{q},\epsilon+\omega_0)G(\mathbf{k},\epsilon). \qquad (34)$$

It is important to note that the one-particle Green's functions appearing in Eqs. (33) and (34) satisfy the well-known Dyson[11] equation

$$G(\mathbf{p},\epsilon)=G_0(\mathbf{p},\epsilon)+G_0(\mathbf{p},\epsilon)\sum(\mathbf{p},\epsilon)G(\mathbf{p},\epsilon), \qquad (35)$$

where $\sum$ represents the sum of all proper self-energy diagrams and the G_0 is the unperturbed free-particle Green's function. In momentum space, G_0 is given by

$$G_0(\mathbf{p},\epsilon)=\frac{\theta(|\mathbf{p}|-k_F)}{\epsilon-\epsilon_p+i\delta}+\frac{\theta(k_F-|\mathbf{p}|)}{\epsilon-\epsilon_p-i\delta}, \qquad (36)$$

where

$$\epsilon_p=\mathbf{p}^2/2m,$$

[11] T. D. Schultz, *Quantum Field Theory and the Many-Body Problem* (Gordon and Breach Science Publishers, Inc., New York, 1964).

and k_F is the unperturbed Fermi momentum. Neglecting the self-energy contribution, we obtain from Eq. (33)

$$Q_{00}^{(w)}(\mathbf{q},\mathbf{q}';\omega_0)$$

$$=(2\pi)^3Q_{00}(\mathbf{q},\omega_0)\delta(\mathbf{q}-\mathbf{q}')-\frac{2Zi}{(2\pi)^4}\frac{v(\mathbf{q}-\mathbf{q}')}{\mathcal{E}_0(\mathbf{q}-\mathbf{q}',0)}$$

$$\times\int d\mathbf{k}\int_{-\infty}^{\infty}d\epsilon[G_0(\mathbf{k}+\mathbf{q},\epsilon+\omega_0)G_0(\mathbf{k}+\mathbf{q}-\mathbf{q}',\epsilon)G_0(\mathbf{k},\epsilon)+G_0(\mathbf{k}+\mathbf{q},\epsilon+\omega_0)G_0(\mathbf{k}+\mathbf{q}',\epsilon+\omega_0)G_0(\mathbf{k},\epsilon)], \quad (37)$$

where[12]

$$Q_{00}(\mathbf{q},\omega_0)=\frac{2i}{(2\pi)^4}\int d\mathbf{k}\int_{-\infty}^{\infty}d\epsilon G_0(\mathbf{k}+\mathbf{q},\epsilon+\omega_0)G_0(\mathbf{k},\epsilon)$$

$$=-\frac{2}{(2\pi)^3}\int_{|\mathbf{k}|<k_F,|\mathbf{k}+\mathbf{q}|>k_F}d\mathbf{k}[(\omega_0+\epsilon_k-\epsilon_{k+q}+i\delta)^{-1}-(\omega_0+\epsilon_{k+q}-\epsilon_k-i\delta)^{-1}]. \quad (38)$$

If we now break up $Q_{00}^{(w)}$ in the following manner:

$$Q_{00}^{(w)}(\mathbf{q},\mathbf{q}';\omega_0)=(2\pi)^3Q_{00}(\mathbf{q},\omega_0)\delta(\mathbf{q}-\mathbf{q}')+\eta(\mathbf{q},\mathbf{q}';\omega_0), \quad (39)$$

we get from Eq. (37), after performing the integration over ϵ,

$$\eta(\mathbf{q},\mathbf{q}';\omega_0)=-2Zv(\mathbf{q}-\mathbf{q}')/(2\pi)^3\mathcal{E}_0(\mathbf{q}-\mathbf{q}',0)$$

$$\times\int d\mathbf{k}\bigg[\frac{\theta(k_F-|\mathbf{k}|)\theta(|\mathbf{k}+\mathbf{q}|-k_F)}{\omega_0-\epsilon_{k+q}+\epsilon_k+i\delta}\left\{\frac{\theta(k_F-|\mathbf{k}+\mathbf{q}-\mathbf{q}'|)}{\omega_0-\epsilon_{k+q}+\epsilon_{k+q-q'}+i\delta}-\frac{\theta(|\mathbf{k}+\mathbf{q}'|-k_F)}{\omega_0-\epsilon_{k+q'}+\epsilon_k+i\delta}\right\}$$

$$+\frac{\theta(|\mathbf{k}|-k_F)\theta(k_F-|\mathbf{k}+\mathbf{q}'|)}{\omega_0-\epsilon_{k+q'}+\epsilon_k-i\delta}\left\{\frac{\theta(k_F-|\mathbf{k}+\mathbf{q}|)}{\omega_0-\epsilon_{k+q}+\epsilon_k-i\delta}-\frac{\theta(|\mathbf{k}+\mathbf{q}'-\mathbf{q}|-k_F)}{\omega_0-\epsilon_{k+q'}+\epsilon_{k+q'-q}-i\delta}\right\}$$

$$+\frac{\theta(k_F-|\mathbf{k}|)\theta(|\mathbf{k}+\mathbf{q}|-k_F)}{\omega_0-\epsilon_{k+q}+\epsilon_k+i\delta}\left\{\frac{\theta(k_F-|\mathbf{k}+\mathbf{q}'|)}{\epsilon_{k+q'}-\epsilon_{k+q}+i\delta}-\frac{\theta(|\mathbf{k}+\mathbf{q}-\mathbf{q}'|-k_F)}{\epsilon_k-\epsilon_{k+q-q'}+i\delta}\right\}$$

$$+\frac{\theta(k_F-|\mathbf{k}|)\theta(|\mathbf{k}+\mathbf{q}'|-k_F)}{\omega_0-\epsilon_{k+q'}+\epsilon_k+i\delta}\left\{\frac{\theta(k_F-|\mathbf{k}+\mathbf{q}|)}{\epsilon_{k+q}-\epsilon_{k+q'}+i\delta}-\frac{\theta(|\mathbf{k}+\mathbf{q}'-\mathbf{q}|-k_F)}{\epsilon_k-\epsilon_{k+q'-q}+i\delta}\right\}$$

$$+\frac{\theta(|\mathbf{k}|-k_F)\theta(k_F-|\mathbf{k}+\mathbf{q}|)}{\omega_0-\epsilon_{k+q}+\epsilon_k-i\delta}\left\{\frac{\theta(k_F-|\mathbf{k}+\mathbf{q}-\mathbf{q}'|)}{\epsilon_k-\epsilon_{k+q-q'}-i\delta}-\frac{\theta(|\mathbf{k}+\mathbf{q}'|-k_F)}{\epsilon_{k+q'}-\epsilon_{k+q}-i\delta}\right\}$$

$$+\frac{\theta(|\mathbf{k}|-k_F)\theta(k_F-|\mathbf{k}+\mathbf{q}'|)}{\omega_0-\epsilon_{k+q'}+\epsilon_k-i\delta}\left\{\frac{\theta(k_F-|\mathbf{k}+\mathbf{q}'-\mathbf{q}|)}{\epsilon_k-\epsilon_{k+q'-q}-i\delta}-\frac{\theta(|\mathbf{k}+\mathbf{q}|-k_F)}{\epsilon_{k+q}-\epsilon_{k+q'}-i\delta}\right\}\bigg]. \quad (40)$$

Since, for an inhomogeneous system of interacting electron gas, the dielectric constant is defined by the relation[2]

$$\mathcal{E}_0(\mathbf{q},\mathbf{q}';\omega_0)=(2\pi)^3\delta(\mathbf{q}-\mathbf{q}')+v(\mathbf{q})Q_{00}^{(w)}(\mathbf{q},\mathbf{q}';\omega_0), \quad (41)$$

one obtains, with the help of Eq. (39),

$$\mathcal{E}_0(\mathbf{q},\mathbf{q}';\omega_0)=(2\pi)^3\delta(\mathbf{q}-\mathbf{q}')[1+v(\mathbf{q})Q_{00}(\mathbf{q},\omega_0)]+v(\mathbf{q})\eta(\mathbf{q},\mathbf{q}';\omega_0)$$

$$=(2\pi)^3\delta(\mathbf{q}-\mathbf{q}')\mathcal{E}_0(\mathbf{q},\omega_0)+v(\mathbf{q})\eta(\mathbf{q},\mathbf{q}';\omega_0), \quad (42)$$

where $\mathcal{E}_0(\mathbf{q},\omega_0)$ is the dielectric constant of the homogeneous system in the random-phase approximation (RPA). The second term in Eq. (42) is the contribution due to the inhomogeneity present in the homogeneous system. This corresponds to the nondiagonal component of $\mathcal{E}_0(\mathbf{q},\mathbf{q}';\omega_0)$, whereas the diagonal part is represented by the first term in Eq. (42).

[12] D. F. Dubois, Ann. Phys. (N.Y.) 7, 174 (1959).

4. EVALUATION OF THE DIELECTRIC CONSTANT

The evaluation of the integrals appearing in Eq. (40) is extremely difficult. As the simplest case, we will restrict ourselves to the region where the collective oscillations dominate. This is the region which is defined by ω_0 greater than both $\pm(\epsilon_{k+q}-\epsilon_k)$ and $\pm(\epsilon_{k+q'}-\epsilon_k)$. In this region, we will be interested in the limit $\mathbf{q}$, $\mathbf{q}'\to 0$.

To proceed, let us now make the transformation $\mathbf{k}+\mathbf{q}\to-\mathbf{k}$ in the first, sixth, and ninth terms of Eq. (40). Similarly, we will be making the transformation $\mathbf{k}+\mathbf{q}'\to-\mathbf{k}$ in the fourth, eighth, and eleventh terms of the same equation. Thus, we get

$$\begin{aligned}\eta_1(\mathbf{q},\mathbf{q}';\omega_0)&=-2Zv(\mathbf{q}-\mathbf{q}')/(2\pi)^3\mathcal{E}_0(\mathbf{q}-\mathbf{q}',0)\\&\times\int d\mathbf{k}\Bigg\{\left[\frac{1}{(\omega_0-\epsilon_k+\epsilon_{k+q})(\omega_0-\epsilon_k+\epsilon_{k+q'})}+\frac{1}{(\omega_0-\epsilon_{k+q}+\epsilon_k)(\omega_0-\epsilon_{k+q'}+\epsilon_k)}\right]\\&\times[\theta(|\mathbf{k}|-k_F)\theta(k_F-|\mathbf{k}+\mathbf{q}|)\theta(k_F-|\mathbf{k}+\mathbf{q}'|)-\theta(k_F-|\mathbf{k}|)\theta(|\mathbf{k}+\mathbf{q}|-k_F)\theta(|\mathbf{k}+\mathbf{q}'|-k_F)]\\&+\left[\frac{1}{(\omega_0-\epsilon_{k+q}+\epsilon_k)}-\frac{1}{(\omega_0-\epsilon_k+\epsilon_{k+q})}\right]P\frac{1}{(\epsilon_{k+q'}-\epsilon_{k+q})}\\&\times[\theta(k_F-|\mathbf{k}|)\theta(|\mathbf{k}+\mathbf{q}|-k_F)\theta(k_F-|\mathbf{k}+\mathbf{q}'|)-\theta(|\mathbf{k}|-k_F)\theta(k_F-|\mathbf{k}+\mathbf{q}|)\theta(|\mathbf{k}+\mathbf{q}'|-k_F)]\\&+\left[\frac{1}{(\omega_0-\epsilon_{k+q'}+\epsilon_k)}-\frac{1}{(\omega_0-\epsilon_k+\epsilon_{k+q'})}\right]P\frac{1}{(\epsilon_{k+q}-\epsilon_{k+q'})}\\&\times\left[\theta(k_F-|\mathbf{k}|)\theta(|\mathbf{k}+\mathbf{q}'|-k_F)\theta(k_F-|\mathbf{k}+\mathbf{q}|)-\theta(|\mathbf{k}|-k_F)\theta(k_F-|\mathbf{k}+\mathbf{q}'|)\theta(|\mathbf{k}+\mathbf{q}|-k_F)\right]\Bigg\},\end{aligned} \tag{43}$$

$$\eta_2(\mathbf{q},\mathbf{q}';\omega_0)=0. \tag{44}$$

Here the symbol P denotes the principal value and η_1 and η_2 represent the real and imaginary parts of η. The fact that the imaginary part η_2 vanishes can be shown by utilizing the property of the δ function. From a detailed analysis as given in the Appendix, one can see that for the region $\frac{1}{2}\pi\leq\alpha\leq\pi$

$$\eta_1(\mathbf{q},\mathbf{q}';\omega_0)\underset{\mathbf{q},\mathbf{q}'\to 0}{=}-\frac{2Zv(\mathbf{q}-\mathbf{q}')}{(2\pi)^3\mathcal{E}_0(\mathbf{q}-\mathbf{q}',0)}\frac{2\pi q^2k_F\cos\alpha}{\omega_0^2}\left[1+\frac{\sin\alpha}{\pi}-\sin\alpha(1-2\cos\alpha)^{1/2}-\sin\alpha\cos\alpha\right]. \tag{45}$$

In the region $0\leq\alpha\leq\frac{1}{2}\pi$, the integral involved in the expression for $\eta_1(\mathbf{q},\mathbf{q}';\omega_0)$ is complicated in structure, and can not be evaluated analytically. In this region, $\eta_1(\mathbf{q},\mathbf{q}';\omega_0)$ is found to be split into two parts, one for the range $0\leq\alpha\leq\frac{1}{3}\pi$ and the other for the range $\frac{1}{3}\pi\leq\alpha\leq\frac{1}{2}\pi$. However, we will be interested in the value of $\eta_1(\mathbf{q},\mathbf{q}';\omega_0)$ for the region $\frac{1}{2}\pi\leq\alpha\leq\pi$. By means of a simple analysis it can be seen that the sum of the contributions from the last three terms in Eq. (45) is small compared to unity. Hence, one can roughly assume the first term to be the leading one, even if there is no justification for neglecting the rest. Denoting the contribution due to the first term in Eq. (45) by $\eta_1^{(1)}$, we have

$$\eta_1^{(1)}(\mathbf{q},\mathbf{q}';\omega_0)\underset{\mathbf{q},\mathbf{q}'\to 0}{=}\left[-2Z\frac{v(\mathbf{q}-\mathbf{q}')}{(2\pi)^3\mathcal{E}_0(\mathbf{q}-\mathbf{q}',0)}\frac{2\pi q^2k_F\cos\alpha}{\omega_0^2}\right], \tag{46}$$

where

$$\cos\alpha=(\mathbf{q}\cdot\mathbf{q}'/q^2).$$

Using the expression for the dielectric constant for the homogeneous system which, in the long-wavelength limit, is given by

$$\mathcal{E}_0(\mathbf{q},0)\underset{\mathbf{q}\to 0}{=}[1+k_{FT}^2/q^2], \tag{47}$$

one can write $\eta_1^{(1)}$ as

$$\eta_1^{(1)}(\mathbf{q},\mathbf{q}';\omega_0)\underset{\mathbf{q},\mathbf{q}'\to 0}{=}\left\{-\frac{\mathbf{q}\cdot\mathbf{q}'}{2m\omega_0^2}Z[1-\{\mathcal{E}_0(\mathbf{q}-\mathbf{q}',0)\}^{-1}]\right\}. \tag{48}$$

Following a similar analysis, it can be shown from Eq. (38) that in the long-wavelength limit we have

$$Q_{00}(\mathbf{q},\omega_0)\underset{\mathbf{q}\to 0}{=}\left\{-\frac{q^2n}{m\omega_0^2}\left[1+\frac{3q^2v_F^2}{5\omega_0^2}+\cdots\right]\right\}, \tag{49}$$

where $n=k_F^3/3\pi^2$ denotes the average electron density in the system of electron gas.

5. DISCUSSION

If one compares our expression for the dielectric constant for the inhomogeneous case with that obtained in the SCF approximation, one finds no discrepancy as far as the diagonal part is concerned. This is because the momentum restriction $|\mathbf{k}+\mathbf{q}|>k_F$ in the real part of $Q_{00}(\mathbf{q},\omega_0)$ occurring in Eq. (38) can be dropped, on the ground that if one violates this restriction in the first integral, the resulting contribution is exactly cancelled by that obtained on violating the restriction in the second integral. But this is not so in the case of the nondiagonal part $\eta(\mathbf{q},\mathbf{q}';\omega_0)$. The expression for the nondiagonal part as obtained by us contains more terms than the corresponding expression in the SCF method. Besides, the very presence of the extra momentum restrictions in the integrals of the nondiagonal part, which are evident in the θ functions, means that the present derivation strictly takes into account the exclusion principle. On evaluating the integrals occuring in Eq. (43) in the limit $\mathbf{q},\mathbf{q}'\to 0$, and under the approximation stated in (A10), one finds that in the region $\frac{1}{2}\pi\leq\alpha\leq\pi$, the first term of our expression for the real part of $\eta(\mathbf{q},\mathbf{q}';\omega_0)$ has the same form as the one evaluated by the SCF method, even though an extra factor of $\frac{1}{2}$ appears in our case. This justifies the validity of the approximation (A10), employed throughout our calculation. It is interesting to note that in the present case the imaginary part of the dielectric constant $\mathcal{E}_0(\mathbf{q},\mathbf{q}';\omega_0)$ vanishes in the region where the plasma oscillations dominate. The vanishing of the imaginary part of the dielectric constant for the homogeneous case is a well-known fact for small values of $\mathbf{q}$, i.e., where the plasmons are well-defined excitations of the system. By analogy, one should expect the same thing to happen to the imaginary part of the dielectric constant for the inhomogeneous case in the limit of small $\mathbf{q}$, $\mathbf{q}'$, which is manifested in our calculations.

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APPENDIX

We wish to evaluate the integrals in Eq. (43) in the limit $\mathbf{q},\mathbf{q}'\to 0$. Let us now make the binomial expansion of the term $(\omega_0\mp\epsilon_k\pm\epsilon_{k+q})^{-1}$ as follows:

$$(\omega_0\mp\epsilon_k\pm\epsilon_{k+q})^{-1}=[\omega_0\pm(\mathbf{k}\cdot\mathbf{q}/m)\pm(q^2/2m)]^{-1}$$
$$\approx\frac{1}{\omega_0^2}\left\{1\mp(\mathbf{k}\cdot\mathbf{q}/m\omega_0)\mp(q^2/2m\omega_0)\,{}^{+}_{+}\,(\mathbf{k}\cdot\mathbf{q}/m\omega_0)^2+\cdots\right\}. \qquad \text{(A1)}$$

Writing the binomial expansion of $(\omega_0\mp\epsilon_k\pm\epsilon_{k+q'})^{-1}$ in a similar fashion and retaining terms up to orders of q^2, q'^2 and qq', Eq. (43) can be written as

$$\eta_1(\mathbf{q},\mathbf{q}';\omega_0)=-\frac{2Zv(\mathbf{q}-\mathbf{q}')}{(2\pi)^3\mathcal{E}_0(\mathbf{q}-\mathbf{q}',0)}[\mathrm{I}+\mathrm{II}+\mathrm{III}], \qquad \text{(A2a)}$$

where

$$\mathrm{I}=\int d\mathbf{k}\left\{\frac{2}{\omega_0^2}[1+(\mathbf{k}\cdot\mathbf{q}/m\omega_0)^2+(\mathbf{k}\cdot\mathbf{q}'/m\omega_0)^2+((\mathbf{k}\cdot\mathbf{q})(\mathbf{k}\cdot\mathbf{q}')/m^2\omega_0^2)]\right.$$
$$\left.\times[\theta(|\mathbf{k}|-k_F)\theta(k_F-|\mathbf{k}+\mathbf{q}|)\theta(k_F-|\mathbf{k}+\mathbf{q}'|)-\theta(k_F-|\mathbf{k}|)\theta(|\mathbf{k}+\mathbf{q}|-k_F)\theta(|\mathbf{k}+\mathbf{q}'|-k_F)]\right\}, \qquad \text{(A2b)}$$

$$\mathrm{II}=\int d\mathbf{k}\left\{\frac{2}{\omega_0}[(\mathbf{k}\cdot\mathbf{q}/m\omega_0)+(q^2/2m\omega_0)]P(\epsilon_{k+q'}-\epsilon_{k+q})^{-1}\right.$$
$$\left.\times[\theta(k_F-|\mathbf{k}|)\theta(|\mathbf{k}+\mathbf{q}|-k_F)\theta(k_F-|\mathbf{k}+\mathbf{q}'|)-\theta(|\mathbf{k}|-k_F)\theta(k_F-|\mathbf{k}+\mathbf{q}|)\theta(|\mathbf{k}+\mathbf{q}'|-k_F)]\right\}, \qquad \text{(A2c)}$$

$$\mathrm{III}=\int d\mathbf{k}\left\{\frac{2}{\omega_0}[(\mathbf{k}\cdot\mathbf{q}'/m\omega_0)+(q'^2/2m\omega_0)]P(\epsilon_{k+q}-\epsilon_{k+q'})^{-1}\right.$$
$$\left.\times[\theta(k_F-|\mathbf{k}|)\theta(|\mathbf{k}+\mathbf{q}'|-k_F)\theta(k_F-|\mathbf{k}+\mathbf{q}|)-\theta(|\mathbf{k}|-k_F)\theta(k_F-|\mathbf{k}+\mathbf{q}'|)\theta(|\mathbf{k}+\mathbf{q}|-k_F)]\right\}. \qquad \text{(A2d)}$$

Let us note that III can be obtained from II by simply replacing $\mathbf{q}$ by $\mathbf{q}'$.

By using the property of the step function $\theta(k_F-|\mathbf{k}+\mathbf{q}|)$ appearing in (A2), one gets

$$k<\{[k_F^2-q^2\sin^2\theta]^{1/2}-q\cos\theta\},$$

where

$$\cos\theta=\mathbf{k}\cdot\mathbf{q}/kq. \tag{A3}$$

By virtue of the step function $\theta(|\mathbf{k}|-k_F)$, one finds from (A3) that in the limit $\mathbf{q}\to 0$, one should have

$$\cos\theta\leq 0,\quad \text{i.e.,}\quad \tfrac{1}{2}\pi\leq\theta\leq\pi. \tag{A4}$$

Making the following binomial expansion of the term $(k_F^2-q^2\sin^2\theta)^{1/2}$ occurring in the right-hand side of Eq. (A3):

$$(k_F^2-q^2\sin^2\theta)^{1/2}\approx k_F-c_0\sin^2\theta, \tag{A5}$$

one obtains

$$\theta(k_F-|\mathbf{k}+\mathbf{q}|)\approx\theta(k_F-|\mathbf{k}|-q\cos\theta-c_0\sin^2\theta), \tag{A6}$$

where c_0 is given by $c_0=q^2/2k_F$.

Similarly, one can show that

$$\theta(|\mathbf{k}+\mathbf{q}|-k_F)\approx\theta(|\mathbf{k}|-k_F+q\cos\theta+c_0\sin^2\theta), \tag{A7}$$

where in this case we will have $\cos\theta\geq 0$, i.e., $0\leq\theta\leq\frac{1}{2}\pi$. In the limit of small $\mathbf{q}$, we can now make Taylor expansions of the step functions as follows[12]:

$$\theta(k_F-|\mathbf{k}|-q\cos\theta-c_0\sin^2\theta)\approx\theta(k_F-|\mathbf{k}|)-(q\cos\theta+c_0\sin^2\theta)\delta(|\mathbf{k}|-k_F)-(q^2/2!)\cos^2\theta\delta'(|\mathbf{k}|-k_F)+O(q^3), \tag{A8}$$

and

$$\theta(|\mathbf{k}|-k_F+q\cos\theta+c_0\sin^2\theta)\approx\theta(|\mathbf{k}|-k_F)+(q\cos\theta+c_0\sin^2\theta)\delta(|\mathbf{k}|-k_F)+(q^2\cos^2\theta/2!)\delta'(|\mathbf{k}|-k_F)+O(q^3). \tag{A9}$$

Since one can also evaluate the integrals in I and II by making Taylor expansions of $\theta(k_F-|\mathbf{k}+\mathbf{q}'|)$ and $\theta(|\mathbf{k}+\mathbf{q}'|-k_F)$, the results of the latter calculations are likely to differ from those of the previous ones. In order to avoid the choice between the two approaches, we will have to assume that in the limit $\mathbf{q}, \mathbf{q}'\to 0$,

$$|\mathbf{q}'|\sim|\mathbf{q}|. \tag{A10}$$

In this approximation, the integrals of III in Eq. (A2d) will be exactly identical with those of II in Eq. (A2c). Keeping terms up to order q^2 and ω_0^{-2}, we obtain from Eq. (A2a), with the help of the Eqs. (A8), (A9), and (A10)'

$$\begin{aligned}\eta_1(\mathbf{q},\mathbf{q}';\omega_0)=&-2Zv(\mathbf{q}-\mathbf{q}')/(2\pi)^3\mathcal{E}_0(\mathbf{q}-\mathbf{q}',0)\\ &\times\int d\mathbf{k}\bigg\{-\frac{2}{\omega_0^2}[\theta(|\mathbf{k}|-k_F)\theta(k_F-|\mathbf{k}+\mathbf{q}'|)\theta(-\mathbf{k}\cdot\mathbf{q}/kq)+\theta(k_F-|\mathbf{k}|)\theta(|\mathbf{k}+\mathbf{q}'|-k_F)\theta(\mathbf{k}\cdot\mathbf{q}/kq)]\\ &\times[(q\cos\theta+c_0\sin^2\theta)\delta(|\mathbf{k}|-k_F)+(q^2\cos^2\theta/2!)\delta'(|\mathbf{k}|-k_F)]+(4/\omega_0^2)P(\cos\theta'-\cos\theta)^{-1}\\ &\times[\theta(k_F-|\mathbf{k}|)\theta(k_F-|\mathbf{k}+\mathbf{q}'|)\theta(\mathbf{k}\cdot\mathbf{q}/kq)+\theta(|\mathbf{k}|-k_F)\theta(|\mathbf{k}+\mathbf{q}'|-k_F)\theta(-\mathbf{k}\cdot\mathbf{q}/kq)]\\ &\times[(q\cos^2\theta+c_0\cos\theta\sin^2\theta+c_0\cos\theta)\delta(|\mathbf{k}|-k_F)+(q^2/2!)\cos^3\theta\delta'(|\mathbf{k}|-k_F)]\bigg\}.\end{aligned} \tag{A11}$$

To evaluate the integrals in (A11) let us choose the polar coordinates with the Z axis along $\mathbf{q}$. If we denote by θ and θ' the angles between the vectors $\mathbf{k}$ and $\mathbf{q}$, $\mathbf{q}'$, respectively, and define α to be the angle between $\mathbf{q}$ and $\mathbf{q}'$, we can write $\cos\theta'$ as

$$\cos\theta'=\cos\theta\cos\alpha+\sin\theta\sin\alpha\cos\phi, \tag{A12}$$

where ϕ is the angle between the planes $(\mathbf{k},\mathbf{q})$ and $(\mathbf{q},\mathbf{q}')$. We write η_1 as

$$\eta_1(\mathbf{q},\mathbf{q}';\omega_0)=-2Z[v(\mathbf{q}-\mathbf{q}')/(2\pi)^3\mathcal{E}_0(\mathbf{q}-\mathbf{q}',0)]J, \tag{A13}$$

where

$$J=\text{I}+\text{II}+\text{III}.$$

With the help of the θ functions one can obtain from Eq. (A11) for the region $0\leq\alpha\leq\frac{1}{2}\pi$,

$$J=\Big\{-\frac{4\pi}{\omega_0^2}\Big[\int_{\pi/2+\alpha}^{\pi}d\theta\,\sin\theta\int_{k_F}^{\gamma(k_F,q,\theta')}dk\,k^2+\int_0^{\pi/2-\alpha}d\theta\,\sin\theta\int_{\gamma(k_F,q,\theta')}^{k_F}dk\,k^2\Big]$$
$$\times[(q\cos\theta+c_0\sin^2\theta)\delta(|\mathbf{k}|-k_F)+(q^2\cos^2\theta/2!)\delta'(|\mathbf{k}|-k_F)]$$
$$+\frac{4}{\omega_0^2}\Big[\int_0^{k_F}dk\,k^2\int_{\pi/2-\alpha}^{\pi/2}d\theta\,\sin\theta\int_{\pi/2+\mu(\theta)}^{3\pi/2-\mu(\theta)}d\phi(\cos\theta'-\cos\theta)^{-1}$$
$$+\int_{k_F}^{\infty}dk\,k^2\int_{\pi/2}^{\pi/2+\alpha}d\theta\,\sin\theta\Big\{\int_0^{\pi/2-\nu(\theta)}d\phi+\int_{3\pi/2+\nu(\theta)}^{2\pi}d\phi\Big\}(\cos\theta'-\cos\theta)^{-1}\Big]$$
$$\times[(q\cos^2\theta+c_0\cos\theta\sin^2\theta+c_0\cos\theta)\delta(|\mathbf{k}|-k_F)+(q^2\cos^3\theta/2!)\delta'(|\mathbf{k}|-k_F)]\Big\},\quad\text{(A14)}$$

and for the region $\frac{1}{2}\pi\leq\alpha\leq\pi$,

$$J=\Big\{-\frac{2}{\omega_0^2}\Big[\int_{\pi/2}^{3\pi/2-\alpha}d\theta\,\sin\theta\int_{\pi/2+\mu(\theta)}^{3\pi/2-\mu(\theta)}d\phi\int_{k_F}^{\gamma(k_F,q,\theta')}dk\,k^2+\int_{\alpha-\pi/2}^{\pi/2}d\theta\,\sin\theta\Big\{\int_0^{\pi/2-\nu(\theta)}d\phi+\int_{3\pi/2+\nu(\theta)}^{2\pi}d\phi\Big\}\int_{\gamma(k_F,q,\theta')}^{k_F}dk\,k^2\Big]$$
$$\times[(q\cos\theta+c_0\sin^2\theta)\delta(|\mathbf{k}|-k_F)+(q^2\cos^2\theta/2!)\delta'(|\mathbf{k}|-k_F)]$$
$$+\frac{4}{\omega_0^2}\Big[\int_0^{k_F}dk\,k^2\int_0^{\alpha-\pi/2}d\theta\,\sin\theta\int_0^{2\pi}d\phi(\cos\theta'-\cos\theta)^{-1}+\int_{k_F}^{\infty}dk\,k^2\int_{3\pi/2-\alpha}^{\pi}d\theta\,\sin\theta\int_0^{2\pi}d\phi(\cos\theta'-\cos\theta)^{-1}\Big]$$
$$\times[(q\cos^2\theta+c_0\cos\theta\sin^2\theta+c_0\cos\theta)\delta(|\mathbf{k}|-k_F)+(q^2\cos^3\theta/2!)\delta'(|\mathbf{k}|-k_F)]\Big\},\quad\text{(A15)}$$

where

$$\cos\alpha=\mathbf{q}\cdot\mathbf{q}'/qq'=\mathbf{q}\cdot\mathbf{q}'/q^2,$$
$$\gamma(k_F,q,\theta')=\{[k_F^2-q^2\sin^2\theta']^{1/2}-q\cos\theta'\},$$
$$\mu(\theta)=\sin^{-1}(\cos\theta\cos\alpha/\sin\theta\sin\alpha),$$
$$\nu(\theta)=\sin^{-1}(-\cos\theta\cos\alpha/\sin\theta\sin\alpha).\quad\text{(A16)}$$

After performing the integrations involved in Eq. (A14), one obtains for the region $0\leq\alpha\leq\frac{1}{2}\pi$,

$$J=(2\pi k_Fq^2/\omega_0^2)\sin\alpha\cos^2\alpha+\Big(-\frac{4q^2k_F}{\omega_0^2}\Big)\int_{\pi/2-\alpha}^{\pi/2}d\theta\frac{\sin\theta\cos\theta}{(b^2-a^2)^{1/2}}(2-3\cos^2\theta)\ln\left|\frac{(b+a)\beta+(b^2-a^2)^{1/2}}{(b+a)\beta-(b^2-a^2)^{1/2}}\right|,\quad\text{for}\quad 0\leq\alpha\leq\tfrac{1}{3}\pi$$
$$=(2\pi k_Fq^2/\omega_0^2)\sin\alpha\cos^2\alpha+\Big(-\frac{4q^2k_F}{\omega_0^2}\Big)\int_{\alpha/2}^{\pi/2}d\theta\frac{\sin\theta\cos\theta(2-3\cos^2\theta)}{(b^2-a^2)^{1/2}}\ln\left|\frac{(b+a)\beta+(b^2-a^2)^{1/2}}{(b+a)\beta-(b^2-a^2)^{1/2}}\right|,\quad\text{for}\quad \tfrac{1}{3}\pi\leq\alpha\leq\tfrac{1}{2}\pi.\quad\text{(A17)}$$

β, a, and b used in Eq. (A17) are given by

$$\beta=-[\sin^2\alpha-\cos^2\theta]^{1/2}/\cos(\theta+\alpha),$$
$$a=\cos\theta(1-\cos\alpha),$$
$$b=\sin\theta\sin\alpha.\quad\text{(A18)}$$

Similarly, for the region $\frac{1}{2}\pi\leq\alpha\leq\pi$, one obtains from Eq. (A15)

$$J=(1/\omega_0^2)[2\pi q^2k_F\cos\alpha+2q^2k_F\sin\alpha\cos\alpha-2\pi q^2k_F\cos\alpha\sin\alpha(1-2\cos\alpha)^{1/2}-2\pi q^2k_F\sin\alpha\cos^2\alpha].\quad\text{(A19)}$$