Theory of a Zeeman Laser. I*

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(Received 4 May 1967)

A theory of a laser subject to a uniform dc magnetic field at any angle to the maser axis is given in extension of a non-Zeeman treatment by Lamb. The electromagnetic field is treated classically for a general state of polarization in a cavity with any desired degree of cavity anisotropy. The active medium consists of thermally moving atoms of varying isotopic abundance which have two electronic states with arbitrary angular momenta and may have hyperfine structure. The self-consistency requirement that a quasistationary field should be sustained by the induced polarization leads to equations which determine the amplitudes and frequencies of multimode oscillation as functions of the laser parameters. A set of computer programs has been written which covers the theory very generally, yielding among other things graphs of intensities and beat frequencies versus time, cavity detuning, or magnetic field strength.

I. INTRODUCTION

IN a recent paper,[1] a multimode theory of an optical maser (henceforth referred to as the scalar theory) was given in which the electromagnetic field was assumed to be classical[2] and plane-polarized, and the effects of atomic degeneracy were ignored. In the present paper, we extend the results of that theory to deal with general states of electric field polarization and cavity frequency and loss anisotropy, a dc magnetic field at arbitrary angles to the maser axis, and an active medium consisting of atoms having two electronic states each of which may have arbitrary angular momenta and hyperfine structure (hfs). We characterize the isotopes in the medium by their fractional abundances and atomic-line centers. Our discussion is aimed specifically, but not inevitably, at the weak-field Zeeman effect (F is assumed to be a good quantum number). Pressure effects and spontaneous emission from the upper to lower maser levels are neglected[3] in this paper.

The calculation differs from that of the scalar theory, notably in that the time integrations involve an averaged density matrix with many elements instead of a pure case with four, and the constitutive relation for the fictitious conductivity is tensorial. It differs, moreover, in that the third-order integrals are evaluated exactly[4] and the form of the amplitude and frequency equations is computer-program-oriented. The coefficients appearing in the laser equations are so complicated that we prefer to leave them in complex form rather than in the real forms used in the scalar theory. These coefficients may be evaluated numerically in a completely general fashion. Explicit theoretical and experimental discussions of the laser equations for various special cases are given in paper II[5] and another paper,[6] respectively.

II. ELECTROMAGNETIC FIELD EQUATIONS

We write Maxwell's equations in mks units as

$$\operatorname{div}\mathbf{D}=0, \qquad \operatorname{curl}\mathbf{E}=-\partial\mathbf{B}/\partial t,$$
$$\operatorname{div}\mathbf{B}=0, \qquad \operatorname{curl}\mathbf{H}=\mathbf{J}+\partial\mathbf{D}/\partial t, \tag{1}$$

where

$$\mathbf{D}=\epsilon_0\mathbf{E}+\mathbf{P}, \qquad \mathbf{B}=\mu_0\mathbf{H}. \tag{2}$$

Here $\mathbf{P}$ is the polarization and $\mathbf{J}$ is the current density. The vector $\mathbf{P}$ will be used to describe the induced atomic polarization of the active medium. It is desirable to provide for different cavity resonant frequencies for linearly polarized radiation along orthogonal Cartesian axes transverse to the maser axis. This may be accomplished using a real symmetric second-rank susceptibility tensor χ_s. It is also desirable to provide for

* The work of two of the authors (M. Sargent III and W. E. Lamb, Jr.) was supported in part by the National Aeronautics and Space Administration, in part by the General Physics Division, Office of Scientific Research, U.S. Air Force, and for M. Sargent in part by Bell Telephone Laboratories. The main results of the paper were given at the American Physical Society meeting, Mexico City, August, 1966. See M. Sargent III, Bull. Am. Phys. Soc. 11, 714 (1966). This paper is based in part on material submitted by Murray Sargent III, in partial fulfillment of the requirements for the degree of Doctor of Philosophy at Yale University.

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[1] W. E. Lamb, Jr., Phys. Rev. **134**, A1429 (1964).

[2] A fully quantum-mechanical generalization of this theory has been given. See M. Scully, W. E. Lamb, Jr. and M. J. Stephen, in *Physics of Quantum Electronics*, edited by P. L. Kelley, B. Lax, and P. E. Tannenwald (McGraw-Hill Book Company, Inc., New York, 1966), p. 759; M. Scully and W. E. Lamb, Jr., Phys Rev. Letters **16**, 853 (1966). However, the semiclassical theory of Ref. 1 is adequate for our purposes.

[3] It is hoped to modify this theory to include pressure effects along the lines of a paper by B. L. Gyorffy and W. E. Lamb, Jr. (to be published). A nonrigorous method of including some of these effects in the present theory is given in Sec. VIII of paper II, Ref. 5.

[4] See Appendix B. The authors are very grateful to B. L. Gyorffy, who demonstrated how to do integrals of a similar form. See B. L. Gyorffy, thesis, Yale University, 1965 (unpublished) and B. L. Gyorffy and W. E. Lamb, Jr., Phys. Rev. (to be published).

[5] M. Sargent III, W. E. Lamb, Jr., and R. L. Fork, following paper, Phys. Rev. **164**, 450 (1967).

[6] W. J. Tomlinson and R. L. Fork, this issue, Phys. Rev. **164**, 466 (1967).

circular birefringence such as could be produced by a Faraday rotator. Accordingly, we introduce a real antisymmetric second-rank tensor χ_a which has the dimensions of a susceptibility. Combining these effects, one has the cavity polarization vector

$$\mathbf{P}^{(\mathrm{cav})}=[\chi_s-\nu^{-1}\chi_a(\partial/\partial t)]\cdot\mathbf{E}. \tag{3}$$

It will be preferable to allow for different effective cavity lengths along the desired component directions of the electric field in its explicit z dependence [see Eq. (7)]. This then requires that the diagonal elements of χ_s are zero. Corresponding cavity losses may be introduced by using the loss current density vector

$$\mathbf{J}^{(\mathrm{loss})}=[\mathfrak{o}_s-\nu^{-1}\mathfrak{o}_a(\partial/\partial t)]\cdot\mathbf{E}, \tag{4}$$

where $\mathfrak{o}_s$ is a real symmetric conductivity tensor and $\mathfrak{o}_a$ is a real antisymmetric tensor with dimensions of a conductivity. Using the fact that a current density is equivalent to the time rate of change of a polarization, (3) and (4) may be combined to give the current density

$$\mathbf{J}=[\mathfrak{o}'-\nu^{-1}\mathfrak{o}''(\partial/\partial t)]\cdot\mathbf{E}, \tag{5}$$

in which $\mathfrak{o}'=\mathfrak{o}_s+\nu\chi_a$ and $\mathfrak{o}''=\mathfrak{o}_a-\nu\chi_s$. The second-rank tensors $\mathfrak{o}'$ and $\mathfrak{o}''$ are real and have in general no symmetry properties.

The wave equation for the electric field becomes

$$\text{curl curl }\mathbf{E}+\mu_0[\mathfrak{o}'-\nu^{-1}\mathfrak{o}''(\partial/\partial t)]\times\partial\mathbf{E}/\partial t+\mu_0\epsilon_0\partial^2\mathbf{E}/\partial t^2=-\mu_0\partial^2\mathbf{P}/\partial t^2. \tag{6}$$

Neglecting the small transverse spatial variation of the field, one may replace the curl curl $\mathbf{E}$ by $-\partial^2\mathbf{E}/\partial z^2$, where the z coordinate lies along the maser axis. The field may then be expanded in the form

$$\mathbf{E}(z,t)=\tfrac{1}{2}\sum_{i=1}^{2}\sum_n\{\mathbf{e}_iE_{ni}(t)\times\exp[-i(\nu_{ni}t+\varphi_{ni}(t))]+\text{c.c.}\}U_{ni}(z), \tag{7}$$

where the $\mathbf{e}_i$ are any two (possibly complex) orthogonal unit vectors in the x-y plane, the amplitudes $E_{ni}(t)$ and phases $\varphi_{ni}(t)$ are real, slowly varying functions of time, and $U_{ni}(z)$ are the real eigenfunctions corresponding to the nth longitudinal normal mode (Fox and Li's TEM_{00n}) for $\mathbf{e}_i$. We will use $U_{ni}(z)=\sin K_nz$ in this paper. The polarization $\mathbf{P}(z,t)$ may similarly be written in the form

$$\mathbf{P}(z,t)=\tfrac{1}{2}\sum_{i=1}^{2}\sum_n\{\mathbf{e}_i\mathcal{P}_{ni}(t)\times\exp[-i(\nu_{ni}t+\varphi_{ni}(t))]+\text{c.c.}\}U_{ni}(z), \tag{8}$$

where $\mathcal{P}_{ni}(t)$ is a complex slowly varying function of time which will be referred to as a complex polarization. Various choices of the basis $\{\mathbf{e}_1,\mathbf{e}_2\}$ will be used after the general self-consistency equations (12) and (13) are derived. It is convenient to introduce here the standard set of Cartesian axes $\{\mathbf{i},\mathbf{j},\mathbf{k}\}$ to describe the laser and $\{\mathbf{i},\mathbf{j}',\mathbf{k}'\}$ to describe the magnetic field as depicted in Fig. 1.

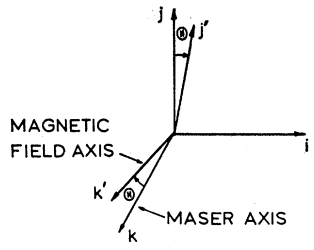

FIG. 1. The basis of mutually orthogonal unit vectors $\{\mathbf{i},\mathbf{j},\mathbf{k}\}$ is chosen to describe the position of the laser such that $\mathbf{k}$ is parallel to and $\mathbf{i}$ and $\mathbf{j}$ perpendicular to the laser axis. A similar basis of mutually orthogonal unit vectors $\{\mathbf{i},\mathbf{j}',\mathbf{k}'\}$ is chosen to describe the magnetic field such that $\mathbf{k}'$ is parallel to and $\mathbf{i}$ and $\mathbf{j}'$ perpendicular to this field. Without loss of generality, the unit vector $\mathbf{i}$ is chosen to be the same for both bases. The bases are referred to as the maser and atomic bases, respectively. The angle $\Theta=\cos^{-1}(\mathbf{k}'\cdot\mathbf{k})$.

To obtain the self-consistency equations, one now substitutes (7) and (8) into (6), projects the result onto the unit vectors $\mathbf{e}_i$, and then onto the Fox-Li mode $U_{ni}(z)$, neglecting terms containing d^2E_{ni}/dt^2, $\ddot{\varphi}_{ni}$, $\sigma'_{ii'}\dot{\varphi}_{ni'}$, $\sigma'_{ii'}\dot{E}_{ni'}$, and $\dot{\varphi}_{ni}\dot{E}_{ni}$. In so doing it is convenient to use the expressions for $\mathbf{E}$ and $\mathbf{P}$ without carrying along the complex conjugates. This procedure is easily validated provided that the equations are linear and real. In our equations neither condition is satisfied in general. However, dropping the complex conjugates amounts to equating the coefficient of the positive frequency term $\exp(-i\nu t)$ to zero separately from that of the negative frequency term $\exp i\nu t$, and to making the rotating-wave approximation in the derivation of $\mathbf{P}$. In these approximations one finds that $\partial/\partial t$ in Eq. (6) simply yields the multiplicative factor $-i\nu$. Thus the conductivity tensor in the constitutive relation (5) for $\mathbf{J}$ has the matrix representation

$$\sigma_n=\begin{pmatrix}\sigma'_{n11} & \sigma'_{n12}+i\sigma''_{n12}\\ \sigma'_{n21}+i\sigma''_{n21} & \sigma'_{n22}\end{pmatrix} \tag{9}$$

in the basis $\{\mathbf{e}_1,\mathbf{e}_2\}$, where we have indicated a possible n dependence of σ. It is convenient to define the anisotropy matrix

$$G_n=\begin{pmatrix}g_{n11} & g_{n12}\\ g_{n21} & g_{n22}\end{pmatrix}=(\epsilon_0\nu)^{-1}\sigma_n+i(2c/\nu)\begin{pmatrix}K_{n1}-K_n & 0\\ 0 & K_{n2}-K_n\end{pmatrix}, \tag{10}$$

where $K_n=n\pi/L$, and L is the length of the cavity. In a representation in which G_n is diagonal, the $\mathrm{Re}(g_{nii})$ are the reciprocals of the cavity Q's and $\frac{1}{2}\nu\ \mathrm{Im}(g_{nii})$ are frequency displacements of the ith polarization from the nonbirefringent cavity frequency $\Omega_n=n\pi c/L$.

Carrying out the substitutions indicated, noting that

$$(\nu_{ni}+\dot{\varphi}_{ni})^2-\Omega_n^2\sim 2\nu(\nu_{ni}+\dot{\varphi}_{ni}-\Omega_n),$$

and multiplying through by

$$-(2\nu)^{-1}\exp[-i(\nu_{ni}t+\varphi_{ni})],$$

one finds for the component equation

$$(\nu_{ni}+\dot{\varphi}_{ni}-\Omega_n)E_{ni}+i[\dot{E}_{ni}+\tfrac{1}{2}\nu\sum_{i'=1}g_{nii'}(\exp i\psi_{nii'}E_{ni'})]$$
$$=-\tfrac{1}{2}(\nu/\epsilon_0)\mathcal{P}_{ni}, \quad (11)$$

where $\psi_{nii'}=\nu_{ni}t+\varphi_{ni}-\nu_{ni'}t-\varphi_{ni'}$ are relative phase angles. Equating the real and imaginary parts separately to zero, one has the self-consistency equations

$$\dot{E}_{ni}+\tfrac{1}{2}\nu\sum_{i'=1}^{2}\mathrm{Im}(ig_{nii'}\exp i\psi_{nii'})E_{ni'}=-\tfrac{1}{2}(\nu/\epsilon_0)\ \mathrm{Im}\mathcal{P}_{ni}, \quad (12a)$$

$$(\nu_{ni}+\dot{\varphi}_{ni}-\Omega_n)E_{ni}+\tfrac{1}{2}\nu\sum_{i'=1}^{2}\mathrm{Re}(ig_{nii'}\exp i\psi_{nii'})E_{ni'}$$
$$=-\tfrac{1}{2}(\nu/\epsilon_0)\ \mathrm{Re}\mathcal{P}_{ni}. \quad (13a)$$

For typographical simplicity we adopt the convention that boldface subscripts index both polarizations for each spatial Fourier mode. Specifically in (12a) and (13a) we replace the subscripts ni by $\mathbf{n}$ and ni' by $\mathbf{n}'$. In this notation we have

$$\dot{E}_{\mathrm{n}}+\tfrac{1}{2}\nu\sum_{i'=1}^{2}\mathrm{Im}(ig_{\mathrm{nn}'}\exp i\psi_{\mathrm{nn}'})E_{\mathrm{n}'}=-\tfrac{1}{2}(\nu/\epsilon_0)\ \mathrm{Im}\mathcal{P}_{\mathrm{n}}, \quad (12b)$$

$$(\nu_{\mathrm{n}}+\dot{\varphi}_{\mathrm{n}}-\Omega_n)E_{\mathrm{n}}+\tfrac{1}{2}\nu\sum_{i'=1}^{2}\mathrm{Re}(ig_{\mathrm{nn}'}\exp i\psi_{\mathrm{nn}'})E_{\mathrm{n}'}$$
$$=-\tfrac{1}{2}(\nu/\epsilon_0)\ \mathrm{Re}\mathcal{P}_{\mathrm{n}}. \quad (13b)$$

If one chooses a set of unit vectors for which the anisotropy matrix is diagonal ($g_{12}=g_{21}=0$), these equations reduce to

$$\dot{E}_{\mathrm{n}}+\tfrac{1}{2}(\nu/Q_{\mathrm{n}})E_{\mathrm{n}}=-\tfrac{1}{2}(\nu/\epsilon_0)\ \mathrm{Im}\mathcal{P}_{\mathrm{n}}, \quad (14)$$

$$(\nu_{\mathrm{n}}+\dot{\varphi}_{\mathrm{n}}-\Omega_{\mathrm{n}})E_{\mathrm{n}}=-\tfrac{1}{2}(\nu/\epsilon_0)\ \mathrm{Re}\mathcal{P}_{\mathrm{n}}, \quad (15)$$

where

$$\Omega_{\mathrm{n}}=\Omega_n+\tfrac{1}{2}\nu\ \mathrm{Im}(g_{\mathrm{nn}}). \quad (16)$$

In particular, if the losses are independent of polarization, (14) and (15) describe the amplitudes and frequencies for any set of orthogonal unit vectors.

Two bases are useful in formulating the present problem: the x-y basis for which

$$\mathbf{e}_1=\mathbf{i}, \qquad \mathbf{e}_2=\mathbf{j}, \quad (17)$$

and the $\pm$ (circularly polarized) basis for which

$$\mathbf{e}_1=\boldsymbol{\epsilon}_-=2^{-1/2}(\mathbf{i}+i\mathbf{j}),$$
$$\mathbf{e}_2=\boldsymbol{\epsilon}_+=2^{-1/2}(\mathbf{i}-i\mathbf{j}). \quad (18)$$

The $\pm$ basis is particularly convenient when the magnetic field is directed along the maser axis, for electric-dipole transitions in which the magnetic quantum number changes by ±1 contribute to $E_{n\pm}$, respectively (changes of 0 are not allowed). This one-to-one correspondence simplifies the laser equations considerably. A similar correspondence occurs for a transverse magnetic field in the x-y representation, for here magnetic quantum number changes of ±1 correspond to E_{nx} and those of 0 to E_{ny} (x is chosen perpendicular to the magnetic field, y parallel—see Fig. 1). One might like to consider a magnetic field at some arbitrary angle to the maser axis as a superposition of two fields, one axial and one transverse, and treat each in its favored representation. Unfortunately, the nonlinear character of the polarization prevents one from transforming the maser equations from one basis to another in any simple fashion and the polarization for the arbitrarily oriented field is best calculated using one representation consistently.

The situation is simpler in the case of the cavity anisotropies, for they enter the laser equations linearly. Thus if G is diagonal in one representation, one may carry out a similarity transformation to find it in any other, and then use (12) and (13) to find the self-consistency equations. In particular, if G is diagonal in the x-y basis and the magnetic field is axial, one may transform G to the $\pm$ basis and thereby simplify the polarization calculation (and the ultimate laser equations). The transformed G matrix $G'=SGS^{-1}$, where the unitary matrix S is the composite of the $\pm$ unit vectors written in row-vector form in the x-y basis

$$S=\tfrac{1}{2}\sqrt{2}\begin{pmatrix}1 & i\\ 1 & -i\end{pmatrix}. \quad (19)$$

Thus, in the circularly-polarized basis, G becomes

$$G'=\tfrac{1}{2}\begin{pmatrix}1 & i\\ 1 & -i\end{pmatrix}\begin{pmatrix}g_{nx} & 0\\ 0 & g_{ny}\end{pmatrix}\begin{pmatrix}1 & 1\\ -i & i\end{pmatrix}$$
$$=\tfrac{1}{2}\begin{pmatrix}g_{nx}+g_{ny} & g_{nx}-g_{ny}\\ g_{nx}-g_{ny} & g_{nx}+g_{ny}\end{pmatrix}, \quad (20)$$

where g_{nx} and g_{ny} are the diagonal elements of G.

On the other hand, if G is diagonal in the circularly polarized basis, it has the form in the x-y basis [the transformation matrix is the conjugate transpose of (19)]

$$G'=\tfrac{1}{2}\begin{pmatrix}g_{n+}+g_{n-} & i(g_{n+}-g_{n-})\\ -i(g_{n+}-g_{n-}) & g_{n+}+g_{n-}\end{pmatrix}, \quad (21)$$

where $g_{n\pm}$ are the diagonal elements of G.

If one should choose the x-y basis at some angle with respect to the axes which diagonalize the anisotropy matrix, the x and y fields will be mixed. As above, one

may find the form of the anisotropy matrix by carrying out a similarity transformation, this time with the orthogonal matrix

$$S=\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}, \tag{22}$$

where θ is the angle of rotation between the chosen axes $\{\mathbf{i}^\dagger, \mathbf{j}^\dagger\}$ and those that diagonalize G, $\{\mathbf{i}, \mathbf{j}\}$. The transformed G is

$$G'=\begin{pmatrix} g_{nx\dagger}\cos^2\theta+g_{ny\dagger}\sin^2\theta & \tfrac{1}{2}(g_{nx\dagger}-g_{ny\dagger})\sin\theta\cos\theta\\ \tfrac{1}{2}(g_{nx\dagger}-g_{ny\dagger})\sin\theta\cos\theta & g_{nx\dagger}\sin^2\theta+g_{ny\dagger}\cos^2\theta\end{pmatrix}, \tag{23}$$

where $g_{nx\dagger}$ and $g_{ny\dagger}$ are the diagonal elements of G. The most general relationships between any two sets orthogonal unit vectors is given in Appendix A.

III. POLARIZATION OF THE MEDIUM

The maser action takes place between two atomic levels a and b (see Fig. 2) which are separated by an electric-dipole transition and are characterized by the total angular momenta J_a and J_b and other quantum numbers n_a and n_b, respectively. The jth isotope of the active medium has nuclear spin I_j, average z component of velocity u_j, and fractional abundance a_j for which

$$\sum_j a_j=1, \tag{24}$$

where j runs over all isotopes present. For the sake of simplicity, we assume the magnetic field does not break down the coupling between I_j and J. If it is desired to consider operations in other regions, one may include a matrix diagonalization or use the Paschen-Back limit in the following.

A basis for a matrix representation consists of the set of eigenvectors

$$|\, nI_jJFm\rangle, \tag{25}$$

where $n=n_a, n_b$; $F=J_a+I_j, \cdots, |\,J_a-I_j\,|$ for $J=J_a$, $F=J_b+I_j, \cdots, |\,J_b-I_j\,|$ for $J=J_b$, and m runs over the corresponding sublevels. For typographical simplicity, we have not subscripted F and m with the isotopic index j and will refrain from so subscripting the resonant frequencies and atomic decay constants (49) and line strengths (31) as well. We assume the decay operator Γ (introduced phenomenologically to describe the radiative decay of the atomic levels) is diagonal in this basis with the elements $\Gamma_{mm}=\gamma_m$. The Hamiltonian $\hbar\mathcal{H}$ has the diagonal elements

$$\hbar W(a',j)=\hbar W(F_a,j)+\mu_B H g_F a' \tag{26}$$

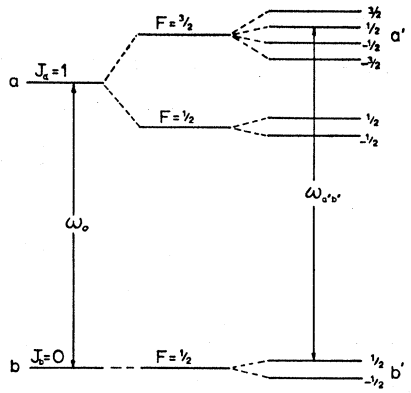

FIG. 2. Possible level diagram showing how the levels a and b of the active medium might be split by hyperfine structure and magnetic field. The nuclear spin of the jth isotope is I_j (here, $\frac{1}{2}$), the total angular momenta are J_a (here, 1) and $J_b(=0)$, respectively; a' and a'' are magnetic quantum numbers for sublevels of a, while b' and b'' are those for b; ω_0 is a zero magnetic field optical frequency between an F level of a and one of b, and $\omega_{a'b'}$ is a frequency between a sublevel of a and one of b. Associated with level a (b), are the g values g_a (g_b) (one for each F value).

and similar elements with a replaced by b, where a' is a magnetic quantum number for a sublevel of a, $\hbar W(F_a, j)$ is the zero-field energy of the F_a level for the jth isotope, g_F is the Lande g factor for this F, H is the magnetic field strength, and μ_B is the Bohr magneton. The off-diagonal elements of the Hamiltonian $\hbar V_{a'b'}$ are the matrix elements of the time-dependent perturbation energy

$$\hbar V_{a'b'}=-\langle n_aI_jJ_aF_aa'\,|\,e\mathbf{E}\cdot\mathbf{r}\,|\,n_bI_jJ_bF_bb'\rangle \tag{27}$$

associated with the optical-frequency electric field. Here e is the charge of the electron, and the position vector $\mathbf{r}$ expressed in the atomic basis (cf. Fig. 1) is

$$\mathbf{r}=x\mathbf{i}+y\mathbf{j}'+z\mathbf{k}'. \tag{28}$$

For the purpose of calculating matrix elements, $\mathbf{r}$ is most conveniently written in the form

$$\mathbf{r}=\tfrac{1}{2}r\sin\theta[(\mathbf{i}-i\mathbf{j}')\exp i\varphi+(\mathbf{i}+i\mathbf{j}')\exp -i\varphi]+r\cos\theta\mathbf{k}', \tag{29}$$

where θ and φ are the polar and aximuthal angles of $\mathbf{r}$ with respect to the atomic axes $\{\mathbf{i}, \mathbf{j}', \mathbf{k}'\}$. Substituting (29) into (27), one has

$$\hbar V_{a'b'}=-\wp_{a'b'}[\mathbf{E}\cdot(\mathbf{i}-i\mathbf{j}')\delta_{a',b'+1}+\mathbf{E}\cdot(\mathbf{i}+i\mathbf{j}')\delta_{a',b'-1}+\mathbf{E}\cdot\mathbf{k}'\delta_{a',b'}], \tag{30}$$

where the electric dipole matrix elements $\wp_{a'b'}$ are given by

$$\begin{aligned}\wp_{a'b'}&=\langle F_aa'\,|\,\tfrac{1}{2}er\sin\theta\exp[\pm i\varphi]\,|\,F_bb'\rangle, && a'=b'\pm1,\ F_a=F_b\pm1, 0\\ &=\langle F_aa'\,|\,er\cos\theta\,|\,F_bb'\rangle, && a'=b',\ F_a=F_b\pm1, 0\\ &=0, && \text{otherwise}\end{aligned} \tag{31}$$

with the explicit values[7]

$$\begin{aligned}
\mathcal{P}_{a'b'} &= \mp\tfrac{1}{2}\mathcal{P}[(F_b \pm a')(F_b \pm a' + 1)]^{1/2} && a' = b' \mp 1 \\
&= \mathcal{P}(F_b^2 - a'^2)^{1/2} && a' = b' && F_a = F_b - 1 \\
&= \tfrac{1}{2}\mathcal{P}[(F_a \mp a')(F_a \pm a' + 1)]^{1/2} && a' = b' \mp 1 \\
&= \mathcal{P}a' && a' = b' && F_a = F_b \\
&= \mp\tfrac{1}{2}\mathcal{P}[(F_a \pm b')(F_a \pm b' + 1)]^{1/2} && a' = b' \pm 1 \\
&= \mathcal{P}(F_a^2 - b'^2)^{1/2} && a' = b' && F_a = F_b + 1,
\end{aligned} \tag{32}$$

and $\mathcal{P}$ is the reduced matrix element $(n_a F_a \,||\, er \,||\, n_b F_b)$. Substituting (7) into (30), one has[8]

$$V_{a\,b'} = -\tfrac{1}{2}\mathcal{P}_{a'b'}\hbar^{-1}\sum_{\mu'} U_{\mu}(z) E_{\mu'} \times \sum_{q=-1}^{1} \delta_{a',b'+q}\{ f_q(p_\mu) \exp[-i(\nu_{\mu'}t + \varphi_{\mu'})] + f_{-q}(p_\mu)^* \exp[+i(\nu_{\mu'}t + \varphi_{\mu'})]\}, \qquad V_{b'a'} = V_{a'b'}^*, \tag{33}$$

where p_μ indexes the two polarizations of μ', q is $a'-b'$, and the "direction cosines" $f_q(p_\mu)$ are given by

$$f_{\pm 1}(p_\mu) = \mathbf{e}(p_\mu)\cdot(\mathbf{i} \mp i\mathbf{j}'), \qquad f_0(p_\mu) = \mathbf{e}(p_\mu)\cdot\mathbf{k}'. \tag{34}$$

The equation of motion for a density matrix

$$\rho(j, \alpha, z_0, t_0, v, t)$$

describing the pure case in which an atom of the jth isotope is excited to the eigenstate α, i.e., $|nFm\rangle$ at place z_0, time t_0, and with z component of velocity v, is

$$\dot{\rho}(j, \alpha, z_0, t_0, v, t) = -i[\mathfrak{IC}, \rho] - \tfrac{1}{2}[\Gamma\rho + \rho\Gamma], \tag{35}$$

where $\hbar\mathfrak{IC}$ is the Hamiltonian. The average electric-dipole moment $\mathbf{p}$ described by this density matrix is

$$\mathbf{p} = \operatorname{spur}(\rho e\mathbf{r}). \tag{36}$$

The macroscopic polarization $\mathbf{P}(z, t)$ is contributed to by all atoms of the medium which arrive at z at time t regardless of j, α, z_0, t_0, and v. Assuming that the excitation mechanism always excites an atom to an eigenstate α, one has

$$\mathbf{P}(z, t) = \sum_j a_j \sum_\alpha \int_{-\infty}^{t} dt_0 \int dz_0 \int dv \lambda_\alpha(j, z_0, t_0, v) \times \operatorname{spur}(\rho e\mathbf{r})\delta[z - z_0 - v(t - t_0)], \tag{37}$$

where $\lambda_\alpha(j, z_0, t_0, v)$ is the number of atoms of isotope j excited to the state α per unit time per unit volume.

One can proceed as in the scalar theory[1] to integrate the equations of motion for the pure case through third order and then to average over z_0, t_0, α, and v. It is simpler, however, to capitalize on the fact that the effective Hamiltonian $\hbar\mathfrak{IC}$ does not depend[9] on z_0, t_0 and α by averaging the pure case matrix over these variables before the integration of the equations of motion. We define the population matrix $\rho(j, z, v, t, \hat{t})$ by

$$\rho(j, z, v, t, \hat{t}) = \sum_\alpha \int_{-\infty}^{\hat{t}} dt_0 \int dz_0 \lambda_\alpha(j, z_0, t_0, v)\rho(j, \alpha, z_0, t_0, v, \hat{t}) \times \delta[z - z_0 - v(t - t_0)]. \tag{38}$$

Here j, z, v, and t label an ensemble of atoms of isotope j with velocity v and the property that at time t the ensemble is at position z. The time dependence of $\rho(j, z, v, t, \hat{t})$ is given by $\hat{t}$ which does not necessarily equal t. To find its differential equation of motion, we differentiate with respect to $\hat{t}$;

$$d\rho/d\hat{t} = \lambda + \sum_\alpha \int_{-\infty}^{\hat{t}} dt_0 \int dz_0 \lambda_\alpha \dot{\rho}(j, \alpha, z_0, t_0, v, \hat{t}) \times \delta[z - z_0 - v(t - t_0)]. \tag{39}$$

Here λ is the excitation operator with elements λ_α which we assume are slowly varying. Substituting (35) for $\dot{\rho}(j, \alpha, z_0, t_0, v, \hat{t})$, evaluating $\lambda_\alpha(j, z_0, t_0, v)$ at $t_0 = t$ and exchanging the order of the Hamiltonian operator and integrations, one has

$$d\rho(j, z, v, t, \hat{t})/d\hat{t} = -i[\mathfrak{IC}, \rho(j, z, v, t, \hat{t})] - \tfrac{1}{2}[\Gamma\rho(j, z, v, t, \hat{t}) + \rho\Gamma] + \lambda(j, z, v, t). \tag{40}$$

The $\mathbf{n}$th spatial Fourier component $P_{\mathbf{n}}(t)$ of the macroscopic polarization $\mathbf{P}(z, t)$ is given by

$$P_{\mathbf{n}}(t) = 2L^{-1}\sum_j a_j \int_0^L dz \sin K_{\mathbf{n}} z \times \int dv \operatorname{spur}[\rho(j, z, v, t, t) e\mathbf{e}(i)\cdot\mathbf{r}], \tag{41}$$

[7] E. U. Condon and G. H. Shortley, *The Theory of Atomic Spectra*, (Cambridge University Press, New York, 1935), p. 63. If more than one pair of F values is involved, one has to include additional Clebsch-Gordon coefficients as given on p. 69 of this reference.

[8] Because $\mathfrak{y}$, $\mathfrak{o}$, and $\mathfrak{d}$ are not available as subscripts, we use μ', ρ', and σ' instead.

[9] What is required is $V(t') = -(\mathcal{P}/\hbar)E[z - v(t - t')]$, which does not depend on t_0. See Ref. 1, page A1432, Eqs. (33) and (34).

in which we have taken $\hat{t}=t$. Defining

$$\rho_{a'b',\mathrm{n}}(t)=2L^{-1}\sum_j a_j\int_0^L dz\,\sin K_\mathrm{n}z \times\int dv\;\rho_{a'b'}(j,z,v,t,t) \quad (42)$$

and using (34), one has the explicit form for $\mathbf{P}_\mathrm{n}(t)$:

$$P_\mathrm{n}(t)=\tfrac{1}{2}\sum_{a'}\sum_{b'}\rho_{a'b',\mathrm{n}}\mathcal{D}_{b'a'}[\delta_{b',a'+1}f_{+1}(i)+\delta_{b',a'-1}f_{-1}(i)+\delta_{b',a'}f_0(i)]+\text{c.c.} \quad (43)$$

Extracting the factor $\exp[-i(\nu_\mathrm{n}t+\varphi_\mathrm{n})]$ as indicated in (8), one has

$$\mathcal{P}_\mathrm{n}(t)=\sum_{a'}\sum_{b'}\rho_{a'b',\mathrm{n}}\exp i(\nu_\mathrm{n}t+\varphi_\mathrm{n})\mathcal{D}_{b'a'}\sum_{k=-1}^{1}f_k(i)^*\delta_{a',b'+k}. \quad (44)$$

IV. INTEGRATION OF THE EQUATIONS OF MOTION

The matrix equation of motion for $\rho(j,z,v,t,\hat{t})$ has components

$$\dot{\rho}_{a'b'}=-(i\omega_{a'b'}+\gamma_{a'b'})\rho_{a'b'}+i\sum_{a''}V_{a''b'}\rho_{a'a''}-i\sum_{b''}V_{a'b''}\rho_{b''b'}, \quad (45)$$

$$\dot{\rho}_{a'a''}=-(i\omega_{a'a''}+\gamma_{a'a''})\rho_{a'a''}+i\sum_{b''}(V_{b''a''}\rho_{a'b''}-V_{a'b''}\rho_{b''a''})+\lambda_{a'}\delta_{a'a''}, \quad (46)$$

$$\dot{\rho}_{b''b'}=-(i\omega_{b''b'}+\gamma_{b''b'})\rho_{b''b'}+i\sum_{a''}(V_{a''b'}\rho_{b''a''}-V_{b''a''}\rho_{a''b'})+\lambda_{b'}\delta_{b'b''}, \quad (47)$$

$$\dot{\rho}_{b'a'}=\dot{\rho}_{a'b'}{}^*, \quad (48)$$

where the dot indicates differentiation with respect to $\hat{t}$, a'' and b'' are magnetic quantum numbers for sublevels of a and b, respectively, and

$$\omega_{\alpha\alpha'}=W(\alpha,j)-W(\alpha',j),$$

$$\gamma_{\alpha\alpha'}=\tfrac{1}{2}(\gamma_\alpha+\gamma_{\alpha'}), \qquad \alpha,\alpha'=a',b',a'',b''. \quad (49)$$

Equations (45)–(48) may be integrated to any desired order in the perturbation $V_{a'b'}$ by the iterative procedure used in the scalar theory.[1] Using (46) and (47), one has in zeroth order

$$\rho_{\alpha\alpha}{}^{(0)}(j,z,v,t,\hat{t})=\lambda_\alpha\int_{-\infty}^{\hat{t}}dt'\exp[-\gamma_\alpha(\hat{t}-t')]=\lambda_\alpha\int_0^\infty d\tau'\exp(-\gamma_\alpha\tau')=\lambda_\alpha/\gamma_\alpha, \quad (50)$$

where $\tau'=\hat{t}-t'$. Here the t' integration extends from the earliest initial time $-\infty$ to the time $\hat{t}$, whereas in the scalar theory it extended from t_0 to t.

Using (45) and (50), one has for the first-order contribution to $\rho_{a'b'}$

$$\rho_{a'b'}{}^{(1)}(j,z,v,t,\hat{t})=i\int_{-\infty}^{\hat{t}}dt'\exp[-(i\omega_{a'b'}+\gamma_{a'b'})(\hat{t}-t')][\rho_{a'a'}{}^{(0)}-\rho_{b'b'}{}^{(0)}]V_{a'b'}(t')$$

$$=iW_j(v)N_{a'b'}(z,t)\int_0^\infty d\tau' V_{a'b'}(t')\exp[-(i\omega_{a'b'}+\gamma_{a'b'})\tau'], \quad (51)$$

where we have taken the excitation rate densities to have the form $\lambda_\alpha=W_j(v)\Lambda_\alpha(z,t)$ and where

$$N_{a'b'}(z,t)=\Lambda_{a'}(z,t)\gamma_{a'}{}^{-1}-\Lambda_{b'}(z,t)\gamma_{b'}{}^{-1} \quad (52)$$

are the unsaturated population inversion densities for the magnetic sublevels a' and b'.

Using (46), (47), and (51), one has for the second-order components $\rho_{a'a''}{}^{(2)}$ and $\rho_{b''b'}{}^{(2)}$:

$$\rho_{a'a''}{}^{(2)}(j,z,v,t,\hat{t})=i\int_0^\infty d\tau'\exp[-(i\omega_{a'a''}+\gamma_{a'a''})\tau']\sum_{b''}[V_{b''a''}(t')\rho_{a'b''}{}^{(1)}(j,z,v,t,t')-V_{a'b''}(t')\rho_{b''a''}{}^{(1)}(j,z,v,t,t')]$$

$$=-W_j(v)\int_0^\infty d\tau'\int_0^\infty d\tau''\exp[-(i\omega_{a'a''}+\gamma_{a'a''})\tau']\sum_{b''}\{V_{b''a''}(t')N_{a'b''}(z,t)V_{a'b''}(t'')\exp[-(i\omega_{a'b''}+\gamma_{a'b''})\tau'']$$
$$-V_{a'b''}(t')N_{b''a''}(z,t)V_{b''a''}(t'')\exp[-(i\omega_{b''a''}+\gamma_{b''a''})\tau'']\}, \quad (53)$$

where $\tau''=t'-t''$ and

$$\rho_{b''b'}{}^{(2)}(j,z,v,t,\hat{t})=-W_j(v)\int_0^\infty d\tau'\int_0^\infty d\tau''\exp[-(i\omega_{b''b'}+\gamma_{b''b'})\tau']$$
$$\times\sum_{a''}\{V_{a''b'}(t')N_{b''a''}(z,t)V_{b''a''}(t'')\exp[-(i\omega_{b''a''}+\gamma_{b''a''})\tau'']$$
$$-V_{b''a''}(t')N_{a''b'}(z,t)V_{a''b'}(t'')\exp[-(i\omega_{a''b'}+\gamma_{a''b'})\tau'']\}. \quad (54)$$

Combining (45) with (53) and (54), one has the third-order components

$$\rho_{a'b'}^{(3)}(j,z,v,t,\hat{t}) = i\int_0^\infty d\tau' \exp[-(i\omega_{a'b'}+\gamma_{a'b'})\tau'][\sum_{a''} V_{a''b'}(t')\rho_{a'a''}^{(2)}(j,z,v,t,t') - \sum_{b''} V_{a'b''}(t')\rho_{b''b'}^{(2)}(j,z,v,t,t')]$$

$$= -iW_j(v)\int_0^\infty d\tau' \int_0^\infty d\tau'' \int_0^\infty d\tau''' \exp[-(i\omega_{a'b'}+\gamma_{a'b'})\tau'] \sum_{a''}\sum_{b''} (V_{a''b'}(t')$$
$$\times \exp[-(i\omega_{a'a''}+\gamma_{a'a''})\tau''] \{V_{b''a''}(t'')N_{a'b''}(z,t) \exp[-(i\omega_{a'b''}+\gamma_{a'b''})\tau'''$$
$$\times V_{a'b''}(t''') - V_{a'b''}(t'')N_{b''a''}(z,t) \exp[-(i\omega_{b''a''}+\gamma_{b''a''})\tau'''] V_{b''a''}(t''')\}$$
$$- V_{a'b''}(t') \exp[-(i\omega_{b''b'}+\gamma_{b''b'})\tau''] \{V_{a''b'}(t'')N_{b''a''}(z,t)$$
$$\times \exp[-(i\omega_{b''a''}+\gamma_{b''a''})\tau'''] V_{b''a''}(t''') - V_{b''a''}(t'')N_{a''b'}(z,t)$$
$$\times \exp[-(i\omega_{a''b'}+\gamma_{a''b'})\tau'''] V_{a''b'}(t''')\}), \tag{55}$$

where $\tau''' = t''-t'''$.

V. FIRST-ORDER THEORY

Except for the summations over magnetic sublevels and the use of the population matrix $\rho(j,z,v,t,\hat{t})$, the perturbation calculation proceeds exactly as in the scalar theory and we shall omit most of the details common to both. Combining (33), (44), (51), setting $\hat{t}=t$, and assuming Maxwellian velocity distributions[10] $W_j(v) = (u_j\pi^{1/2})^{-1} \exp[-(v/u_j)^2]$, one has for the complex polarization (44)

$$\mathcal{P}_n^{(1)} = -(\hbar K)^{-1}\sum_j (a_j/u_j)\sum_{a'}\sum_{b'} (\wp_{a'b'})^2 \sum_{k=-1}^{1} f_k(i)^* \delta_{a',b'+k} \sum_{\mu'} E_{\mu'} \exp i\psi_{n\mu'}$$
$$\times \sum_{q=-1}^{1} f_q(p_\mu)\delta_{a',b'+q} N_{a'b'(n-\mu')} Z[\gamma_{a'b'}+i(\omega_{a'b'}-\nu_{\mu'})], \tag{56}$$

where the relative phase angle

$$\psi_{n\mu'} = (\nu_n-\nu_{\mu'})t+\varphi_n-\varphi_{\mu'}, \tag{57}$$

the plasma dispersion function

$$Z(v) = iKu_j\int_0^\infty d\tau \exp[-v\tau-\tfrac{1}{4}K^2u_j^2\tau^2], \tag{58}$$

and the spatial Fourier components of the population inversion density

$$N_{a'b'(n-\mu')}(t) = L^{-1}\int_0^L dz N_{a'b'}(z,t) \cos(K_n-K_{\mu'})z. \tag{59}$$

We will often use the notation

$$\bar{N}_{a'b'} \equiv N_{a'b'(0)}(t) = L^{-1}\int_0^L dz N_{a'b'}(z,t). \tag{59'}$$

Using $\delta_{a',b'+q}$ to do the sum over q and interchanging the sum over μ' with the summations over a' and b', one has

$$\mathcal{P}_n^{(1)} = -(\hbar K)^{-1}\sum_{\mu'} E_{\mu'}(\exp i\psi_{n\mu'}) \sum_j (a_j/u_j) \sum_{k=-1}^{1} f_k(i)^* f_k(p_\mu) \sum_{a'}\sum_{b'} \delta_{a',b'+k} (\wp_{a'b'})^2 N_{a'b'(n-\mu')} Z[\gamma_{a'b'}+i(\omega_{a'b'}-\nu_{\mu'})]. \tag{60}$$

Neglecting terms with rapidly varying phase angles and using the self-consistency equations (12) and (13), one has

$$\dot{E}_n = \sum_{i'} \mathrm{Im}(\alpha_{nn'} \exp i\psi_{nn'}) E_{n'}, \tag{61}$$

$$\nu_n+\dot{\varphi}_n = \Omega_n + \sum_{i'} \mathrm{Re}(\alpha_{nn'} \exp i\psi_{nn'}) E_{n'}E_n^{-1}, \tag{62}$$

[10] A different velocity distribution will result in a function other than the plasma dispersion function. A substitution can be made in the final answers if desired.

where $\mathbf{n}\equiv ni$, $\mathbf{n}'\equiv ni'$, $i, i'=1, 2$, and

$$\alpha_{\mathrm{nn}'}=-\tfrac{1}{2}i\nu g_{\mathrm{nn}'}+\tfrac{1}{2}(\nu/\epsilon_0\hbar K)\sum_j(a_j/u_j)\sum_{k=-1}^{1}f_k(i)^*f_k(i')\sum_{a'}\sum_{b'}\delta_{a',b'+k}(\wp_{a'b'})^2N_{a'b'(\mathrm{n}-\mathrm{n}')}Z[\gamma_{a'b'}+i(\omega_{a'b'}-\nu_{\mathrm{n}'})]. \quad (63)$$

For zero magnetic field and any value of cavity tuning, this equation may be used to determine the minimum values of the population inversions $N_{a'b'(\mathrm{n}-\mathrm{n}')}$ for which oscillations can occur. The problem is simplified if we assume that $N_{a'b'(\mathrm{n}-\mathrm{n}')}$ has the form

$$N_{a'b'(\mathrm{n}-\mathrm{n}')}=\bar{N}D_{a',b'(\mathrm{n}-\mathrm{n}')}, \quad (64)$$

where $\bar{N}$ measures the level of excitation and the distribution $D_{a',b'(\mathrm{n}-\mathrm{n}')}$ is independent of excitation level and has the same form as $N_{a'b'(\mathrm{n}-\mu')}$ of (59). Then as $\bar{N}$ is increased from zero, it will reach a value $\bar{N}(\Omega_n)$ at which oscillations just begin. The minimum value of $\bar{N}(\Omega_n)$ as Ω_n is varied will be denoted by $\bar{N}_T$, corresponding to threshold. In the scalar theory, $\bar{N}_T$ was the excitation required for threshold oscillations when the cavity frequency Ω_n was tuned to the peak ω of the atomic resonance curve (central tuning). Here with a number of isotopes, $\bar{N}_T$ will correspond to some less obvious tuning which can be evaluated numerically.

In general, the threshold oscillations will have a polarization specified by the unit vector $\mathbf{e}'_1$ (not necessarily equal to $\mathbf{e}_1$) which maximizes the imaginary part of α'_{n11}, the α coefficient corresponding to $\mathbf{e}'_1$. The relations between this unit vector and the chosen basis $\{\mathbf{e}_1, \mathbf{e}_2\}$ are given in Appendix A along with the maximization procedure. Identifying the mode with polarization $\mathbf{e}'_1$ by the subscript $\mathbf{h}$ (highest Q), the starting condition is

$$\mathrm{Im}\,\alpha_{\mathrm{hh}}=0. \quad (65)$$

Combining this with (63), one has

$$\mathrm{Re}g_{\mathrm{hh}}=(\epsilon_0\hbar K)^{-1}\sum_j(a_j/u_j)\sum_k|f_k(p_h)|^2\sum_{a'}\sum_{b'}\delta_{a',b'+k}\bar{N}(\Omega_n)D_{a',b'(0)}(\wp_{a'b'})^2Z_i[\gamma_{a'b'}+i(\omega_{0j}-\nu_{\mathrm{h}})], \quad (66)$$

where ω_{0j} is the frequency at line center in zero magnetic field of the jth isotope.

It is useful to use the starting condition (66) to express $\alpha_{\mathrm{nn}'}$ (and other coefficients which will appear later) in terms of the ratio

$$\mathfrak{N}=\bar{N}/\bar{N}_T, \quad (67)$$

called the "relative excitation." Eliminating the factor $(\epsilon_0\hbar K)^{-1}$ in (63) and using (66) and (67), one has

$$\alpha_{\mathrm{nn}'}=-\tfrac{1}{2}i\nu g_{\mathrm{nn}'}+\tfrac{1}{2}\nu\,\mathrm{Re}(g_{\mathrm{hh}})\mathfrak{N}\frac{\sum_j(a_j/u_j)\sum_k f_k(i)^*f_k(i')\sum_{a'}\sum_{b'}\delta_{a',b'+k}(\wp_{a'b'})^2D_{a',b'(\mathrm{n}-\mathrm{n}')}Z[\gamma_{a'b'}+i(\omega_{a'b'}-\nu_{\mathrm{n}'})]}{\sum_j(a_j/u_j)\sum_k|f_k(p_h)|^2\sum_{a'}\sum_{b'}\delta_{a',b'+k}(\wp_{a'b'})^2D_{a',b'(0)}Z_i[\gamma_{a'b'}+i(\omega_{0j}-\nu_{\mathrm{h}t})]}, \quad (68)$$

where $\nu_{\mathrm{h}t}$ is the oscillation frequency corresponding to $\bar{N}_T$. Other coefficients ($\theta_{\mathrm{n}\mu'\rho'\sigma'}$) which will appear shortly can similarly be expressed in terms of $\mathfrak{N}$.

VI. HIGHER-ORDER TERMS

In evaluating higher-order terms, it is worth noting that only the positive-frequency terms of $V_{a'b'}$ given by (33) lead to resonant contributions to the density-matrix elements. That this is true is immediately revealed by formal integrations of (45), (46), and (47). Similarly, only the negative-frequency terms of $V_{b'a'}$ lead to resonant contributions to the density-matrix elements. Remembering these simplifications, we substitute appropriate terms of the form (33) and its complex conjugate into (53), interchange the dummy summation indices μ' and ρ' in the terms resulting from the second term of (53), and find

$$\begin{aligned}\rho_{a'a''}^{(2)}(j,z,v,t)=&-\tfrac{1}{8}[u_j(\sqrt{\pi})\hbar^2]^{-1}\{\exp[-(v/u_j)^2]\}\sum_{b''}\sum_{\mu'}\sum_{\rho'}E_{\mu'}E_{\rho'}(\exp i\psi_{\mu'\rho'})\\
&\times\cos[(K_{\mathrm{n}}-K_{\mu'})z]\wp_{a''b''}\wp_{a'b''}\sum_{q=-1}^{1}f_q(p_\mu)^*\delta_{a'',b''+q}\sum_{r=-1}^{1}f_r(p_\rho)\delta_{a',b''+r}\mathfrak{D}(\omega_{a'a''}-\nu_{\mu'}+\nu_{\rho'})\\
&\times\{N_{a'b''}\mathfrak{D}(\omega_{a'b''}-\nu_{\rho'}-Kv)-N_{b''a''}\mathfrak{D}(\omega_{b''a''}+\nu_{\mu'}-Kv)+N_{a'b''}\mathfrak{D}(\omega_{a'b''}-\nu_{\rho'}+Kv)\\
&-N_{b''a''}\mathfrak{D}(\omega_{b''a''}+\nu_{\mu'}+Kv)\}, \quad (69)\end{aligned}$$

where

$$\mathfrak{D}(\omega_{\alpha\alpha'}-\nu)=[\gamma_{\alpha\alpha'}+i(\omega_{\alpha\alpha'}-\nu)]^{-1}. \quad (70)$$

TABLE I. This table defines the N_{tw} in (74) in terms of variables like $N_{a'b'(n-\mu')}$, which is defined by (59). The simpler forms of the N_{tw} obtain if $K_n \approx K_{n'}$ because the only combinations of the mode indices μ, ρ, σ, and n corresponding to significant contributions to the amplitude- and frequency-determining equations are those for which $n=\mu-\rho+\sigma$. Other combinations correspond to rapidly varying contributions [see Eqs. (72), (81), and (82)] which average to zero in time. For reasons given in Sec. II of paper II (Ref. 5), the simpler forms obtain for arbitrary birefringence when the magnetic field is axial.

V_{tw}	$w=1$	2	3
$t=1$	$N_{a'b''(n-\mu'-\rho'+\sigma')}=N_{a'b'',2(\sigma'-\rho')}$	$N_{a'b''(n-\mu'+\rho'-\sigma')}=\tilde{N}_{a'b''}$	$N_{a'b''(n+\mu'-\rho'-\sigma')}=N_{a'b'',2(\mu'-\rho')}$
2	$N_{b''a''(n-\mu'+\rho'-\sigma')}=\tilde{N}_{b''a''}$	$N_{b''a''(n-\mu'-\rho'+\sigma')}=N_{b''a'',2(\sigma'-\rho')}$	$N_{b''a''(n+\mu'-\rho'-\sigma')}=N_{b''a'',2(\mu'-\rho')}$
3	$N_{b''a''(n-\mu'+\rho'-\sigma')}=\tilde{N}_{b''a''}$	$N_{b''a''(n+\mu'-\rho'-\sigma')}=N_{b''a'',2(\mu'-\rho')}$	$N_{b''a''(n-\mu'-\rho'+\sigma')}=N_{b''a'',2(\sigma'-\rho')}$
4	$N_{a''b'(n+\mu'-\rho'-\sigma')}=N_{a''b',2(\mu'-\rho')}$	$N_{a''b'(n-\mu'+\rho'-\sigma')}=\tilde{N}_{a''b'}$	$N_{a''b'(n-\mu'-\rho'+\sigma')}=N_{a''b',2(\sigma'-\rho')}$
x_w	$\tau'''-\tau'$	$\tau'''+\tau'$	$\tau'''+2\tau''+\tau'$

Integrating over velocity and exchanging the sums over μ' and ρ' with the sum over b'', one has

$$\rho_{a'a''}{}^{(2)}(j,z,t)=\tfrac{1}{8}(\hbar^2 K u_j)^{-1}\sum_{\mu'}\sum_{\rho'}E_{\mu'}E_{\rho'}(\exp i\psi_{\mu'\rho'})\cos[(K_{\mu'}-K_{\rho'})z]\sum_{q=-1}^{1}f_q(p_\mu)^*\sum_{r=-1}^{1}f_r(p_\rho)\sum_{b''}\delta_{a'',b''+q}\delta_{a',b''+r}$$
$$\times\wp_{a''b''}\wp_{a'b''}\mathfrak{D}(\omega_{a'a''}-\nu_{\mu'}+\nu_{\rho'})[N_{a'b''}iZ(\omega_{a'b''}-\nu_{\rho'})+N_{a''b''}iZ(\omega_{b''a''}+\nu_{\mu'})]. \tag{71}$$

One can obtain similar expressions for $\rho_{b''b'}{}^{(2)}$ using (54) in place of (53).

To find the third-order complex polarization $\mathcal{P}_n{}^{(3)}(t)$, we substitute appropriate variants of $V_{a'b'}$ as given by (33) into Eq. (55). In so doing, increased compactness results from using the dummy summation indices μ', ρ', and σ' in $V_{a''b'}$, $V_{b''a''}$, and $V_{a'b''}$, respectively, because the relative phase angles $\psi_{n\mu'\rho'\sigma'}$ all have the same form:

$$\psi_{n\mu'\rho'\sigma'}=(\nu_n-\nu_{\mu'}+\nu_{\rho'}-\nu_{\sigma'})t+\varphi_n-\varphi_{\mu'}+\varphi_{\rho'}-\varphi_{\sigma'}. \tag{72}$$

We then interchange summations over magnetic sublevel indices a', b', a'', and b'' with those over mode indices μ', ρ', and σ' and project on the nth mode $U_n(z)$. Using (44), we find

$$\mathcal{P}_n{}^{(3)}(t)=\tfrac{1}{16}(\hbar^3 K)^{-1}\sum_{\mu'}\sum_{\rho'}\sum_{\sigma'}E_{\mu'}E_{\rho'}E_{\sigma'}(\exp i\psi_{n\mu'\rho'\sigma'})\sum_j(a_j/u_j)\sum_k f_k(i)^*\sum_q f_q(p_\mu)\sum_r f_r(p_\rho)^*\sum_s f_s(p_\sigma)$$
$$\times\sum_{a'}\sum_{b'}\sum_{a''}\sum_{b''}\delta_{a',b'+k}\delta_{a'',b'+q}\delta_{b'',a''-r}\delta_{a',b''+s}\wp_{b'a'}\wp_{a''b'}\wp_{a'''b''}\wp_{a'b''}\sum_{t=1}^{4}\sum_{w=1}^{3}T_{tw}, \tag{73}$$

where

$$T_{tw}=iKu_jN_{tw}\int_0^\infty d\tau'\int_0^\infty d\tau''\int_0^\infty d\tau'''\exp\{-[v_{t1}\tau'+v_{t2}\tau''+v_{t3}\tau'''+\tfrac{1}{4}(Ku_j)^2x_w^2]\} \tag{74}$$

and where N_{tw} and x_w are defined in Table I and v_{tk} in Table II. The T_{tw} may be conveniently evaluated[4] in terms of the plasma dispersion function $Z(v)$ defined by (58):

$$T_{t1}=[N_{t1}/v_{t2}][Z(v_{t1})+Z(v_{t3})]/(v_{t1}+v_{t3}), \tag{75}$$

$$T_{t2}=-[N_{t2}/v_{t2}][Z(v_{t1})-Z(v_{t3})]/(v_{t1}-v_{t3}), \tag{76}$$

$$T_{t3}=\tfrac{1}{2}\left[\frac{N_{t3}}{v_{t1}-v_{t3}}\right]\left[\frac{Z(v_{t1})-Z(\frac{1}{2}v_{t2})}{v_{t1}-\frac{1}{2}v_{t2}}-\frac{Z(v_{t3})-Z(\frac{1}{2}v_{t2})}{v_{t3}-\frac{1}{2}v_{t2}}\right]. \tag{77}$$

If any of the denominators in T_{t2} and T_{t3} approach zero, the numerators do also, the limit being a derivative of Z with respect to v. Thus for $v_{t1}=v_{t3}$, $T_{t2}=(N_{t2}/v_{t2})dZ(v_1)/dv_1$, etc.

As the atomic velocity $u_j\to 0$, $Z(v)\to i(Ku_j/v)$, and (75), (76), and (77) yield

$$T_{t1},\ T_{t2},\ T_{t3}\to iKu_jN_{t1}/(v_{t1}v_{t2}v_{t3}), \tag{78}$$

as they should according to Eq. (74).

For large Doppler broadening ($Ku_j \gg$ atomic decay rates and various beat frequencies), $Z(v)=i(\sqrt{\pi})$ and T_{tw} reduces to the "Doppler limit" of the scalar theory

$$T_{t1}=2i(\sqrt{\pi})N_{t1}[v_{t2}(v_{t1}+v_{t3})]^{-1},\qquad T_{t2}\approx T_{t3}\approx 0. \tag{79}$$

TABLE II. This table defines the v_{tk} appearing in the third-order integrals (74).

v_{tk}	1	2	3
$t=1$	$\gamma_{a'b'}+i(\omega_{a'b'}-\nu_{\mu'}+\nu_{\rho'}-\nu_{\sigma'})$	$\gamma_{a'a''}+i(\omega_{a'a''}+\nu_{\rho'}-\nu_{\sigma'})$	$\gamma_{a'b''}+i(\omega_{a'b''}-\nu_{\sigma'})$
2	$\gamma_{a'b'}+i(\omega_{a'b'}-\nu_{\mu'}+\nu_{\rho'}-\nu_{\sigma'})$	$\gamma_{a'a''}+i(\omega_{a'a''}+\nu_{\rho'}-\nu_{\sigma'})$	$\gamma_{b''a''}+i(\omega_{b''a''}+\nu_{\rho'})$
3	$\gamma_{a'b'}+i(\omega_{a'b'}-\nu_{\mu'}+\nu_{\rho'}-\nu_{\sigma'})$	$\gamma_{b''b'}+i(\omega_{b''b'}+\nu_{\rho'}-\nu_{\mu'})$	$\gamma_{b''a''}+i(\omega_{b''a''}+\nu_{\rho'})$
4	$\gamma_{a'b'}+i(\omega_{a'b'}-\nu_{\mu'}+\nu_{\rho'}-\nu_{\sigma'})$	$\gamma_{b''b'}+i(\omega_{b''b'}+\nu_{\rho'}-\nu_{\mu'})$	$\gamma_{a''b'}+i(\omega_{a''b'}-\nu_{\mu'})$

Substituting (79) into (73) and writing the result in a form more closely resembling Eq. (76) of Ref. 1, one has

$$\begin{aligned}\mathcal{P}_{\rm n}^{(3)}(t)=&\tfrac{1}{8}i\pi^{1/2}(\hbar^3K)^{-1}\sum_{\mu'}\sum_{\rho'}\sum_{\sigma'}E_{\mu'}E_{\rho'}E_{\sigma'}\exp(i\psi_{{\rm n}\mu'\rho'\sigma'})\sum_j(a_j/u_j)\sum_k f_k(i)^*\sum_q f_q(p_\mu)\sum_r f_r(p_\rho)^*\sum_s f_s(p_\sigma)\\
&\times\sum_{a'}\sum_{b'}\sum_{a''}\sum_{b''}\delta_{a',b'+k}\delta_{a'',b'+q}\delta_{b'',a''-r}\delta_{a',b''+s}\wp_{b'a'}\wp_{a''b'}\wp_{a''b''}\wp_{a'b''}\\
&\times\{\mathfrak{D}(\omega_{a'a''}+\nu_{\rho'}-\nu_{\sigma'})[N_{a'b''(\rho'-\sigma'+\mu'-{\rm n})}\mathfrak{D}(\omega_{a'b'}-\nu_{\mu'}+\nu_{\rho'}-2\nu_{\sigma'}+\omega_{a'b''})\\
&-N_{b''a''(\sigma'-\rho'+\mu'-{\rm n})}\mathfrak{D}(\omega_{a'b'}-\nu_{\mu'}-\nu_{\sigma'}+2\nu_{\rho'}+\omega_{b''a''})]\\
&-\mathfrak{D}(\omega_{b''b'}+\nu_{\rho'}-\nu_{\mu'})[N_{a''b'(\rho'-\mu'+\sigma'-{\rm n})}\mathfrak{D}(\omega_{a'b'}-\nu_{\sigma'}+\nu_{\rho'}-2\nu_{\mu'}+\omega_{a''b'})\\
&-N_{b''a''(\mu'-\rho'+\sigma'-{\rm n})}\mathfrak{D}(\omega_{a'b'}-\nu_{\sigma'}-\nu_{\mu'}+2\nu_{\rho'}+\omega_{b''a''})]\},\end{aligned}\tag{80}$$

where, for example,

$$\mathfrak{D}(\omega_{a'b'}+\omega_{a'b''}-2\nu)=[\gamma_{a'b'}+\gamma_{a'b''}+i(\omega_{a'b'}+\omega_{a'b''}-2\nu)]^{-1}.$$

VII. GENERAL AMPLITUDE- AND FREQUENCY-DETERMINING EQUATIONS

Adding the real and imaginary parts of third-order terms given by (73) to (61) and (62) according to (12) and (13), one has (to third order) the completely general form of the amplitude- and frequency-determining equations

$$\dot{E}_{\rm n}=\operatorname{Im}\{\sum_{i'}\alpha_{{\rm n}n'}E_{\rm n'}\exp i\psi_{{\rm n}n'}-\sum_{\mu'}\sum_{\rho'}\sum_{\sigma'}\theta_{{\rm n}\mu'\rho'\sigma'}E_{\mu'}E_{\rho'}E_{\sigma'}\exp i\psi_{{\rm n}\mu'\rho'\sigma'}\},\tag{81}$$

$$\nu_{\rm n}+\dot{\varphi}_{\rm n}=\Omega_n+E_{\rm n}^{-1}\operatorname{Re}\{\sum_{i'}\alpha_{{\rm n}n'}E_{\rm n'}\exp i\psi_{{\rm n}n'}-\sum_{\mu'}\sum_{\rho'}\sum_{\sigma'}\theta_{{\rm n}\mu'\rho'\sigma'}E_{\mu'}E_{\rho'}E_{\sigma'}\exp i\psi_{{\rm n}\mu'\rho'\sigma'}\},\tag{82}$$

where

$$\begin{aligned}\theta_{{\rm n}\mu'\rho'\sigma'}=&\tfrac{1}{32}\nu(\epsilon_0\hbar^3K)^{-1}\sum_j(a_j/u_j)\sum_{k=-1}^{1}f_k(i)^*\sum_{q=-1}^{1}f_q(p_\mu)\sum_{r=-1}^{1}f_r(p_\rho)^*\sum_{s=-1}^{1}f_s(p_\sigma)\sum_{a'}\sum_{b'}\sum_{a''}\sum_{b''}\delta_{a',b'+k}\delta_{a'',b'+q}\\
&\times\delta_{b'',a''-r}\delta_{a',b''+s}\wp_{b'a'}\wp_{a''b'}\wp_{a''b''}\wp_{a'b''}\sum_{t=1}^{4}\sum_{w=1}^{3}T_{tw},\end{aligned}\tag{83}$$

and various symbols are defined by equations as indicated: $\alpha_{nn'}$, (63); $\psi_{nn'}$, (57); $\psi_{n\mu'\rho'\sigma'}$, (72); a_j, (24); p_μ and f_q, (34); $\wp_{a'b'}$, (32); T_{tw}, (74) through (79) and Table I.

VIII. DESCRIPTION OF THE PROGRAMS

The computer programs[11] written in FORTRAN IV are depicted schematically in Fig. 3. The heart of the programs consists of the subroutines COEFF and INTGRL, which are used to calculate α and θ given the number of Fox and Li modes, the nuclear spins, atomic angular momenta, decay constants, g factors, population inversion densities, magnetic field strength and inclination, cavity detuning and anisotropy, etc.

A large class of steady-state solutions of the laser equations have been calculated algebraically and may be evaluated using INTEN and INTEN1. Any solutions may be calculated by numerical integration using PREDCT, a predictor-corrector routine and RUNGE, a Runge-Kutta routine which sets up the required back values for PREDCT. Subroutine FTION defines the differential equations for the laser and, if desired, for servomechanisms and other devices used with the laser.

Subroutine SWEEP is used to control the computation of one or more values of the intensities and frequencies

[11] Further details about these programs are contained in the Ph.D. thesis of Murray Sargent, which may be obtained from University Microfilms, Inc., and the programs themselves may be obtained from M. Sargent at Bell Laboratories, Inc., Holmdel, New Jersey.

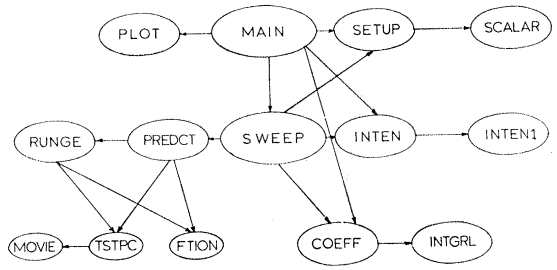

FIG. 3. Diagram of computer programs showing links between subroutines. The programs are used as follows: MAIN and SETUP to read in data and initialize parameters; SCALAR to calculate Clebsch-Gordon coefficients; SWEEP to vary one of several parameters such as cavity detuning; INTEN and INTEN1 to calculate certain steady-state solutions to (81) and (82); COEFF and INTGRL to calculate α and θ; PREDCT, RUNGE, and FTION to integrate (81) and (82); TSTPC and MOVIE to print, plot, and make movies of the solutions; and PLOT to plot α, θ, E_n, $\nu_n+\dot{\varphi}_n$, etc., as functions of the parameter varied in SWEEP. A number of additional subprograms are not shown, such as mathematical functions and auxiliary plotting routines.

of oscillation for which some parameter such as field strength or inclination is varied. When integrations are used in the sweep of a parameter, the α and θ may be held constant for a given point in the sweep until the desired number of steps of the integration has been taken, or they may be interpolated linearly up to the next point in the sweep. The latter choice corresponds quite closely to what is done experimentally. For a particular value of the sweeping parameter, SWEEP may be used to iterate the calculation in a self-consistent fashion.

TSTPC is used to print and plot results of time integrations and with the help of MOVIE to make computer movies of the solutions. PLOT controls the graphing of coefficients, intensities, etc., versus the sweeping parameter. The laser parameters are either calculated or read in as data in MAIN and SETUP. MAIN is also used to read in values of logical variables which control the flow of the program during execution and to choose any unspecified initial conditions for integrations of the equations.

IX. DISCUSSION

Although specific connections between the present and previous calculations are more conveniently discussed in paper II, it is desirable to make some comments of a general nature here. Among the rather large number of papers dealing with a laser in a magnetic field, those that come closest to describing the amplitudes and frequencies of laser oscillation experimentally are all generalizations of the single-mode scalar theory.[1] It is these that we discuss here more or less in order of publication.

Fork and Sargent[12,13] gave amplitude- and frequency-determining equations for axial magnetic fields and a $J=0\leftrightarrow J=1$ transition. Rozanov and Tulub[14] gave similar equations without the electric quadrupole contribution[13] which they believed could be neglected because of collision effects. Inasmuch as the quadrupole terms increase the coupling terms and it has been observed experimentally[15] that the coupling is still stronger than predicted, this assumption seems unjustified. D'Yakonov[16] gave a theory of a laser valid in a weak axial magnetic field in which the medium consisted of atoms of arbitrary angular momenta and the cavity could have an x-y Q anisotropy. Culshaw and Kannelaud[17] also analyzed this problem but used a $J=\frac{1}{2}\leftrightarrow J=\frac{1}{2}$ transition and assumed the electric field was plane-polarized at an angle to the axis of highest Q. For sufficiently weak magnetic fields they found that such a state of oscillation could be realized.

Heer and Graft[18] outlined an extension of the scalar theory which dealt with a magnetic field at any angle to the maser axis and atoms with arbitrary angular momenta, hfs, and isotopic abundance. Their amplitude and frequency equations have a form general enough to describe the operation of amplifiers, ring lasers, and standing-wave lasers. However, they did not evaluate the coefficients in these equations (or in any special cases), and they assumed the cavity losses were isotropic. In another calculation for arbitrary field orientation and angular momenta, D'Yakonov and Perel[19] evaluated the coefficients for transverse and axial fields in which they included the effects of spontaneous emission neglected in our analysis. They assumed that $g_a=g_b$, which considerably simplifies the calculation but is valid only for a limited number of atoms in nonzero magnetic fields, and they did not give any numerical results.

Culshaw and Kannelaud[20] analyzed a laser in transverse and axial magnetic fields for the $0\leftrightarrow 1$ transition. Durand[21,22] also treated a laser in a transverse field for this transition. His amplitude- and frequency-determining equations were derived however, from rate equations whose velocity dependence was introduced in the nonrigorous although plausible fashion described[23]

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[13] R. L. Fork and M. Sargent III, in *Proceedings of the International Conference on the Physics of Quantum Electronics*, edited by P. L. Kelley, B. Lax, and P. E. Tannenwald (McGraw-Hill Book Company, Inc., New York, 1966), pp. 611–619.
[14] N. N. Rozanov and A. V. Tulub, Dokl. Akad. Nauk SSSR **165**, 1280 (1965) [English transl.: Soviet Phys.—Doklady **10**, 1209 (1966)].
[15] R. L. Fork and W. J. Tomlinson, IEEE J. Quant. Electron. **QE-2**, xxiii (1966); R. L. Fork, W. J. Tomlinson, and L. J. Heilos, Appl. Phys. Letters **8**, 162 (1966); H. deLang, and G. Bouwhis, Phys. Letters **20**, 383 (1966); H. deLang, thesis, State University of Utrecht, 1966, especially pp. 57–75 (unpublished).
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[19] M. I. D'Yakonov and V. E. Perel, Opt. i Spektroskopiya **20**, 472 (1966) [English transl: Opt. Spectry. (USSR) **20**, 257 (1966)]. See also the review article by M. I. D'Yakonov and S. A. Fridrikhov, Usp. Fiz. Nauk **90**, 565 (1966) [English transl: Soviet Phys.—Usp. **9**, 837 (1967)].
[20] W. Culshaw and J. Kannelaud, Phys. Rev. **145**, 258 (1966); **156**, 308 (1967).
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[22] G. Durand, IEEE J. Quant. Electron. **QE-2**, 448 (1966).
[23] W. E. Lamb, Jr., Ref. 1, p. A1448.

in the scalar theory. This method may be interpreted in terms of "hole burning."[24] D'Yakonov and Perel[25] gave a qualitative description of axial and transverse field operation using a hole-burning model.

In a zero-field calculation with atoms of arbitrary angular momenta, Polder and Van Haeringen[26] gave an expression for $\epsilon^{-1}d\epsilon/dt$, where the eccentricity

$$\epsilon=(E_+-E_-)/(E_++E_-).$$

In another zero-field calculation with atoms of arbitrary angular momenta, Doyle and White[27] analyzed a laser with cavity anisotropy represented by (in our notation) a diagonal anisotropy matrix. This precludes certain interesting configurations (see paper II, Sec. V). Pelikan[28] analyzed a laser in an axial magnetic field and included the x-y Q anisotropy. His treatment of this anisotropy is more in the spirit of our work than of D'Yakonov.[16] Sargent *et al.* [29] gave a discussion of this problem in terms of the present theory. Velzel[30] has indicated how to include cavity anisotropy of a somewhat more general nature in the laser equations.

The present calculation gives the amplitude- and frequency-determining equations for the multimode operation of a laser in an arbitrarily oriented magnetic field with possibly anisotropic cavity loss and resonance and an active medium consisting of atoms which may have arbitrary angular momenta, isotopic abundance, and hfs. Computer programs have been written for analyzing these equations very generally, yielding graphs of intensities and frequencies of oscillation versus time, cavity detuning, magnetic field strength, or any other laser parameter. As illustrated in the following paper, this facility allows the results of the theory to be graphed either in the most convenient theoretical form or in virtually any way the experimentalist observes his apparatus in the laboratory.

ACKNOWLEDGMENTS

We would like to thank J. Winocur and W. J. Tomlinson for calling a number of errors and omissions to our attention.

APPENDIX A

In general, any two orthogonal unit vectors $\{\mathbf{e}'_1, \mathbf{e}'_2\}$ are related (apart from a possible phase factor) to the unit vectors $\{\mathbf{e}_1, \mathbf{e}_2\}$ of our initial cavity basis by equations

$$\mathbf{e}'_1=\mathbf{e}_1\cos\theta+\mathbf{e}_2\sin\theta\exp(i\beta),$$
$$\mathbf{e}'_2=-\mathbf{e}_1\sin\theta\exp(-i\beta)+\mathbf{e}_2\cos\theta. \quad \text{(A1)}$$

These equations correspond to the similarity transformation

$$S=\begin{pmatrix}\cos\theta & \sin\theta\exp(i\beta)\\ -\sin\theta\exp(-i\beta) & \cos\theta\end{pmatrix}. \quad \text{(A2)}$$

As indicated in the latter part of Sec. II, the anisotropy matrix G in the $\{\mathbf{e}_1, \mathbf{e}_2\}$ basis is transformed to

$$G'=SGS^{-1} \quad \text{(A3)}$$

in the $\{\mathbf{e}'_1, \mathbf{e}'_2\}$ basis. One of the diagonal elements of G' is

$$g'_{11}=g_{11}\cos^2\theta+[g_{21}\exp(i\beta)+g_{12}\exp(-i\beta)]\times\sin\theta\cos\theta+g_{22}\sin^2\theta. \quad \text{(A4)}$$

The polarization of threshold oscillations will be specified by that unit vector $\mathbf{e}'_1$ which maximizes the imaginary part of α'_{n11}, the α coefficient [Eq. (63)] corresponding to $\mathbf{e}'_1$. Using (A1), (A4), and (63), one has

$$\alpha'_{n11}=-\tfrac{1}{2}i\nu\{g_{11}\cos^2\theta+\tfrac{1}{2}[g_{21}\exp(i\beta)+g_{12}\exp(-i\beta)]\sin^2\theta+g_{22}\sin^2\theta\}+\tfrac{1}{2}(\nu/\epsilon_0\hbar K)\times\sum_j(a_j/u_j)\sum_{k=-1}^{1}|\cos\theta f_k(1)+\sin\theta\exp(i\beta)f_k(2)|^2\sum_{a'}\sum_{b'}\delta_{a',b'+k}(\wp_{a'b'})^2\bar{N}_{a'b'}Z[\gamma_{a'b'}+i(\omega_{a'b'}-\nu'_{n1})]. \quad \text{(A5)}$$

The imaginary part of α'_{n11} may be written as

$$\mathrm{Im}(\alpha'_{n11})=a\cos^2\theta+\tfrac{1}{2}(b\cos\beta+c\sin\beta)\sin2\theta+d\sin^2\theta, \quad \text{(A6)}$$

where

$$a=-\tfrac{1}{2}\nu\,\mathrm{Re}g_{11}+\mathcal{G}(1,1)\equiv\mathrm{Im}\,\alpha_{n11},$$
$$b=-\tfrac{1}{2}\nu(\mathrm{Re}g_{12}+\mathrm{Re}g_{21})+2\mathrm{Re}[\mathcal{G}(1,2)],$$
$$c=-\tfrac{1}{2}\nu(\mathrm{Im}g_{12}-\mathrm{Im}g_{21})-2\mathrm{Im}[\mathcal{G}(1,2)],$$
$$d=-\tfrac{1}{2}\nu\,\mathrm{Re}g_{22}+\mathcal{G}(2,2)\equiv\mathrm{Im}\,\alpha_{n22}, \quad \text{(A7)}$$

and the gain parts $\mathcal{G}(i,i')$ of α'_{n11} are given by

$$\mathcal{G}(i,i')=\tfrac{1}{2}(\nu/\epsilon_0\hbar K)\sum_{k=-1}^{1}f_k(i)^*f_k(i')\sum_j(a_j/u_j)\times\sum_{a'}\sum_{b'}\delta_{a',b'+k}(\wp_{a'b'})^2\times\bar{N}_{a'b'}Z_i[\gamma_{a'b'}+i(\omega_{a'b'}-\nu'_{n1})]. \quad \text{(A8)}$$

[24] W. R. Bennett, Jr., Appl. Opt. Suppl. **1**, 24 (1962), especially pp. 53–59.
[25] M. I. D'Yakonov and V. I. Perel, Zh. Eksperim i Teor. Fiz. **50**, 448 (1966) [English transl.: Soviet Phys.—JETP **23**, 298 (1966)].
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[28] H. Pelikan, Phys. Letters **21**, 652 (1966). Because of a sign error this work predicts that phase locking between circularly polarized modes of equal amplitude does not occur. See Sec. V of paper II; Z. Physik **201**, 523 (1967).
[29] M. Sargent III, W. E. Lamb, Jr., W. J. Tomlinson, and R. L. Fork, Bull. Am. Phys. Soc. **12**, 90 (1967).
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To maximize $\mathrm{Im}\,\alpha'_{n11}$ one must solve the equations

$$\partial(\mathrm{Im}\,\alpha'_{n11})/\partial\beta=0 \tag{A9}$$

and

$$\partial(\mathrm{Im}\,\alpha'_{n11})/\partial\theta=0 \tag{A10}$$

simultaneously and choose the solution yielding the largest $\mathrm{Im}\,\alpha'_{n11}$. Equation (A9) gives

$$\partial(\mathrm{Im}\,\alpha'_{n11})/\partial\beta=\tfrac{1}{2}\sin2\theta(c\cos\beta-b\sin\beta)=0, \tag{A11}$$

which implies either

$$\sin2\theta=0, \text{ i.e.,} \quad \theta=0 \quad \text{or} \quad \tfrac{1}{2}\pi, \tag{A12}$$

or

$$\tan\beta=c/b. \tag{A13}$$

Equation (A10) gives

$$\partial(\mathrm{Im}\,\alpha'_{n11})/\partial\theta$$
$$=-(a-d)\sin2\theta+(b\cos\beta+c\sin\beta)\cos2\theta=0 \tag{A14}$$

or

$$\tan2\theta=(b\cos\beta+c\sin\beta)/(a-d)$$
$$=(b+c\tan\beta)\cos\beta/(a-d). \tag{A15}$$

Equation (A12) implies either the identity transformation ($\theta=0$) or a simple interchange ($\theta=\frac{1}{2}\pi$) of $\mathbf{e}_1$ and $\mathbf{e}_2$ with the introduction of the phase factors $\exp(\pm i\beta)$. The resulting $\mathrm{Im}\,\alpha'_{n11}$ are $\mathrm{Im}\,\alpha_{n11}$ and $\mathrm{Im}\,\alpha_{n22}$, corresponding to the polarizations of $\mathbf{e}_1$ and $\mathbf{e}_2$, respectively. Combining (A15) and (A13), one has

$$\tan2\theta=\pm(b^2+c^2)^{1/2}/(a-d), \tag{A16}$$

where the sign is the same as that of $\cos\beta$. Noting that (A6) can be written as

$$\mathrm{Im}\,\alpha'_{n11}=\tfrac{1}{2}(a+d)\pm\tfrac{1}{2}(\tan^2 2\theta+1)^{-1/2}$$
$$\times[(a-d)+\cos\beta(b+c\tan\beta)\tan2\theta], \tag{A17}$$

we substitute (A13) and (A15) into (A17) and find

$$\mathrm{Im}\,\alpha'_{n11}=\tfrac{1}{2}(a+d)+\tfrac{1}{2}\,|\,[b^2+c^2+(a-d)^2]^{1/2}\,|, \tag{A18}$$

where we have kept only that root of (A17) which yields the larger $\mathrm{Im}\,\alpha'_{n11}$. This value for $\mathrm{Im}\,\alpha'_{n11}$ is greater than or equal to the solutions a and d implied by (A12). Thus the polarization of threshold oscillations is specified by $\mathbf{e}'_1$ of (A1) in which β is determined by (A13) and θ by (A16).

APPENDIX B

Integrals such as

$$T_\pm=iKu\int_0^\infty d\tau'\int_0^\infty d\tau''\int_0^\infty d\tau'''$$
$$\times\exp\{-[v_1\tau'+v_2\tau''+v_3\tau'''+\tfrac{1}{4}(Ku)^2(\tau'''\pm\tau')^2]\} \tag{B1}$$

may be evaluated in terms of the plasma dispersion function Z of (58). This can be accomplished by making the coordinate transformation

$$x_1=\tau'''-\tau',$$
$$x_2=\tau'''+\tau',$$
$$x_3=\tau'', \tag{B2}$$

with inverse

$$\tau'=\tfrac{1}{2}(x_2-x_1),$$
$$\tau''=x_3,$$
$$\tau'''=\tfrac{1}{2}(x_2+x_1), \tag{B3}$$

and Jacobian

$$\mathcal{J}=\frac{\partial(\tau',\tau'',\tau''')}{\partial(x_1,x_2,x_3)}=\begin{vmatrix}-\frac{1}{2} & \frac{1}{2} & 0\\ 0 & 0 & 1\\ \frac{1}{2} & \frac{1}{2} & 0\end{vmatrix}=\tfrac{1}{2}. \tag{B4}$$

The integration over x_3 may be done immediately, yielding the multiplicative factor v_2^{-1}. The variable x_2 varies between 0 and ∞ as τ' and τ'' vary between 0 and ∞. Choosing these limits for x_2, x_1 varies between $-x_2$ and $+x_2$. Thus one has

$$T_2\equiv T_+=\tfrac{1}{2}i(Ku/v_2)\int_0^\infty dx_2\int_{-x_2}^{x_2}dx_1\exp\{-[\tfrac{1}{2}(v_3-v_1)x_1+\tfrac{1}{2}(v_1+v_3)x_2+\tfrac{1}{4}(Ku)^2x_2^2]\}. \tag{B5}$$

After integrating over x_2, one has

$$T_2\equiv+iKu[v_2(v_1-v_3)]^{-1}\int_0^\infty dx_2\exp\{-[\tfrac{1}{2}(v_3-v_1)x_1+\tfrac{1}{2}(v_1+v_3)x_2+\tfrac{1}{4}(Ku)^2x_2^2]\}\,|_{x_1=-x_2}^{x_1=x_2}$$
$$=-[v_2(v_1-v_3)]^{-1}[Z(v_1)-Z(v_3)]. \tag{B6}$$

If

$$v_1\longrightarrow v_3, \qquad T_1\longrightarrow -v_2^{-1}dZ(v_1)/dv_1. \tag{B7}$$

Next we consider

$$T_1\equiv T_-=\tfrac{1}{2}i(Ku/v_2)\int_0^\infty dx_2\int_{-x_2}^{x_2}dx_1\exp\{-[\tfrac{1}{2}(v_3-v_1)x_1+\tfrac{1}{2}(v_1+v_3)x_2+\tfrac{1}{4}(Ku)^2x_1^2]\}. \tag{B8}$$

Interchanging the integrations over x_1 and x_2, and using (58) one has

$$T_1=\tfrac{1}{2}i(Ku/v_2)\int_0^\infty dx_1\int_{x_1}^\infty dx_2\{\exp\{-[\tfrac{1}{2}(v_3-v_1)x_1+\tfrac{1}{2}(v_1+v_3)x_2+\tfrac{1}{4}(Ku)^2x_1^2]\}$$
$$+\exp\{-[\tfrac{1}{2}(v_1-v_3)x_1+\tfrac{1}{2}(v_1+v_3)x_2+\tfrac{1}{4}(Ku)^2x_1^2]\}\}$$
$$=[v_2(v_1+v_3)]^{-1}[Z(v_1)+Z(v_3)]. \tag{B9}$$

An integral of the form

$$T_3=iKu\int_0^\infty d\tau'\int_0^\infty d\tau''\int_0^\infty d\tau'''\exp\{-[v_1\tau'+v_2\tau''+v_3\tau'''+\tfrac{1}{4}(Ku)^2(\tau'+2\tau''+\tau''')^2]\} \tag{B10}$$

may be similarly evaluated after completing the coordinate transformation

$$x_1=\tau'''-\tau',$$
$$x_2=\tau'''+\tau',$$
$$x_3=\tau'''+2\tau''+\tau'. \tag{B11}$$

This transformation has the inverse

$$\tau'=\tfrac{1}{2}(x_2-x_1),$$
$$\tau''=\tfrac{1}{2}(x_3-x_2),$$
$$\tau'''=\tfrac{1}{2}(x_2+x_1), \tag{B12}$$

and Jacobian

$$\mathcal{J}=\frac{\partial(\tau',\tau'',\tau''')}{\partial(x_1,x_2,x_3)}=\begin{vmatrix}-\frac{1}{2} & \frac{1}{2} & 0\\ 0 & -\frac{1}{2} & \frac{1}{2}\\ \frac{1}{2} & \frac{1}{2} & 0\end{vmatrix}=\tfrac{1}{4}. \tag{B13}$$

The transformed T_3 is

$$T_3=\tfrac{1}{4}iKu\int_0^\infty dx_3\int_0^{x_2}dx_2\int_{-x_2}^{x_2}dx_1\exp\{-[\tfrac{1}{2}(v_3-v_1)x_1+\tfrac{1}{2}(v_1-v_2+v_3)x_2+\tfrac{1}{2}v_2x_3+\tfrac{1}{4}(Ku)^2x_3^2]\}$$
$$=-\tfrac{1}{2}iKu(v_1-v_3)^{-1}\int_0^\infty dx_3\int_0^{x_3}dx_2\{\exp\{-[(v_1-\tfrac{1}{2}v_2)x_2+\tfrac{1}{2}v_2x_3+\tfrac{1}{4}(Ku)^2x_3^2]\}$$
$$-\exp-[(v_3-\tfrac{1}{2}v_2)x_2+\tfrac{1}{2}v_2x_3+\tfrac{1}{4}(Ku)^2x_3^2]\}$$
$$=\tfrac{1}{2}(v_1-v_3)^{-1}\left[\frac{Z(v_1)-Z(\frac{1}{2}v_2)}{v_1-\frac{1}{2}v_2}-\frac{Z(v_3)-Z(\frac{1}{2}v_2)}{v_3-\frac{1}{2}v_2}\right]. \tag{B14}$$

If $v_1=v_3$, this reduces to

$$T_3=\tfrac{1}{2}(v_1-\tfrac{1}{2}v_2)^{-1}\left[dZ(v_1)/dv_1-\frac{Z(v_1)-Z(\frac{1}{2}v_2)}{v_1-\frac{1}{2}v_2}\right]. \tag{B15}$$