Quantized Circulation and Heat Torques

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(Received 24 July 1967)

It is shown that there will exist a torque on a cylindrically symmetric, uniformly heated body immersed in liquid helium if and only if there exists trapped circulation about the body. The resultant torque will be quantized if the circulation is. For a heat input of 10 mW, a single quantum of circulation would yield a torque of 5×10^{-7} dyn cm, which is of reasonable magnitude.

I. INTRODUCTION

RECENTLY,[1] it has been pointed out that the existence of heat-exchange forces in superfluids could be experimentally checked by measuring the torque on a nonuniformly heated cylinder suspended in a flow of helium.

In the present analysis, we wish to show that the existence of quantization of circulation in superfluid helium may be detected by measuring the torque on a uniformly heated cylinder immersed in helium.

The magnitudes of the experimental quantities involved lead to the hope that a straightforward demonstration of quantized circulation may be possible.

In the next section, we will review some of the features of heat-exchange torques previously explained.[1] We will then take account of quantized circulation and calculate the torque on a cylinder about which a vortex is trapped.

II. HEAT-EXCHANGE TORQUES

As shown in a previous analysis,[1] the two-fluid model for superfluid helium implies the existence of torques and forces on heated objects immersed in helium. Landau and Lifshitz[2] mention the existence of such forces, which are solely due to the peculiarities of two-fluid hydrodynamics with conservation of mass and energy.

As in ordinary hydrodynamics, the relevant quantity for calculating forces due to fluid flow is the momentum-flux density tensor π_{ij}, which for the two-fluid case is given by[2]

$$\pi_{ij}=\rho_n v_i^n v_j^n+\rho_s v_i^s v_j^s+p\delta_{ij}. \tag{1}$$

Here, ρ_n and ρ_s are the normal-fluid and super-fluid mass densities, $\mathbf{v}^n$ and $\mathbf{v}^s$ their velocities, and p the pressure.

The hydrodynamic equations for the fluids are well known. The normal fluid is assumed to be a classical, viscous, Navier-Stokes fluid; the superfluid undergoes potential flow.

The boundary conditions at a solid surface are dictated by the conservation laws and the physical attributes ascribed to the fluids. For the normal fluid, we assume its tangential component vanishes on a solid surface, as though it were a classical viscous fluid. In addition, the conservation of mass demands that

$$(\rho_n v_l^n+\rho_s v_l^s)dS_l=0, \tag{2}$$

while conservation of entropy demands that

$$\rho_n v_l^n s_n T dS_l=KdS, \tag{3}$$

where s_n is the entropy per gram of the normal fluid at temperature T, and $K\ dS$ is the heat flow from the surface element dS.

If we have a solid body immersed in helium, the torque exerted on the body by the flow fields of the helium is given by the expression

$$\tau_i=\tfrac{1}{2}\epsilon_{ijk}\oint r_j\pi_{kl}dS_l, \tag{4}$$

where $\pi_{kl}dS_l$ is the force on the surface element dS_l in the k direction and ϵ_{ijk} is the Levi-Civita symbol.

The expression for the torque follows solely from the meaning of the momentum-flux density tensor, with no assumptions necessary. We use the summation convention for tensor indices.

When we use the assumed boundary conditions on the solid surface, the torque becomes

$$\tau_i=\tfrac{1}{2}\epsilon_{ijk}\oint r_j p dS_k+(2s_nT)^{-1}\epsilon_{ijk}\oint r_j(v_k^n-v_k^s)KdS, \tag{5}$$

where the integrals are over the surface of the body in question. The first term in the torque is the usual Bernoulli term, familiar in ordinary hydrodynamics. The second term is the heat-exchange torque which is entirely peculiar to the two-fluid model.

We omit further mention of the Bernoulli term, because we are going to consider an object with cylindrical symmetry. To be specific, let us imagine a long cylinder immersed in helium. That is, we imagine that a line integral encircling the cylinder cannot be deformed to a point without leaving the helium.

For such a cylinder, the torque is given by

$$\tau=La^2(s_nT)^{-1}\int_0^{2\pi}(v_\theta^n-v_\theta^s)K(\theta)d\theta, \tag{6}$$

[1] R. Penney, Phys. Fluids 10, 2147 (1967).
[2] L. Landau and E. Lifshitz, *Fluid Mechanics* (Addison-Wesley Publishing Company, Reading, Massachusetts, 1959), Ch. XVI.

where a is the cylinder radius, L its length, $K(\theta)$ the heat per unit area as a function of position on the cylinder, $v_\theta{}^n$ and $v_\theta{}^s$ the tangential velocity components.

Thus, if one knew the velocity fields, and the heat flow, one could readily calculate the torque. We will assume $v_\theta{}^n$ vanishes on the solid surface, as mentioned before. It is then obvious that, if the cylinder is uniformly heated, there will be no torque unless there exists circulation for the superfluid velocity field.

That, is, for $K(\theta)$ independent of θ,

$$\tau = La^2(s_nT)^{-1}K\int_0^{2\pi} v_\theta{}^s d\theta, \tag{7}$$

so the torque vanishes unless there is trapped circulation around the cylinder. If that circulation is quantized, so is the resultant torque.

III. QUANTIZED CIRCULATION

As is well known, for a simply connected region, the equations

$$\nabla\cdot\mathbf{v}=0, \quad \nabla\times\mathbf{v}=0, \quad \mathbf{v}_\perp=0 \tag{8}$$

imply that $\mathbf{v}=0$. However, for fluid flow around a cylindrical region, there may be an arbitrary circulation around that region which still satisfies the hydrodynamic problem.

In particular, for the superfluid component of helium, the lack of uniqueness of the hydrodynamic problem allows the superfluid to have a nonzero circulation around cylindrical regions within the helium, without violating the curl-free condition.

In particular, Onsager[3] suggested that the circulation of the superfluid velocity field should be quantized,

$$M\oint \mathbf{v}^s\cdot d\mathbf{l} = Nh, \tag{9}$$

by analogy with the Bohr quantization conditions. Here M is the mass of a helium atom, N is an integer, and h is Planck's constant.

For a cylindrical, uniformly heated solid body around which has become trapped a circulation of superfluid, there will be a torque of magnitude

$$\tau = La^2(s_nT)^{-1}KNh(2\pi Ma)^{-1}\times 2\pi, \tag{10}$$

which simplifies to

$$\tau = NhQ/MTs_n, \tag{11}$$

where Q is the heat per unit time (uniformly) dissipated on the cylindrical surface.

Since s_n is relatively independent of temperature, the torque due to circulation varies essentially as T^{-1}. As an example of the magnitude of the effect, suppose T to be 2°K, and Q to be 10 mW. Then, we obtain

$$\tau = 5\times 10^{-7} \text{ dyn cm/quantum.} \tag{12}$$

With a relatively realistic fiber suspension,[4] a restoring constant of 10^{-5} dyn cm rad^{-1} may be used, and a deflection of about 3 deg of arc would be obtained for one quantum of circulation. Without minimizing the experimental difficulties, we may safely suggest that the magnitude of the torque so calculated indicates a feasible experiment.

IV. PHYSICAL ORIGINS OF THE TORQUE

The expression for the torque on a uniformly heated cylinder may be simply understood without detailed calculation. If we suppose our cylinder initially unheated, with a superfluid vortex trapped around it, the superfluid velocity is given by

$$\mathbf{v}_s = Nh\hat{u}_\theta/M\mathrm{r}, \tag{13}$$

where $\hat{u}_\theta$ is the unit vector in the direction tangential to the cylinder surface.

If we now suddenly heat some surface element of the cylinder, normal fluid will stream from the surface, and the superfluid must stream in to conserve mass at the solid surface. In so doing, the superfluid must give up its forward momentum in the θ direction. In so doing, the superfluid imparts a tangential force on the cylinder in the direction of its circulation.

The important feature of the torque due to circulation is that it exists even though the solid body is cylindrically symmetric and uniformly heated. The vortex circulation itself is the agency which determines the lack of symmetry of the situation.

In principle, one should be able to rotate a bucket of helium, cool below the λ point, thus locking in the circulation, and detect the trapped circulation about the cylinder after the bucket is stopped.

V. CONCLUSIONS

We have shown that the two-fluid hydrodynamic model for helium leads to the existence of heat-exhange torques on a solid, cylindrically symmetric object which is uniformly heated, if and only if circulation exists about the object. Thus, the insertion of heat as an extra probe of the dynamics of superfluids should allow one to study the quantization of circulation.

Observation of circulation torques having their expected magnitude, quantization, and temperature dependence would corroborate many of the important concepts involved in understanding superfluids.

ACKNOWLEDGMENTS

We wish to thank T. K. Hunt, H. W. Jackson, and A. D. Brailsford for many helpful suggestions on the theoretical and experimental aspects of the proposed experiment.

[3] F. London, *Superfluids* (John Wiley & Sons, Inc. New York, 1954), Vol. II, p. 151.

[4] T. K. Hunt (private communication).