Intrinsic Resistive Transition in Narrow Superconducting Channels*

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We present a theory of current-reducing fluctuations in narrow superconducting channels. The theory is based on the Ginzburg-Landau equation, and is constructed in analogy with the droplet model of the condensation of a supersaturated vapor. Our theory suggests that large and improbable fluctuations are important, and cause measurable departures from the mean-field critical current. The calculation leads to a concrete result for the intrinsic resistive transition. This transition is predicted to occur at a temperature slightly lower than the bulk transition temperature, and to have a width which is observable but smaller than those measured in recent experiments.

1. INTRODUCTION

A SUPERCONDUCTING channel can be driven into a resistive state by applying a suitably large current. Recent experiments[1,2] on strips of very small cross section have suggested measurable departures from the simplest mean-field theory of the maximum supercurrent.[3] These experiments also suggest that the resistive transition has a finite width, even under ideal conditions. It seems clear that fluctuations of some kind must be responsible for these phenomena. There is a well-known procedure for analyzing fluctuations in statistical systems, and our purpose in what follows is to bring this procedure to bear on the problem of critical supercurrents.

Basically, we assert that any state of nonzero supercurrent is metastable in the sense that there is a topologically accessible fluctuation which leads to a state of lower current and, therefore, lower free energy. This conception of metastability owes much to the droplet model of the condensation of a supersaturated vapor.[4] In the droplet model, the metastable vapor phase persists until, because of a very unlikely statistical fluctuation, a liquid droplet large enough to nucleate the condensation occurs. A simple calculation of the free-energy barrier opposing the formation of the critical droplet provides a rough understanding of experimental evidence. Very recently, Langer and Fisher proposed an analog of the droplet model for calculating the rate of current-decreasing fluctuations in superfluid helium very near the λ point.[5] The following analysis for the superconductor has been developed in parallel with the work on liquid helium.

Two main ideas underlie our calculation. The first gives meaning to the (only superficially paradoxical) concept of the intrinsic resistance of a superconductor, and also justifies the use of a simple model. In essence, the idea is as follows. Consider two points in a piece of superconductor. A voltage can be maintained between them only if the corresponding difference in the phase of the order parameter increases steadily with time.[6] In general, this implies a continually increasing current. A steady current is possible, however, if fluctuations in the interior of the superconductor reduce the phase difference at the same average rate as the voltage increases it. We calculate the resistance by equating these two rates.

One important consequence of this method of calculation is, as will be shown, that observable resistivities are produced by fluctuations whose free energies are an order of magnitude larger than the characteristic thermal energy, k_BT. A knowledge of the detailed form of the fluctuations is not needed for this conclusion which shows that, just as in the droplet model, the relevant fluctuations are extremely improbable and play no role in determining the bulk thermodynamic properties of the system. Because of their large energies, these fluctuations have a spatial extent much larger than the range of nonlocality associated with the paired state of the superconductor. They, therefore, should be adequately described by a phenomenological mean-field theory, i.e., the Ginzburg-Landau equation.

Our second idea permits a detailed study of the fluctuations discussed above. We assume that the relevant states of the system can be described by a complex-valued order parameter $\psi(\mathbf{r})$, and that the *a priori* probability that the system will be found in the state described by $\psi(\mathbf{r})$ is proportional to the Boltzmann factor, $\exp(-F/k_BT)$, where F, the Ginzburg-Landau free energy, is a functional of $\psi(\mathbf{r})$. The statistical fluctuations of the system, caused by interactions with a constant-temperature bath, may be visualized as a continuous random motion of the system point ψ in a function space of continuous functions $\psi(\mathbf{r})$ satisfying

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[1] R. D. Parks and R. P. Groff, Phys. Rev. Letters **18**, 342 (1967).
[2] T. K. Hunt and J. E. Mercereau, Phys. Rev. Letters **18**, 551 (1967).
[3] J. Bardeen, Rev. Mod. Phys. **34**, 667 (1962).
[4] J. Frenkel, *Kinetic Theory of Liquids* (Dover Publications, Inc., New York, 1955), Chap. 7.
[5] J. S. Langer and M. E. Fisher, Phys. Rev. Letters **19**, 560 (1967).
[6] B. D. Josephson, Advan. Phys. **14**, 419 (1965); P. W. Anderson, Rev. Mod. Phys. **38**, 298 (1966).

appropriate boundary conditions.[7] The neighborhood of each point in this function space is visited with a frequency proportional to the stated Boltzmann factor; thus the various stable and metastable (current carrying) states of the superconductor must correspond to local minima of the functional $F\{\psi(\mathbf{r})\}$. That is, these states are solutions of the Ginzburg-Landau equation:

$$[\delta/\delta\psi(\mathbf{r})]F=0. \tag{1.1}$$

Consider now a system in a state very near some ψ which locates a minimum, but not the absolute minimum of F. In order to pass from ψ to a neighboring minimum, say ψ', of lower free energy, the system point must move through regions of higher free energy and thus lower statistical weight. In terms of the topography of $F\{\psi\}$, it is clear that the most probable (least improbable) fluctuation which can carry the system from ψ to ψ' corresponds to the lowest saddle point, say $\bar{\psi}(\mathbf{r})$, between these two minima. The height of the free-energy barrier at $\bar{\psi}$, relative to the minimum at ψ, determines the rate at which the transition from ψ to ψ' can take place.

The point we wish to emphasize is that $\bar{\psi}(\mathbf{r})$, being a stationary point of F, must also satisfy the Ginzburg-Landau equation (1.1). This criterion for the transition-nucleating fluctuation has been discussed in more detail in an earlier paper on the droplet model[8] and was very useful in the recent work on critical velocities in helium.[5] For the helium problem, it was argued that $\bar{\psi}(\mathbf{r})$ must describe a vortex ring of a size determined by the initial state of superfluid flow. In a superconductor, however, it turns out that homogeneous nucleation of resistive transitions by vortex rings will be appreciable only at temperatures unobservably close to the critical point T_c.[9] On the other hand, because of the large correlation lengths which occur in superconductors, it is feasible to construct superconducting samples which are so narrow that the linear dimensions of the cross section are small compared to the (temperature-dependent) correlation length and the penetration depth. In this case, the variation of the order parameter ψ is effectively restricted to one dimension. The saddle-point fluctuation $\bar{\psi}$, being confined to a narrow channel, has a smaller free energy than a bulk fluctuation, and hence gives rise to an observable resistivity at temperatures appreciably lower than T_c. A final advantage of the one-dimensional geometry is that the Ginzburg-Landau equation is exactly soluble. We shall therefore confine our attention throughout the rest of this paper to this effectively one-dimensional situation.[10]

In outline, the program of this paper is as follows. In the next section we establish the connection between the steady-state voltage and the rate of fluctuation. In Sec. 3 we calculate the height of the free-energy barrier for one-dimensional fluctuations. Some features of the arguments contained in these two sections are expanded in three appendices.

Section 4 is devoted to a summary of the quantitative predictions of the theory and a comparison with existing experiments. The agreement with current experimental information is not spectacular, the main discrepancy being that the theory predicts sharper resistive transitions than have been reported. Ours is, however, a theory of the *homogeneous* nucleation of a resistive transition. The presence of special sites favoring the nucleation would of course broaden the transition. This may have been the case in some experimental specimens. We do not, however, discuss inhomogeneous nucleation in this paper because special (and less than unique) assumptions about the nature of the nucleating sites would be needed.

2. FLUCTUATIONS AND RESISTANCE

In this section we shall discuss the character of the fluctuations required to nucleate resistive transitions in a superconductor, and establish a relation between the free energy of these fluctuations and the measured resistivity.

Our analysis will be based on a Ginzburg-Landau free-energy functional of the form[11]

$$F\{\psi(\mathbf{r})\}=\int d\mathbf{r}\,[\,|\nabla\psi|^2-\alpha\,|\psi|^2+\tfrac{1}{2}\beta\,|\psi|^4], \tag{2.1}$$

which implies a choice of normalization and units for the order parameter ψ. Since we shall be concerned only with superconducting samples narrow compared to the penetration depth and carrying small currents, the magnetic field generated by the currents is of no importance and has been omitted in (2.1).[12] The

[7] A similar picture has been discussed by W. A. Little, Phys. Rev. **156**, 396 (1967). Little's work is very similar in spirit to ours but differs in several important respects. Specifically, we differ with regard to the mechanism relating fluctuations to resistivity. We also differ in our descriptions of the detailed nature of the fluctuations.

[8] J. S. Langer, Ann. Phys. (N.Y.) **41**, 108 (1967).

[9] The relevant calculation has been performed by J. W. Wilkins (unpublished).

[10] It should be noted that there is a very serious problem of principle concerning the self-consistency of our phenomenological treatment in the one-dimensional limit. We refer to the papers of T. M. Rice [Phys. Rev. **140**, A1889 (1965)] and P. Hohenberg [Phys. Rev. **158**, 383 (1967)], where it is shown that the conventional sort of long-range order cannot occur in one or two-dimensional superfluids. It may be argued that our present theory is consistent with this theorem because it predicts only a smooth transition in which the resistivity never goes exactly to zero. We do require, however, that it be possible to define a locally meaningful order parameter $\psi(\mathbf{r})$, perhaps analogous to the local (vector) magnetization in a one-dimensional Heisenberg model. We do not claim to see any rigorous justification of this assumption.

[11] For a review of the Ginzburg-Landau theory, see P. G. de Gennes, *Superconductivity of Metals and Alloys* (W. A. Benjamin, Inc., New York, 1966), Chaps. 6 and 7.

[12] In the main body of this paper we work in the gauge in which no magnetic field is described by no vector potential, as is possible for the simply connected pieces of superconductor we are considering. The neglect of the magnetic field is justified in Appendix A.

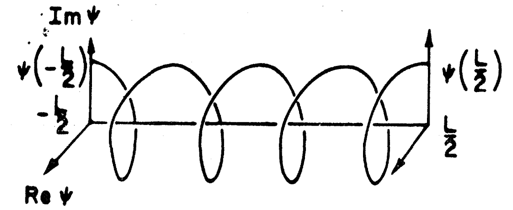

FIG. 1. The order parameter ψ for a uniform current-carrying state.

stationarity condition, Eq. (1.1), may be written

$$-\nabla^2\psi-\alpha\psi+\beta\,|\,\psi\,|^2\psi=0, \qquad (2.2)$$

as long as the boundary conditions imposed upon ψ are such that the usual surface integrals vanish. By making the transformation

$$\nabla\rightarrow\nabla-(ie^*/\hbar c)\mathbf{A}$$

in (2.1), and looking at the term linear in $\mathbf{A}$, we see that (for small $\mathbf{A}$) the electrical current density is

$$\mathcal{J}=(e^*/i\hbar)(\psi^*\nabla\psi-\psi\nabla\psi^*)\equiv(2e^*/\hbar)J. \qquad (2.3)$$

Here e^* is the charge of a pair ($e^*=2e$).

The usual uniform constant-current solution of (2.2) is

$$\psi_k=f_k\exp(ikx), \qquad f_k{}^2=(\alpha-k^2)/\beta, \qquad (2.4)$$

where x is measured in the direction of the current and k is an allowed wave vector subject to, say, periodic boundary conditions in the x direction. The current density (in reduced units) associated with ψ_k is

$$J=kf_k{}^2=k(\alpha-k^2)/\beta. \qquad (2.5)$$

This current has its maximum value at $k=k_c=(\alpha/3)^{1/2}$, which yields the well-known mean-field critical current:

$$J_c=2\alpha^{3/2}/3\sqrt{3}\beta\propto(T_c-T)^{3/2}. \qquad (2.6)$$

It is a simple matter to show that, for $k<k_c$, each of the ψ_k's given by (2.4) locate an isolated minimum of $F\{\psi\}$ in the space of functions $\psi(x)$. That is, the system must overcome some sort of free-energy barrier in order to pass continuously from one ψ_k to another.[13] One very general property of the fluctuation in ψ as it passes this barrier has been emphasized by Little.[7] The situation is best illustrated by a picture of the complex quantity ψ as a function of x. In particular, the ψ_k given in Eq. (2.4) may be represented by the helix shown in Fig. 1. The picture has been drawn for the case $\psi(-\frac{1}{2}L)=\psi(\frac{1}{2}L)$, where L is the length of the sample in the direction of the current. The important point is that, if ψ is to lose a wavelength—if the helix is to lose a loop while its ends are held fixed—then the amplitude of $\psi(x)$ must pass through zero at at least one place along the x axis. Unlike Little, we shall discover that the top of the free-energy barrier does not occur at a $\psi(x)$ which actually vanishes somewhere. But we shall find it very useful to visualize the current-changing fluctuations of ψ as those which annihilate or create loops in Fig. 1.

It is also useful to visualize the effect of an applied voltage in terms of the helix in Fig. 1. The order parameter ψ must have a time dependence $\exp(-2i\mu t)$, where μ is the chemical potential.[6] A small voltage across the sample means that the local chemical potential has a spatially varying part. One thus has the relation

$$(2e/\hbar)\Delta V=(\partial/\partial t)\Delta(\arg\psi), \qquad (2.7)$$

where ΔV is the potential difference between two points, and $\Delta(\arg\psi)$ is the corresponding difference in the phase of ψ. It follows that a constant voltage implies a steady increase with time of this relative phase. In Fig. 1, the helix tightens uniformly, and the current grows.

As is discussed in Sec. 1, a steady state of the superconductor should be achieved when the tightening of the helix due to the applied voltage is balanced, on the average, by the random loss of loops due to fluctuations. It remains for us to calculate the rate of these fluctuations. Although we shall not complete this calculation until we have investigated the fluctuations in detail in Sec. 3, we can learn a great deal from the following general arguments.

Let us assume that the free-energy barrier opposing transitions between states ψ_k and $\psi_{k-2\pi/L}$ has a height $\delta F_0(k)$. Note, however, that the transition $k\rightarrow k-2\pi/L$ is more probable than the reverse because it is a transition to a state of lower free energy. Specifically, the free energy for uniform states is

$$F\{\psi_k\}=\sigma L[(k^2-\alpha)f_k{}^2+\tfrac{1}{2}\beta f_k{}^4], \qquad (2.8)$$

where σ is the cross-sectional area of the sample. Because F is already stationary with respect to variations of f_k, we have

$$dF/dk=\partial F/\partial k=2\sigma Lkf_k{}^2=2\sigma LJ, \qquad (2.9)$$

so that the difference in free energy between neighboring states is

$$\delta F_1=(dF/dk)(2\pi/L)=4\pi\sigma J. \qquad (2.10)$$

We can then write the transition rates in the form

$$\text{Rate}\left(k\rightleftarrows k-\frac{2\pi}{L}\right)\cong\frac{\sigma Ln}{\tau}\exp\left(-\frac{\delta F_0}{k_BT}\pm\frac{\delta F_1}{2k_BT}\right), \qquad (2.11)$$

where τ^{-1} is some characteristic rate for microscopic processes and n is the density of conduction electrons. The prefactor on the right-hand side of (2.11) is supposed to be a rough estimate of the basic rate at which the system point ψ moves through its function space, driven in some unspecified manner by interactions with

[13] For $k\geq k_c$ this barrier disappears. A demonstration of this fact is presented at the beginning of Appendix C.

a constant-temperature bath. As we shall see, our final results will depend only logarithmically on, and thus be happily insensitive to, the value chosen for this microscopic rate.

For present purposes, it will be sufficient simply to take the difference between the two rates given by Eq. (2.11) in order to obtain an estimate of the rate at which $\Delta(\arg\bar{\psi})$ decreases due to fluctuations.[14] Using Eq. (2.7), we obtain

$$\frac{2e}{\hbar}\Delta V=\frac{4\pi n\sigma L}{\tau}\sinh\left(\frac{\delta F_1}{2k_BT}\right)\exp\left(-\frac{\delta F_0}{k_BT}\right). \quad (2.12)$$

From (2.12) we deduce a superconducting resistivity:

$$\rho_s\equiv\frac{\Delta V}{L\mathcal{J}}=\frac{hn\sigma}{e\tau}\frac{1}{\mathcal{J}}\sinh\left(\frac{\mathcal{J}}{2\mathcal{J}_1}\right)\exp\left(-\frac{\delta F_0}{k_BT}\right). \quad (2.13)$$

Here we have introduced a quantity with the dimensions of a current density:

$$\mathcal{J}_1\equiv 2ek_BT/h, \quad (2.14)$$

in order to exhibit explicitly the $\mathcal{J}$ dependence of δF_1 as given by Eq. (2.10).

One natural, but hardly compulsory, choice for the characteristic time τ is the relaxation time which determines the resistivity in the normal state:

$$\rho_n=m/ne^2\tau, \quad (2.15)$$

where m is the mass of the electron. Using (2.15), we rewrite (2.13) in the form

$$\frac{\rho_s}{\rho_n}=\frac{hn^2e\sigma}{m\mathcal{J}}\sinh\left(\frac{\mathcal{J}}{2\mathcal{J}_1}\right)\exp\left(-\frac{\delta F_0}{k_BT}\right). \quad (2.16)$$

Equation (2.16) is the central result of our theory.

Of special interest is the resistive transition in the limit of zero current, that is, the $\mathcal{J}\to 0$ limit of (2.16).

$$\lim_{J\to 0}\left(\frac{\rho_s}{\rho_n}\right)=\frac{h^2n^2\sigma^2}{4mk_BT}\exp\left(-\frac{\delta F_0\,(\mathcal{J}=0)}{k_BT}\right). \quad (2.17)$$

For tin microstrips of the sort used in the experiments of Parks and Groff[1] ($n=5\times 10^{22}$ cm^{-3}, $\sigma\cong 10^{-19}$ cm^2, $T_c=3.7$°K), the prefactor on the right-hand side of (2.17) is of the order 10^{17}. This means that fluctuations with δF_0 as large as 40 k_B should give rise to observable resistivities. As mentioned in Sec. 1, the fact that the resistivity is governed by very large and improbable fluctuations is an important justification of our use of a phenomenological model.

3. THE FREE-ENERGY BARRIER

We turn now to a detailed calculation of the saddle-point fluctuation $\bar{\psi}(x)$ and its associated free energy.

[14] See Appendix B for a more systematic statistical analysis of the way in which the current decays due to fluctuations.

Our problem is to find a function $\bar{\psi}(x)$ which solves the Ginzburg-Landau equation (2.2) and satisfies the following two criteria:

(1) $\bar{\psi}(x)$ must be "close to" $\psi_k(x)$, as given by Eq. (2.4), in the sense that $F\{\bar{\psi}\}$ must differ from $F\{\psi_k\}$ by only an amount independent of L, the length of the system.

(2) $\bar{\psi}(x)$ must locate a saddle point of $F(\psi\}$, such that $F\{\psi\}$ is nondecreasing as one moves away from $\bar{\psi}$ in all but one direction in the function space.

The first criterion assures us that we shall be calculating intrinsic properties of the system, i.e., that the free-energy barrier will not become prohibitively large in the limit of a very long channel. The second criterion requires that we choose at least the locally optimum path across the barrier.[15]

In general, solutions of the one-dimensional version of (2.2) can be written in the form

$$\bar{\psi}(x)=f(x)\exp[i\phi(x)], \quad (3.1)$$

where f and ϕ are real functions of x. Substituting (3.1) into (2.2) and taking real and imaginary parts of the resulting equation, we obtain

$$d^2f/dx^2-f(d\phi/dx)^2+\alpha f-\beta f^3=0, \quad (3.2)$$

and

$$dJ/dx=0, \quad (3.3)$$

where, as in (2.3),

$$J=f^2(d\phi/dx). \quad (3.4)$$

The qualitative behavior of the solutions of (3.2–3.4), and indeed the method of quantitative solution, is suggested by the following analogy with the classical motion of a particle in a central force field. Let f and ϕ be, respectively, the radial and angular coordinates of the position of a particle of unit mass; and let x be the time. Then the first two terms in (3.2) are the radial acceleration of the particle; and (3.3) is equivalent to the statement of conservation of angular momentum. In fact, after using (3.4) to eliminate $d\phi/dx$ in favor of J, Eq. (3.2) can be put in the form

$$d^2f/dx^2=-d/df[J^2/2f^2+\tfrac{1}{2}\alpha f^2-\tfrac{1}{4}\beta f^4]$$
$$\equiv-(d/df)U_{\text{eff}}(f). \quad (3.5)$$

Here the quantity in square brackets plays the role of an effective radial potential and includes a centrifugal barrier. The shape of $U_{\text{eff}}(f)$ for $J<J_c$ is illustrated in Fig. 2.

For $J<J_c$, $U_{\text{eff}}(f)$ has two stationary points, denoted by f_0 and f_0' in the figure. (For $J\geq J_c$, these points merge and then disappear.) In the mechanical

[15] We shall not claim to provide a mathematically rigorous proof that our $\bar{\psi}$ is actually the lowest maximum point on *all possible* paths joining the initial and final states. As shown in Sec. 3 and Appendix C, our $\bar{\psi}$ seems to be sensible from a physical point of view; and this leads us to believe that it may satisfy the much more stringent mathematical requirement.

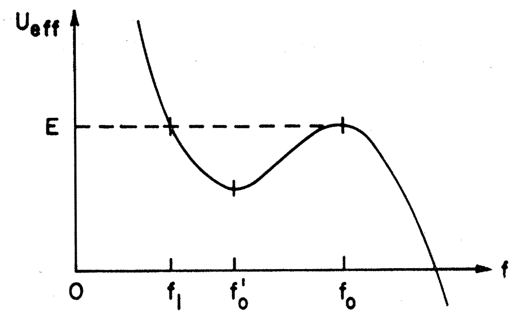

FIG. 2. The effective potential U_{eff} for $J<J_c$.

analog, circular orbits are possible at the two radii f_0 and f_0'. These are the two solutions of constant amplitude, and phase increasing linearly with x, that we find when we solve Eqs. (2.4) and (2.5) for f_k and k at fixed J. It is a simple matter to check that it is the mechanically unstable orbit at f_0, rather than the stable one at f_0', which has a lower free energy F, and must therefore describe the thermodynamically metastable current-carrying state of the superconductor.

The mechanical analogy at once suggests a non-uniform solution $f(x)$ which satisfies the first criterion stated above in that it differs from the uniform solution f_0 only within a finite region along the length L. Imagine a particle performing circular motion in the potential well of Fig. 2 with an initial radius infinitesimally less than but arbitrarily close to f_0. The particle will spend most of its time near f_0, but, in the course of an arbitrarily long interval of time, it will spiral in once to the point labeled f_1 and then return to f_0.

To obtain this solution, we use the conservation of "energy," namely that:

$$d/dx[\tfrac{1}{2}(df/dx)^2+U_{\mathrm{eff}}(f)]\equiv dE/dx=0. \quad (3.6)$$

From the constancy of E we get

$$x=\int_{f_1}^{f}\frac{df}{\{2[E-U_{\mathrm{eff}}(f)]\}^{1/2}}. \quad (3.7)$$

We have required that $f=f_1$ at $x=0$, so that the fluctuation is centered at the origin of coordinates. The value of E that interests us is

$$E=U_{\mathrm{eff}}(f_0)=U_{\mathrm{eff}}(f_1). \quad (3.8)$$

Substituting for U_{eff} from (3.5) and introducing dimensionless units according to

$$f^2=(\alpha/\beta)u,\qquad E=(\alpha^2/2\beta)\epsilon,$$
$$J=jJ_c=(4\alpha^3/27\beta^2)^{1/2}j, \quad (3.9)$$

we find

$$(2\alpha)^{1/2}x=\int_{u_1}^{u}\frac{du}{[u^3-2u^2+2\epsilon u-(8/27)j^2]^{1/2}}. \quad (3.10)$$

From the remarks of the last paragraph and Fig. 2, it follows that the cubic in the denominator of (3.10) vanishes linearly at $u=u_1=\beta f_1^2/\alpha$ and quadratically at $u=u_0=\beta f_0^2/\alpha$. Thus the denominator in (3.10) is

$$[(u-u_1)(u_0-u)^2]^{1/2}.$$

Comparing this form with (3.10), one obtains three equations,

$$2u_0+u_1=2,\qquad u_0^2+2u_0u_1=2\epsilon,\qquad u_1u_0^2=(8/27)j^2, \quad (3.11)$$

which determine the three quantities u_0, u_1, and ϵ in terms of $j=J/J_c$.

Once the denominator in (3.10) is expressed in terms of u_0 and u_1 as just described, the integral is elementary and leads to the result

$$u(x)=(\beta/\alpha)f^2(x)=u_0-\Delta\,\mathrm{sech}^2[x(\tfrac{1}{2}\alpha\Delta)^{1/2}], \quad (3.12)$$

where $\Delta\equiv u_0-u_1$ and satisfies

$$(2+\Delta)^2(1-\Delta)=4j^2, \quad (3.13)$$

as may be deduced from Eqs. (3.11). Equation (3.12) describes a solution which is identical to the uniform solution u_0 over almost all of the sample and which carries the identical current. In a region near $x=0$ the amplitude diminishes and, by virtue of Eq. (3.4), the phase varies more rapidly with position. It is plausible that, from this state of locally diminished amplitude, the system will run downhill in free energy through a configuration in which the amplitude vanishes somewhere, and finally will achieve the configuration in which one less loop in ψ occurs across the length L.

A more detailed picture of how the transition occurs at the saddle point may be obtained by examining $F\{\psi\}$ in the neighborhood of $\bar{\psi}$. Such an analysis is in fact necessary in order to verify that (3.12) satisfies the second criterion stated at the beginning of this section. Because the mathematics is lengthy and only indirectly relevant to our final results, this analysis has been relegated to Appendix C. It turns out—not unexpectedly—that the saddle point will satisfy the second criterion as long as the width of the channel is less than the temperature-dependent correlation length.

Before inserting (3.12) into (2.1) to compute a free energy, we must pay some attention to the question of boundary conditions. If the fluctuation-induced resistivity is indeed an intrinsic property of the system, then the boundary conditions imposed on $\bar{\psi}(x)$ should have little effect on the physical predictions of the theory.[16] We shall assume that the fluctuation rate which interests us is the same as that which determines the spontaneous decay of current in a very large superconducting ring, and shall therefore apply periodic boundary conditions. We believe that this assumption is justifiable in situations where a voltage is imposed across the superconducting sample by external (normal)

[16] However, experimentalists should be wary of boundary effects such as nucleation of current-reducing fluctuations at a normal-to-superconducting interface.

circuitry. According to the discussion in the last section. the phase difference across the sample is determined as a function of time by the applied voltage. The time in which a fluctuation occurs, being an atomic time, is much smaller than the interval between fluctuations, which is related to the voltage as in Eq. (2.16). Thus it seems reasonable to assume that the phases at the end points of the sample remain fixed for the duration of a fluctuation.

Because the phase increases more rapidly as the amplitude decreases, the $\Delta\phi$ associated with the localized amplitude fluctuation, Eq. (3.12), will not be the same as that which would occur for the uniform state with the same current. This total phase difference may be computed by writing

$$\Delta\phi=\int_{-L/2}^{L/2} dx(d\phi/dx)=J\int_{-L/2}^{L/2} dxf^{-2}, \quad (3.14)$$

where we have used (3.4). It is convenient to express this integral in terms of the variables introduced in (3.9) and to write it in the following form:

$$\begin{aligned}\Delta\phi&=\frac{2J\beta}{\alpha}\int_0^{L/2}\frac{dx}{u}=\frac{J\beta L}{\alpha u_0}+\frac{2J\beta}{\alpha}\int_0^{L/2}(u^{-1}-u_0^{-1})dx\\&=\frac{J\beta L}{\alpha u_0}+\frac{2J\beta}{u_0\alpha(2\alpha)^{1/2}}\int_{u_1}^{u_0}\frac{du}{u(u-u_1)^{1/2}}\\&=\frac{JL}{f_0^2}+2\tan^{-1}\left[\frac{3\Delta}{2(1-\Delta)}\right]^{1/2}. \quad (3.15)\end{aligned}$$

Here we have used Eq. (3.10) to transform the variable of integration from x to u. The quantity Δ appearing in this formula is to be obtained from Eq. (3.13). Equations (3.11) have been used in deriving the final form of this result.

The first term in (3.15) is the phase change that would occur if the uniform portion of $\bar{\psi}(x)$ far from the amplitude fluctuation extended over the whole length of the sample. The second term is the phase change associated with the region of fluctuation. Note that this second term is positive and, according to (3.13), varies between π and zero as J goes from zero to J_c. If we wish to compare this $\bar{\psi}(x)$ to a uniform state with the same $\Delta\phi$, i.e., the same number of loops in the helix, then it follows that $\bar{\psi}(x)$ must have a slightly smaller current J than this uniform state in order to compensate for the increased phase change in (3.15).

To be specific, let k_i be the wave number of the uniform state with current J_i and the same $\Delta\phi$ as $\bar{\psi}$; that is, $\Delta\phi=k_iL$. Also identify $k_0\equiv J/f_0^2$ as the wave number appropriate to the uniform part of $\bar{\psi}$. Then, equating $\Delta\phi$ for the two states, we have

$$\delta k\equiv k_i-k_0=(2/L)\tan^{-1}\left[\frac{3\Delta}{2(1-\Delta)}\right]^{1/2}, \quad (3.16)$$

which, in turn, determines a current decrement $\delta J=J_i-J$. We conclude that the current-reducing fluctuation starts with the uniform state k_i, passes through $\bar{\psi}$ with a partially reduced current, and finally reaches $k_i-(2\pi/L)$ after ψ has passed through zero somewhere. The energetically most economical current-increasing fluctuation travels the same route in reverse.

We may now compute the free-energy barrier for the transition $k_i\rightarrow k_i-(2\pi/L)$. Since both the initial uniform state and $\bar{\psi}$ are solutions of the Ginzburg-Landau equation, we may use (2.2) to simplify the integrand in (2.1). This yields, quite generally,

$$F=-\tfrac{1}{2}(\beta\sigma)\int dxf^4. \quad (3.17)$$

The height of the barrier is therefore

$$\delta F=-\tfrac{1}{2}(\beta\sigma)\int dx[f^4(x,J)-f_i^4(J_i)], \quad (3.18)$$

where f_i is the constant amplitude f_k at $k=k_i$ as given by Eq. (2.4). Because δk is of order L^{-1}, we need keep only terms linear in δk. We make the expansion

$$f_i^4(J_i)=f_0^4(J)+(\partial f_0^4/\partial k_0)\delta k+\cdots. \quad (3.19)$$

From Eqs. (2.4) and (2.5), we have

$$\partial f_0^4/\partial k_0=(d/dk_0)[(\alpha-k_0^2)/\beta]^2=-4J/\beta, \quad (3.20)$$

so that δF takes the form

$$\delta F=\tfrac{1}{2}(\beta\sigma)\int dx[f_0^4(J)-f^4(x,J)]-4J\sigma\tan^{-1}\left[\frac{3\Delta}{2(1-\Delta)}\right]^{1/2}. \quad (3.21)$$

Here we have used the explicit expression for δk given in (3.16). Finally, we may substitute (3.12) to evaluate the integral in (3.21). The integration is straightforward, and after a few manipulations involving Eqs. (3.11), we obtain

$$\delta F=(8\sqrt{2}/3)(\alpha^2/2\beta)(\sigma/\sqrt{\alpha})\{\sqrt{\Delta}-(\sqrt{\tfrac{2}{3}})(J/J_c)\times\tan^{-1}[3\Delta/2(1-\Delta)]^{1/2}\}. \quad (3.22)$$

A useful way of expressing the prefactor in (3.22) in terms of interesting experimental quantities is to note that the advantage in free energy per unit volume enjoyed by the superconducting state (when there is no current) relative to the normal state, g_n-g_s, is $(\alpha^2/2\beta)$. Also, in our units the temperature-dependent coherence length $\xi(T)$ is $\alpha^{-1/2}$. Thus

$$\delta F=(8\sqrt{2}/3)(g_n-g_s)\sigma\xi(T)\times\{\sqrt{\Delta}-(\sqrt{\tfrac{2}{3}})(J/J_c)\tan^{-1}[3\Delta/2(1-\Delta)]^{1/2}\}. \quad (3.23)$$

This expression shows that the volume of the fluctuating region is the product of the cross-sectional area and the coherence length. Within this region the system approaches the normal state, in agreement with Little's conjecture.[7]

In order to complete the derivation of Eq. (2.16), we must consider the current-increasing transition, $k_i-(2\pi/L)\rightarrow k_i$. The free-energy barrier to this transition, say $\delta F'$, must be larger than δF by just the amount calculated in Eq. (2.10):

$$\delta F'=\delta F+4\pi\sigma J$$
$$=\delta F+(16\pi\sigma/3\sqrt{3})(\alpha^2/2\beta)\alpha^{-1/2}(J/J_c), \quad (3.24)$$

where we have used Eq. (2.6) for J_c in order to bring this expression into a form comparable to (3.22). Then, comparing (3.22) and (3.24) to the exponent in (2.11), we have

$$\delta F_0=(8\sqrt{2}/3)(\alpha^2/2\beta)(\sigma/\sqrt{\alpha})\{\sqrt{\Delta}+\sqrt{(\tfrac{2}{3})}(J/J_c)(\tfrac{1}{2}\pi-\tan^{-1}[3\Delta/2(1-\Delta)]^{1/2})\}$$
$$=(8\sqrt{2}/3)(g_n-g_s)\sigma\xi(T)\{\sqrt{\Delta}+\sqrt{(\tfrac{2}{3})}(\mathcal{J}/\mathcal{J}c)(\tfrac{1}{2}\pi-\tan^{-1}[3\Delta/2(1-\Delta)]^{1/2})\}. \quad (3.25)$$

Equation (3.25) and Eq. (3.13) for Δ, and Eqs. (2.14) and (2.16) constitute a complete summary of our theoretical results.

4. INTERPRETATION

As a first step in the interpretation of this theory, we examine the resistive transition in the limit of zero current. From Eqs. (2.17) and (3.25) we have

$$\rho_s(\mathcal{J}\rightarrow 0)/\rho_n=\exp[\gamma-(8\sqrt{2}/3k_BT)(g_n-g_s)\sigma\xi(T)], \quad (4.1)$$

where

$$\gamma\equiv \ln(\hbar^2n^2\sigma^2/4mk_BT) \quad (4.2)$$

is the number which we estimated in Sec. 2 to be about 40 for typical experimental microstrips. We define $T_c-\Delta T_c$ to be the temperature at which $\rho_s\cong\rho_n$. The quantity ΔT_c always will be very much smaller than T_c, so that we may replace the explicit T in (4.1) by T_c. The remaining temperature dependence comes from

$$(g_n-g_s)\xi(T)\propto(\Delta T)^{3/2}, \quad (4.3)$$

where $\Delta T\equiv T_c-T$. Thus we can write

$$\rho_s(0)/\rho_n\cong\exp\{\gamma[1-(\Delta T/\Delta T_c)^{3/2}]\}. \quad (4.4)$$

The total resistivity of the system should be obtained from (4.4) by considering the superconducting electrons and normal electrons (excitation gas) as if they were connected in parallel. For $\Delta T>\Delta T_c$, the superfluid very nearly shorts out the circuit, and $\rho\cong\rho_s$. For $\Delta T<\Delta T_c$, on the other hand, $\rho_s\gg\rho_n$ and $\rho\cong\rho_n$.

The explicit formula for ΔT_c can be obtained from[17]

$$g_n-g_s=[4\pi^2/7\zeta(3)]N(0)k_B^2(\Delta T)^2$$
$$=4.7N(0)k_B^2(\Delta T)^2, \quad (4.5)$$

where $N(0)$ is the density of states at the Fermi surface. To evaluate $N(0)$, one uses the coefficient of T in the normal specific heat, $\tfrac{2}{3}\pi^2k_B^2N(0)$. The correlation length is given by[18]

$$\xi(T)=0.85(\xi_0lT_c/\Delta T)^{1/2}, \quad (4.6)$$

appropriate in the case of short mean free path l. The temperature-independent coherence length is

$$\xi_0=0.18(\hbar v_F/k_BT_c), \quad (4.7)$$

where v_F is the velocity of electrons at the Fermi surface. Finally, we have

$$(\Delta T_c/T_c)^{3/2}=0.437\gamma[\tfrac{2}{3}\pi^2N(0)k_B^2]^{-1}[k_B/\sigma(\xi_0l)^{1/2}T_c]. \quad (4.8)$$

To compare these predictions with the results of Parks and Groff,[1] we choose l to be the thickness of the film, 580 Å. For tin, $\xi_0\cong 2.3\times10^{-5}$ cm, and the specific-heat coefficient is 1080 erg/deg² cm³. Using $\gamma\cong 40$, we obtain $\Delta T_c\cong 4.3°\times10^{-3}$ °K, which is to be compared with the experimental result of ΔT_c between 15 and 20×10^{-3} °K. According to Eq. (4.4), our theoretical transition should have a width of about $2\Delta T_c/3\gamma\cong 10^{-4}$ °K, whereas the experimentally observed transitions are roughly 50 times broader than this.

The discrepancy in the value of ΔT_c is actually more serious than it looks. It cannot be ascribed to uncertainty in the fundamental fluctuation rate discussed

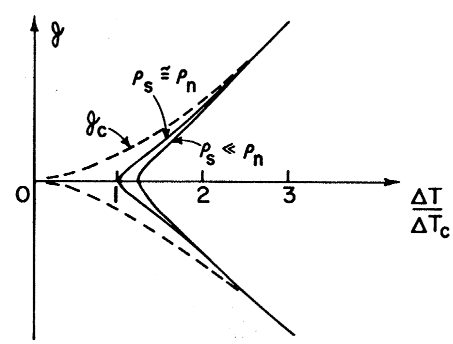

FIG. 3. Qualitative curves of constant ρ_s/ρ_n in the current-temperature plane.

[17] For example, see A. A. Abrikosov, L. P. Gorkov, and I. E. Dzyaloshinski, *Methods of Quantum Field Theory in Statistical Physics* (Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1963), p. 306, Eq. (36.9). The formula quoted by de Gennes [Ref. 11, p. 173, Eqs. (6.3) and (6.5)] seems to be in error by a factor of ζ (3).

[18] P. G. de Gennes, Ref. 11, p. 225, Table 7-1.

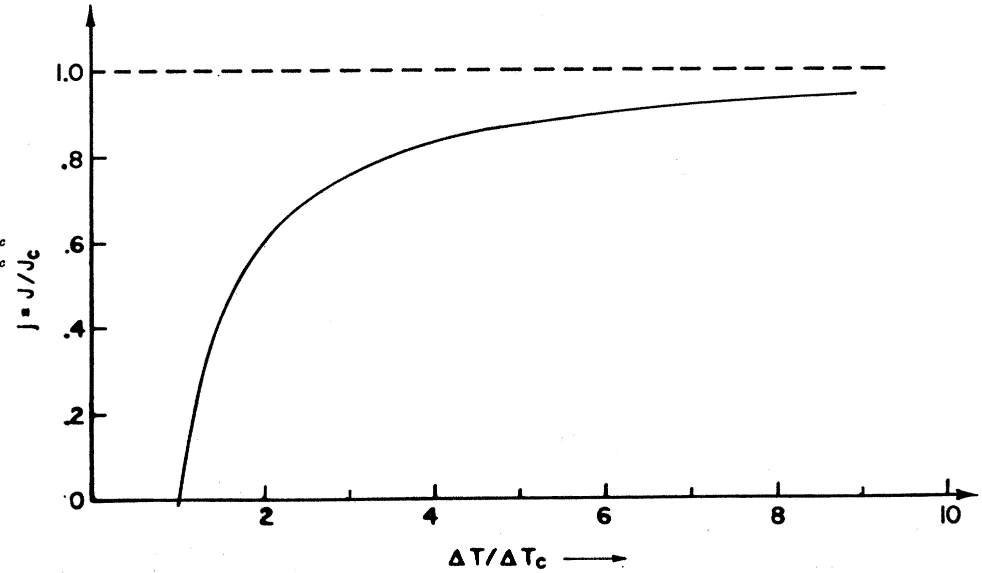

FIG. 4. Graph of $j=J/J_c$ as a function of $\Delta T/\Delta T_c$ for $\rho_s=\rho_n$ and $\gamma=40$.

following Eq. (2.11). In order to increase ΔT_c by the required factor of about 4, we should have to increase γ from 40 to 320, which means increasing $\exp(\gamma)$ from about 10^{17} to 10^{136}. It seems most likely to us, therefore, that the observed resistivities may be governed by inhomogeneities that nucleate fluctuations at relatively low activation energies. Such inhomogeneities also might tend to broaden the transitions.

The complete information contained in Eq. (2.16) is best displayed as a series of contours for constant ρ_s/ρ_n plotted in the $\mathcal{J}$-*versus*-T plane. Qualitative features of such a plot are shown in Fig. 3. Note that, for all values of ρ_s/ρ_n, $\mathcal{J}(T)$ approaches $\mathcal{J}_c(T)\propto(\Delta T)^{3/2}$ as ΔT becomes large compared to ΔT_c. The curves for observable values of ρ_s/ρ_n all lie within a very narrow region, reflecting the fact that the resistive transition is relatively sharp. In fact, one can choose one of these curves, say $\rho_s=\rho_n$, to represent the critical current as a function of temperature, with the understanding that the transition is actually slightly smeared out in the immediate neighborhood of this curve.

In constructing contours of constant ρ_s/ρ_n, it is convenient to introduce a dimensionless variable related to the temperature:

$$y\equiv(\Delta T/\Delta T_c)^{3/2}. \tag{4.9}$$

In terms of y and the quantity $j\equiv\mathcal{J}/\mathcal{J}_c$ introduced earlier, the combination of Eqs. (2.16) and (3.25) becomes

$$\rho_s/\rho_n=(\sqrt{6}/\pi\gamma yj)\sinh(\pi\gamma yj/\sqrt{6})\exp(\gamma-\delta F_0/k_BT_c), \tag{4.10}$$

where

$$\delta F_0/k_BT_c=\gamma y\sqrt{\Delta}+(\pi\gamma yj/\sqrt{6})\times\{1-(2/\pi)\tan^{-1}[3\Delta/2(1-\Delta)]^{1/2}\}, \tag{4.11}$$

and Δ is a function of j alone. Taking the logarithm of both sides and rearranging terms, we obtain

$$yG(j)-1=\gamma^{-1}\ln(\sqrt{6}\rho_n/2\pi\gamma\rho_s jy)+\gamma^{-1}\ln[1-\exp(-2\pi\gamma jy/\sqrt{6})], \tag{4.12}$$

where

$$G(j)=\sqrt{\Delta}-(2j/\sqrt{6})\tan^{-1}[3\Delta/2(1-\Delta)]^{1/2}. \tag{4.13}$$

Equation (4.12) has been written in a form which suggests a suitable iterative method of solution. As a first approximation, the entire right-hand side may be set to zero; and the results will be accurate to about 5% throughout the range of interesting values of y and j. More accurate solutions may be obtained by iterations including only the first term on the right-hand side. The last term is negligible except in the region $j\gtrsim\gamma^{-1}\cong 2\times10^{-2}$, which is probably too small to be seen experimentally and does not show up on the graph of our numerical solution. (In fact, j vanishes with infinite slope in this region.) The graph of j versus $\Delta T/\Delta T_c$ for $\rho_n=\rho_s$ and $\gamma=40$ is shown in Fig. 4.

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APPENDIX A: THE MAGNETIC FIELD

Magnetic fields have been ignored in the main body of this paper. Their neglect is justified here. We show that the magnetic field generated by a supercurrent flowing in a channel narrow compared to the penetration depth modifies the order parameter insignificantly. We also establish the perhaps less obvious proposition that the magnetic-field energy makes a negligible contribution to the free-energy barrier opposing the nucleation of a state of lower current.

The free-energy functional (2.1) is modified by the inclusion of a magnetic field $\mathbf{H}$, described by a vector potential $\mathbf{A}(\nabla\times\mathbf{A}=\mathbf{H})$, and acquires the form

$$F\{\psi(r),\mathbf{A}(r)\}=\int d\mathbf{r}[\,|(\nabla-(ie^*/\hbar c)\mathbf{A})\psi|^2-\alpha|\psi|^2+\tfrac{1}{2}\beta|\psi|^4]+(8\pi)^{-1}\int d\mathbf{r}(\nabla\times\mathbf{A})^2. \quad \text{(A1)}$$

The requirement of stationarity with respect to variations of ψ and $\mathbf{A}$ leads to well-known equations

$$(i^{-1}\nabla-(e^*/\hbar_c)\mathbf{A})^2\psi-\alpha\psi+\beta|\psi|^2\psi=0, \quad \text{(A2)}$$

$$\nabla\times(\nabla\times\mathbf{A})=(4\pi/c)(2e^*/\hbar)J, \quad \text{(A3)}$$

where the current density J is given by

$$\mathbf{J}=\tfrac{1}{2}[\psi^*\nabla\psi-\psi\nabla\psi^*-(2e^*i/\hbar c)\mathbf{A}|\psi|^2]. \quad \text{(A4)}$$

We note in passing that the current $\mathbf{J}$ is only conserved at the stationary points of (A1). [$\nabla\cdot\mathbf{J}=0$ only by virtue of (A3).] Since our analysis requires the calculation of F only at stationary points, the nonconservation of J away from these points does not trouble us.

We shall establish the unimportance of the magnetic field for the case of interest by making an iterative solution of (A2–A4) about the solution (2.5) used in the text. In order to keep geometrical complications to a minimum we shall consider an infinite plate of thickness $2d\ll\lambda(T)$, $\xi(T)$. We take the y axis to be normal to the plate, which fills the region $-d<y<d$, and imagine that the current flows in the x direction, i.e., $J=J_x(y)$. The magnetic field is then in the z direction, $H=H_z(y)$. We choose the vector potential to have only an x component, $A=A_x(y)$. The zeroth-order solution of (A2–A4) is

$$A^{(0)}=0;\qquad \psi^{(0)}=f_k\exp(ikx),\qquad [f_k{}^2=(\alpha-k^2)/\beta];$$
$$J^{(0)}=kf_k{}^2. \quad \text{(A5)}$$

In next order we substitute $J^{(0)}$ into (A3), which equation reduces to

$$(d^2A/dy^2)=-(4\pi/c)(2e^*/\hbar)J^{(0)} \quad \text{(A6)}$$

and has the solution

$$A^{(1)}=(4\pi e^*/\hbar c)J^{(0)}y^2. \quad \text{(A7)}$$

We substitute (A7) into (A2) and linearize about $\psi=\psi^{(0)}$. Writing $\psi=\psi^{(0)}(1+\epsilon)$ we obtain

$$-(d^2\epsilon/dy^2)+2(\alpha-k^2)\epsilon=(4ek/\hbar c)A^{(1)}-(4e^2/\hbar^2c^2)[A^{(1)}]^2\equiv h(y), \quad \text{(A8)}$$

where we have been free to chose ϵ real and a function only of y. The boundary condition[11] to be satisfied by ϵ is

$$d\epsilon/dy\,|_{y=\pm d}=0. \quad \text{(A9)}$$

A complete set of states obeying this boundary condition is made up of

$$\epsilon_n(y)=\cos(n\pi x/d),\qquad n=0,1,2,\cdots. \quad \text{(A10)}$$

Making an expansion of the form $\epsilon(y)=\sum_n C_n\epsilon_n(y)$, we get

$$C_0=(2dE_0)^{-1}\int_{-d}^{d}h(y)dy,$$

$$C_n=(dE_n)^{-1}\int_{-d}^{d}\cos(n\pi y/d)h(y)dy,\qquad n=1,2,\cdots, \quad \text{(A11)}$$

where $h(y)$ is the inhomogeneous part of (A8) and

$$E_n=(n^2\pi^2/d^2)+2(\alpha-k^2),\qquad n=0,1,2,\cdots. \quad \text{(A12)}$$

Now, we are interested in $k^2\lesssim\alpha/3=(3\xi^2(T))^{-1}$. Consideration of (A11) shows that for $d<\xi(T)$, C_0 is the largest coefficient, because $|C_1/C_0|\propto(d/\xi)^2$. This expresses the obvious fact that a variation of the magnitude of the order parameter cannot take place in a distance small compared to $\xi(T)$. The term in (A8) linear in the vector potential may also be seen to make the largest contribution to C_0. One then finds

$$C_0\propto(e^{*2}/\hbar^2c^2)\xi(T)d^2J^{(0)}. \quad \text{(A13)}$$

The criterion for the smallness of the correction [$C_0\ll1$] is thus

$$J^{(0)}\ll(\hbar^2c^2/e^{*2})[\xi(T)d^2]^{-1}. \quad \text{(A14)}$$

Introducing the mean-field maximum current given by (2.6) and the penetration depth, which in the units we are using is given by

$$\lambda=(8\pi e^{*2}\alpha/\hbar^2c^2\beta)^{-1/2}, \quad \text{(A15)}$$

one finds that (A14) may be expressed as

$$J^{(0)}\ll(\lambda/d)^2J_c. \quad \text{(A16)}$$

For $d<\lambda$, (A16) is always satisfied, showing that the correction to (A5) is negligible.

Finally, we examine the contribution of the field energy to the height of the barrier discussed in Sec. 3. The inductance of a thin wire per unit length is $(2c^2)^{-1}(1+K)$, where K is a factor which depends weakly (logarithmically) on the dimensions of the specimen. The field energy associated with a current density $\mathcal{J}=(2e^*J/\hbar)$ is then

$$F_H=\tfrac{1}{2}L[(1+K)/2c^2]\sigma^2(2e^*J/\hbar)^2. \quad \text{(A17)}$$

Here L is the length of the sample, and σ is the cross-sectional area. At the midpoint of the fluctuation J is decreased by δJ, where from (2.5), (3.16), and (3.11),

$$L\delta J = L(\partial J/\partial k)_{k_0}\delta k = (\alpha/\beta)(3u_0-2)L\delta k$$
$$= (2\alpha/\beta)\Delta \tan^{-1}[3\Delta/2(1-\Delta)]^{1/2}. \qquad (A18)$$

The change in field energy may be expressed in terms of the penetration depth (A16) and the quantities discussed above (3.23) as follows:

$$\delta F_H = -\frac{(1+K)}{6\pi\sqrt{3}}\frac{\sigma^2}{\lambda^2}(g_n-g_s)\xi(T)\frac{J}{J_c}\Delta \tan^{-1}\left[\frac{3\Delta}{2(1-\Delta)}\right]^{1/2}. \qquad (A19)$$

This expression is to be compared with (3.23). In the region where fluctuations are important, $J<J_c$ and $\Delta\to 1$. The quantity (A19) is then seen to be smaller than (3.23) by a factor $(\sigma/\lambda^2)(J/J_c)$. Since by hypothesis the cross-sectional area σ is smaller than λ^2, (A19) represents a small correction to (3.23).

APPENDIX B: DERIVATION OF EQ. (2.12)

Equation (2.12) may be derived more systematically than in the text by setting up and solving a master equation for the time rate of change of $\Delta(\arg\psi)=kL$. To couch the argument in statistical terms, define $P(k,t)$ to be the probability that the system is in state ψ_k at time t. Let $T(k\to k')$ represent the rate at which a system in state ψ_k makes transitions to $\psi_{k'}$. Then the master equation is

$$(\Delta t)^{-1}[P(k,t+\Delta t)-P(k,t)]$$
$$= T(k+2\pi/L\to k)P(k+2\pi/L,t)$$
$$+T(k-2\pi/L\to k)P(k-2\pi/L,t)$$
$$-T(k\to k+2\pi/L)P(k,t)$$
$$-T(k\to k-2\pi/L)P(k,t). \qquad (B1)$$

Here we have assumed that only the transitions $k\to k\pm(2\pi/L)$ are allowed.

From Eq. (2.11), we have

$$T(k\rightleftarrows k-2\pi/L) = (\sigma Ln/\tau)\exp(-\delta F_0/k_BT\pm\delta F_1/2k_BT). \qquad (B2)$$

It is important to note that the exponent in (B2) is a slowly varying function of k in the sense that it does not change appreciably over intervals of order L^{-1}.

At this point we must make some ansatz for $P(k,t)$. In thermal equilibrium the states of the statistical ensemble are distributed with probability

$$P\propto \exp[-F(k)/k_BT],$$

where the free energy $F(k)$ is proportional to L. Because the distributions of interest in our nonequilibrium problem should be qualitatively similar to the equilibrium distribution, we write

$$P(k,t)=\exp[Ls(k,t)]. \qquad (B3)$$

The function $s(k,t)$ is assumed to be slowly varying; but note that P may vary rapidly because of the explicit L in the exponent.

Inserting (B2) and (B3) into (B1), and dropping terms of order L^{-1} in the exponents, we obtain

$$(L\Delta t)^{-1}\{\exp[Ls(k,t+\Delta t)-Ls(k,t)]-1\}$$
$$=(\sigma n/\tau)\exp(-\delta F_0/k_BT)\{\exp(\delta F_1/2k_BT)$$
$$\times[\exp(2\pi(\partial s/\partial k))-1]+\exp(-\delta F_1/2k_BT)$$
$$\times[\exp(-2\pi(\partial s/\partial k))-1]\}. \qquad (B4)$$

In the limit $\Delta t\to 0$, the left-hand side of (B4) becomes simply $\partial s/\partial t$. To make a similar simplification of the right-hand side, note that the most probable value of k, say $\bar{k}$, occurs at $\partial s/\partial k=0$. Restricting ourselves to values of k near $\bar{k}$, we may linearize the right-hand side. The resulting equation is

$$\partial s/\partial t=\Gamma(k)\partial s/\partial k, \qquad (B5)$$

where

$$\Gamma(k)=(4\pi\sigma n/\tau)\exp(-\delta F_0/k_BT)\sinh(\delta F_1/2k_BT). \qquad (B6)$$

The maximum of $s(k)$ at $\bar{k}$, of course, locates a very sharp peak in $P(k,t)$. Equation (B5) can now be used to determine how this peak moves as a function of time. Suppose that the peak moves from $\bar{k}$ to $\bar{k}+\Delta\bar{k}$ in a time Δt. The criterion $\partial s/\partial k=0$ at $\bar{k}$ implies

$$\Delta(\partial s/\partial k)=(\partial^2 s/\partial k^2)\Delta\bar{k}+(\partial^2 s/\partial k\partial t)\Delta t=0. \qquad (B7)$$

But, from (B5) we find that

$$\partial^2 s/\partial k\partial t=\Gamma(k)\partial^2 s/\partial k^2, \qquad (B8)$$

(again using $\partial s/\partial k=0$ at $\bar{k}$), so that

$$(d\bar{k}/dt)_{\text{fluct.}}=-\Gamma(k). \qquad (B9)$$

In these terms, Eq. (2.7) is

$$(2e/\hbar)\Delta V=-L(d\bar{k}/dt)_{\text{fluct.}}=L\Gamma(k), \qquad (B10)$$

which is the same as Eq. (2.12).

APPENDIX C: PROPERTIES OF THE STATIONARY POINTS OF $F\{\psi(\mathbf{r})\}$

Several of the arguments in the main text of this paper depend on mathematical properties of the functional $F\{\psi(\mathbf{r})\}$ in the neighborhood of its stationary points in the function space. This Appendix is devoted to an investigation of these properties.

We consider first the neighborhood of the uniform stationary points, ψ_k, described by Eq. (2.4). Let the function $\nu(\mathbf{r})$ represent a deviation from the point $\psi_k(\mathbf{r})$. That is,

$$\psi(\mathbf{r})=f_k\exp(i\mathbf{k}\cdot\mathbf{r})+\nu(\mathbf{r}). \qquad (C1)$$

Then expand $F\{\psi\}$ out to terms quadratic in $\nu(\mathbf{r})$:

$$F\{\psi(\mathbf{r})\}=F\{\psi_k\}+Q\{\nu(\mathbf{r})\}+\cdots, \tag{C2}$$

where

$$Q\{\nu(\mathbf{r})\}=\int d\mathbf{r}\{\,|\nabla\nu|^2-k^2|\nu|^2+\tfrac{1}{2}(\alpha-k^2)\times[\nu\exp(-i\mathbf{k}\cdot\mathbf{r})+\nu^*\exp(i\mathbf{k}\cdot\mathbf{r})]^2\}. \tag{C3}$$

$Q\{\nu\}$ is a quadratic form in $\nu(\mathbf{r})$ whose eigenvalues are the characteristic curvatures of $F\{\psi\}$ at the point $\psi_\mathbf{k}$. In order for F to have a local minimum at $\psi_\mathbf{k}$, all of these eigenvalues must be positive.

The relevant eigenvalue equation is

$$-\nabla^2\nu-k^2\nu+(\alpha-k^2)[\nu+\nu^*\exp(2i\mathbf{k}\cdot\mathbf{r})]=\lambda\nu, \tag{C4}$$

λ being the desired eigenvalue. This equation is simplified by writing

$$\nu=\exp(i\mathbf{k}\cdot\mathbf{r})[u_1(\mathbf{r})+iu_2(\mathbf{r})], \tag{C5}$$

where u_1 and u_2 are both real functions. Taking the real and imaginary parts of (C4), we obtain

$$-\nabla^2u_1+2\mathbf{k}\cdot\nabla u_2+2(\alpha-k^2)u_1=\lambda u_1,$$
$$-\nabla^2u_2-2\mathbf{k}\cdot\nabla u_1=\lambda u_2. \tag{C6}$$

These coupled equations have plane-wave solutions of the form

$$u_1=\mathrm{Re}[a_1\exp(i\mathbf{p}\cdot\mathbf{r})],\qquad u_2=\mathrm{Re}[a_2\exp(i\mathbf{p}\cdot\mathbf{r})], \tag{C7}$$

and the resulting secular equation yields the eigenvalues

$$\lambda_\mathbf{p}=p^2+\alpha-k^2\pm[(\alpha-k^2)+4(\mathbf{k}\cdot\mathbf{p})^2]^{1/2}. \tag{C8}$$

It is a simple matter to check that $\lambda_\mathbf{p}$ is non-negative for all $\mathbf{p}$ unless $k^2\geq\frac{1}{3}\alpha$. Thus the stability condition is identical to the criterion for the Ginzburg-Landau critical current.

We now must perform a similar analysis in the neighborhood of the stationary point $\bar{\psi}(x)$ discussed in Sec. 3. In analogy with (C1), we write

$$\psi(\mathbf{r})=\bar{\psi}(x)+\nu(\mathbf{r}), \tag{C9}$$

where x measures distance in the direction of the current. We shall consider the full three-dimensional variation of ψ in order to test our assumptions concerning one-dimensional variation of the saddle-point fluctuation $\bar{\psi}(x)$.

Just as in Eqs. (C2) and (C3) above, we insert (C9) into $F\{\psi\}$ and examine the terms quadratic in $\nu(\mathbf{r})$. The eigenvalue equation analogous to (C4) is

$$-\nabla^2\nu-\alpha\nu+2\beta|\bar{\psi}|^2\nu+\beta\bar{\psi}^2\nu^*=\lambda\nu. \tag{C10}$$

Because $\bar{\psi}$ is a function of x only, Eq. (C10) will have solutions of the form

$$\nu(\mathbf{r})=\tilde{\nu}(x)\cos(q_yy)\cos(q_zz), \tag{C11}$$

where q_y and q_z are components of a transverse wave vector $\mathbf{q}$ whose allowed values are determined by the boundary conditions imposed at the sides of the sample. Inserting (C11) into (C10), we have

$$-d^2\tilde{\nu}/dx^2-\alpha\tilde{\nu}+2\beta|\bar{\psi}|^2\tilde{\nu}+\beta\bar{\psi}^2\tilde{\nu}^*=\epsilon\tilde{\nu}, \tag{C12}$$

where

$$\lambda=\epsilon+q^2. \tag{C13}$$

The first step in solving (C12) is to write

$$\tilde{\nu}(x)=\bar{\psi}(x)w(x), \tag{C14}$$

so that the eigenvalue equation becomes

$$-d^2w/dx^2-2(d/dx\ln\bar{\psi})dw/dx+\beta|\bar{\psi}|^2(w+w^*)=\epsilon w. \tag{C15}$$

From Eqs. (3.1) and (3.4), we have

$$|\bar{\psi}|^2=f^2(x) \tag{C16}$$

and

$$d/dx\ln\bar{\psi}=f^{-1}(df/dx)+i(J/f^2(x)). \tag{C17}$$

Then, writing

$$w=w_1+iw_2, \tag{C18}$$

with w_1 and w_2 both real, Eq. (C15) becomes two coupled equations in real variables:

$$-\frac{d^2w_1}{dx^2}+\frac{2}{f}\frac{df}{dx}\frac{dw_1}{dx}+2\beta f^2w_1+\frac{2J}{f^2}\frac{dw_2}{dx}=\epsilon w_1, \tag{C19}$$

and

$$-\frac{d^2w_2}{dx^2}+\frac{2}{f}\frac{df}{dx}\frac{dw_2}{dx}-\frac{2J}{f^2}\frac{dw_1}{dx}=\epsilon w_2. \tag{C20}$$

The function $f(x)$ to be used in these equations is that given in Eq. (3.12).

We have not succeeded in solving Eqs. (C19) and (C20), although an analytic solution may very well be possible. We are able, however, to find exact solutions in the limit $J\rightarrow0$. Remember that the fluctuation $\bar{\psi}(x)$ extends over a coherence length $\xi(T)\propto\alpha^{-1/2}$, independent of the current J. Thus the limit $J\rightarrow0$ is appropriate whenever $\xi(T)$ is much smaller than the wavelength of the current-carrying state ψ_k. In this case we have

$$f(x)=(\alpha/\beta)^{1/2}|\tanh(\tfrac{1}{2}\alpha)^{1/2}x|. \tag{C21}$$

Equations (C19) and (C20) become

$$-d^2w_1/d\zeta^2-(4/\sinh2\zeta)(dw_1/d\zeta)+4(\tanh^2\zeta)w_1=\epsilon'w_1, \tag{C22}$$

and

$$-d^2w_2/d\zeta^2-(4/\sinh2\zeta)(dw_2/d\zeta)=\epsilon'w_2, \tag{C23}$$

where the new independent variable is

$$\zeta=(\tfrac{1}{2}\alpha)^{1/2}x, \tag{C24}$$

and

$$\epsilon'\equiv2\epsilon/\alpha. \tag{C25}$$

The most important simplification is that (C22) and

(C23) are no longer coupled and may be treated separately.

The simpler and, in fact, more interesting of the equations is (C23). To solve this, we make the transformation

$$t = \sinh\zeta, \qquad (C26)$$

so that (C23) becomes

$$(1+t^2)(d^2w_2/dt^2)+(t+2/t)(dw_2/dt)+\epsilon' w_2=0. \qquad (C27)$$

The solutions of (C27) are finite polynomials in t^2, multiplied by either t^{-1} or t^0. The only two acceptable solutions are

$$w_{2,-1}=t^{-1}=(\sinh\zeta)^{-1}, \qquad \epsilon'=-1, \qquad (C28)$$

and

$$w_{2,0}=\text{const}, \qquad \epsilon'=0. \qquad (C29)$$

All other solutions contain higher powers of t, and, therefore, do not belong to the function space because the corresponding $\nu(\mathbf{r})$ is not normalizable. The eigenstate $w_{2,0}$ given in Eq. (C29) is also not a localized function but is normalizable because, like the plane waves in Eq. (C7), it remains finite at infinity. Clearly (C29) locates the bottom of a continuum of stable deviations $\bar{\nu}$ with eigenvalues starting at $\epsilon=0$.

Equation (C28) exhibits the only eigenstate of Eq. (C12) which has a negative eigenvalue. The corresponding unstable deviation from $\bar{\psi}$ has the form

$$\bar{\nu}(x) \propto i\bar{\psi}(x)\, \text{csch}(\tfrac{1}{2}\alpha)^{1/2}x, \qquad (C30)$$

which must describe the path followed by $\psi(\mathbf{r})$ in passing over the peak of the free-energy barrier at $\bar{\psi}$. To understand the implications of (C30), note first that $\bar{\nu}$ is localized in the region of the fluctuation and, within that region, is everywhere ninety degrees out of phase with $\bar{\psi}$. Also note that $w_{2,-1}$ changes sign at the center of the fluctuation, $x=0$, and that $\bar{\nu}$ remains finite there only because $\bar{\psi}$ vanishes. Finally, note that $\bar{\psi}$ changes phase by π across the fluctuation, as determined by Eq. (3.15).

The above observations enable us to draw a sequence of functions ψ illustrating the steps in the transition between uniform states at, say, $k=2\pi/L$ and $k=0$. In Fig. 5 we show real and imaginary parts of ψ for five different stages in the transition. In accord with the arguments in Sec. 3, we have drawn $\bar{\psi}$ with a wave number $k_0=\pi/L$ everywhere except in the region of the fluctuation, labeled $\xi(T)$, where the phase change of π takes place. The functions labeled $\bar{\psi}\pm\nu$ represent ψ, respectively, just after and just before it passes over the saddle point at $\bar{\psi}$. Within $\xi(T)$, we have, to a good approximation,

$$\bar{\psi}(x)=(\alpha/\beta)^{1/2}\tanh(\tfrac{1}{2}\alpha)^{1/2}x, \qquad (C31)$$

and

$$\bar{\nu}(x) \propto i\, \text{sech}(\tfrac{1}{2}\alpha)^{1/2}x, \qquad (C32)$$

so that $\bar{\nu}$ has exactly the desired effect of bringing $\bar{\psi}$ through the x axis.

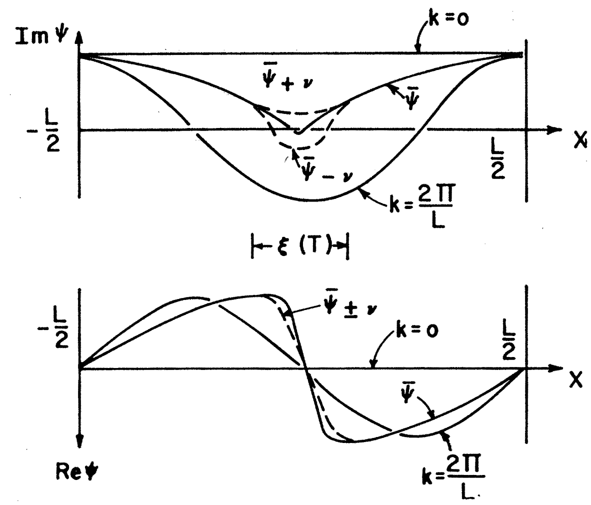

FIG. 5. Successive steps in the continuous deformation of ψ as it makes the transition from the uniform state with $k=2\pi/L$, through $\bar{\psi}$, to the state with $k=0$.

We return now to Eq. (C22), which determines the eigenstates of (C15) (for $J=0$) which are in phase with $\bar{\psi}$. To solve (C22), we write

$$w_1=\text{sech}\zeta\, v(t), \qquad t=\sinh\zeta. \qquad (C33)$$

Then (C22) becomes

$$(1+t^2)(d^2v/dt^2)-(t-2/t)(dv/dt)+(\epsilon'-3)v=0. \qquad (C34)$$

The solutions of (C34) are similar to those of (C27), but there are now three acceptable solutions:

$$w_{1,0}=(\cosh\zeta\,\sinh\zeta)^{-1}, \qquad \epsilon'=0; \qquad (C35)$$

$$w_{1,3}=(\cosh\zeta)^{-1}, \qquad \epsilon'=3; \qquad (C36)$$

and

$$w_{1,4}=(\cosh\zeta)^{-1}[(\sinh\zeta)^{-1}-2\sinh\zeta], \qquad \epsilon'=4. \qquad (C37)$$

Equation (C35) is of some interest because it expresses the translational symmetry of the problem. The fluctuation ψ can be placed anywhere along the x axis with no change in energy; thus deviations of the form $\Delta x d\bar{\psi}/dx$ must be—and are—exact eigenstates of (C12) with $\epsilon=0$. Equation (C35) is just a special case of this general observation.

Equation (C36) describes a stable localized distortion of $\bar{\psi}$ which is of no special concern to us. The deviation described by (C37) is not localized, however, and must represent the bottom of a second continuum of stable, plane-wave-like fluctuations. Note that the two continua, starting at $\epsilon=0$ [Eq. (C29)] and $\epsilon=2\alpha$ [Eq. (C37)] correspond exactly to the two continua of states determined by Eq. (C8) with $k=0$.

Finally, we must consider the transverse fluctuations described by Eq. (C11). Since only $w_{2,-1}$ in Eq. (C28) has a negative eigenvalue, it is the only state that need be considered in looking for unstable transverse deviations from $\bar{\psi}$. Thus we have, in Eq. (C13),

$$\lambda_{-1,q} = -\tfrac{1}{2}\alpha + q^2. \quad \text{(C38)}$$

Let $d_{\max}$ be the largest linear dimension of the sample perpendicular to the direction of the current. The Ginzburg-Landau boundary condition is that ψ have zero gradient normal to a surface through which no supercurrent flows. Thus the smallest allowed q other than zero is $\pi/d_{\max}$, and this mode will be unstable if

$$d_{\max} > \pi(2/\alpha)^{1/2} = \pi\sqrt{2}\xi(T). \quad \text{(C39)}$$

Equation (C39) is our criterion for a superconducting channel to be effectively one-dimensional.

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Electron-Tunneling Measurements on Lanthanum and Lanthanum-Lutetium Alloy Films

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Results of electron-tunneling measurements on evaporated films of fcc lanthanum and dhcp lanthanum-lutetium alloys are presented. The ratio of the zero-temperature energy gap to the temperature at which the energy gap vanishes for both the pure lanthanum and lanthanum-lutetium alloy samples varied from 3.41 to 3.58. If one fits the lowest-temperature data for the energy gap with a curve of the BCS temperature dependence, the values at intermediate temperatures fall below the weak-coupling BCS prediction. The conductance maxima for the (pure La)-Al_2O_3-Al diodes are larger than predicted by the weak-coupling BCS theory. The conductance maxima for the lutetium alloy samples are more nearly equal to the weak-coupling BCS values than are those of the pure samples. They were not significantly altered by the presence of small zero-voltage anomalies. Hence zero-voltage anomalies are not enhanced at temperatures below the superconducting transition temperature. No change in conductance as large as 0.1% was observed which could be associated with the second energy gap predicted by the multiband-superconductor theory of Kuper, Jensen, and Hamilton. Kondo's multiband-superconductor theory is consistent with the experimental results. It is shown that if the f band in Kondo's theory is approximately 20 meV or more higher than the Fermi level, then Kondo's theory reduces to a single-parameter theory having a gap equation identical in form to the BCS gap equation.

I. INTRODUCTION

THERE has been considerable speculation concerning the mechanism for superconductivity in lanthanum.[1–3] Similar elements such as scandium, yttrium, and lutetium are not superconducting down to 0.2°K while lanthanum is superconducting at around 6°K. It has been suggested that the Cooper pairs can make transitions to nearby f states and that these transitions are responsible for lanthanum's relatively high transition temperature.

Electron-tunneling measurements can provide information to test these theories. Previous measurements[4] gave an anomalously low ratio of the zero temperature energy gap $2\Delta(0)$ to transition temperature kT_c of 1.65±0.15. The measurements on face-centered cubic (fcc) lanthanum films and double hexagonal close-packed (dhcp) lanthanum-lutetium alloy films reported here give values for this ratio varying from 3.41 to 3.58. This value is in the range expected on the basis of BCS theory. The difference between the two results is due to sample preparation and will be discussed in Sec. II. Hauser's measurements[5] on evaporated lanthanum films are in reasonable agreement with those reported here, but the values he has obtained for the energy gap are smaller. Recently, Levinstein *et al.*[6] have performed measurements on bulk samples using the technique of point tunneling. Their average value of $2\Delta(0)/kT_c$ was 3.7, though they measured values for the ratio as low as 3.3. Section II describes the method of sample preparation and measuring technique, and gives the experimental results. Section III discusses the results in terms of the multiband-superconductor theories. In particular, it is shown that Kondo's theory reduces to a single-parameter theory (like the BCS theory) if the f band is not too close to the Fermi level.

[1] J. Kondo, Progr. Theoret. Phys. (Kyoto) **29**, 1 (1963).
[2] C. G. Kuper, M. Anthony Jensen, and D. C. Hamilton, Phys. Rev. **134**, A15 (1964).
[3] D. C. Hamilton and M. Anthony Jensen, Phys. Rev. Letters **11**, 205 (1963).
[4] A. S. Edelstein and A. M. Toxen, Phys. Rev. Letters **17**, 196 (1966).
[5] J. J. Hauser, Phys. Rev. Letters **17**, 921 (1966).
[6] H. J. Levinstein, V. G. Chirba, and J. E. Kunzler, Phys. Letters **24A**, 362 (1967).