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# Phonon-Quasiparticle Interactions in Dilute Solutions of $He^3$ in Superfluid $He^4$: I. Phonon Thermal Conductivity and Ultrasonic Attenuation*

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Expressing the interaction between long-wavelength phonons and $He^3$ quasiparticles in dilute solutions of $He^3$ in $He^4$ entirely in terms of macroscopic observables, we calculate the attenuation of first sound due to the $He^3$ viscosity, and the rate of scattering of phonons from the $He^3$. The results for the attenuation are in excellent agreement with recent experiments.In addition, we calculate the thermal conduction due to phonons for $T \lesssim 0.6°K$; the conductivity is limited at very low temperatures by ultrasonic attenuation, while at higher temperatures it is predominantly limited by elastic phonon-quasiparticle scattering.

## I. INTRODUCTION

The transport properties of dilute solutions of $He^3$ in $He^4$ at temperatures below 0.6°K are the result of a number of competing scattering mechanisms involving $He^3$ quasiparticles and phonons. Khalatnikov and Zharkov[1,2] initially considered the transport theory of this region; however, recent experimental[3–5] and theoretical [6–10] work has now provided more complete knowledge of the nature of the $He^3$-$He^3$ and the $He^3$-phonon interactions in the solutions. It is the purpose of this paper to apply this information to a description of the thermal conductivity and ultrasonic attenuation as determined by these interactions.

The structure of the thermal conductivity illustrates very clearly how various interaction mechanisms come into play in determining the transport properties of the solutions. First of all, transport by the $He^3$ quasiparticles and phonons occur in "parallel"; thus the thermal conductivity $\kappa$ is of the form

$$\kappa=\kappa_3+\kappa_{\mathrm{ph}}, \tag{1}$$

where $\kappa_3$ is the $He^3$ thermal conductivity and $\kappa_{\mathrm{ph}}$ is the phonon thermal conductivity. $\kappa_3$ is limited either by boundary scattering or by scattering from other $He^3$ quasiparticles; below 0.6°K, scattering from phonons has a negligible effect on $He^3$ heat transport. One may think of boundary and quasiparticle scattering as impedances in "series." Thus if $\kappa_{3,\mathrm{b}}$ is the $He^3$ thermal conductivity when only boundary scattering is considered, and $\kappa_{3-3}$ is the $He^3$ thermal conductivity when only quasiparticle interactions are taken into account, then

$$\kappa_3^{-1}\approx\kappa_{3,\mathrm{b}}^{-1}+\kappa_{3-3}^{-1}. \tag{2}$$

Phonon transport is similarly limited, below 0.6°K, by boundary scattering and by scattering and attenuation of the phonons by the $He^3$ quasiparticles; thus we have

$$\kappa_{\mathrm{ph}}^{-1}\approx\kappa_{\mathrm{ph,b}}^{-1}+\kappa_{\mathrm{ph,qp}}^{-1}, \tag{3}$$

where $\kappa_{\mathrm{ph,b}}^{-1}$ is the resistivity due to boundary scattering, and $\kappa_{\mathrm{ph,qp}}^{-1}$ that due to interaction of the phonons with the quasiparticles.

At the lowest temperatures both $He^3$ and phonon transport are limited only by scattering from the walls of the container. Phonon and quasiparticle mean free paths are then of the order of the container size $d$, so that

$$\kappa_{3,\mathrm{b}}\sim C_3 v_f d \tag{4}$$

and

$$\kappa_{\mathrm{ph,b}}\sim C_{\mathrm{ph}} s d, \tag{5}$$

where $C_3 \sim T$ is the $He^3$ specific heat per unit volume, $C_{\mathrm{ph}} \sim T^3$ is the phonon specific heat per unit volume, $v_f$ is the Fermi velocity, and $s$ is the first sound velocity. The exact coefficients in (4) and (5) depend on the nature of the scattering from the walls, specular reflection having much less effect than diffuse reflection on the heat current. The coefficient in $\kappa_{\mathrm{ph,b}}$ is expected to decrease with increasing temperature, since the shorter the wavelengths of the thermal phonons the more the average reflection will be diffuse.[11] On the other hand, one expects the coefficient in $\kappa_{3,\mathrm{b}}$ to remain fairly constant with temperature.

Since $C_3 \gg C_{\mathrm{ph}}$ at the lowest temperatures, $\kappa_{3,\mathrm{b}}$ is much larger than $\kappa_{\mathrm{ph,b}}$; thus $\kappa\approx\kappa_{3,\mathrm{b}}\sim T$. As the temperature is raised, the first effect to change this picture is $He^3$-$He^3$ scattering. The consequences of this inter-

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[1] I. M. Khalatnikov and V. Zharkov, Zh. Eskperim. i Teor. Fiz. 32, 1108 (1957) [English transl.: Soviet Phys.—JETP **5**, 905 (1957)].

[2] V. N. Zharkov, Zh. Eskperim. i Teor. Fiz. 33, 929 (1957) [English. transl.: Soviet Phys.—JETP **6**, 714 (1958)].

[3] A. C. Anderson, D. O. Edwards, R. Roach, R. J. Sarwinski, and J. C. Wheatley, Phys. Rev. Letters **17**, 367 (1966).

[4] W. R. Abel, R. T. Johnson, J. C. Wheatley, and W. Zimmermann, Jr., Phys. Rev. Letters **18**, 737 (1967).

[5] B. M. Abraham, Y. Eckstein, J. B. Ketterson, and J. H. Vignos, Phys. Rev. Letters **17**, 1254 (1967).

[6] J. Bardeen, G. Baym, and D. Pines, Phys. Rev. Letters **17**, 372 (1966); referred to hereafter as BBP.

[7] J. Bardeen, G. Baym, and D. Pines, Phys. Rev. **156**, 207 (1967).

[8] C. Ebner, Phys. Rev. **156**, 222 (1967).

[9] G. Baym, Phys. Rev. Letters **17**, 952 (1966).

[10] G. Baym, Phys. Rev. Letters **18**, 71 (1967).

[11] H. A. Fairbank and J. Wilks, Proc. Roy. Soc. (London) **A231**, 545 (1955).

action can be found via the usual Fermi-liquid expression[12,13] for $\kappa_{3-3}$:

$$\kappa_{3-3}=\frac{4\pi^2}{3T}\frac{(\hbar v_f)^3}{m^*\langle W\sin^2(\theta/2)/\cos(\theta/2)\rangle}, \tag{6}$$

where $m^*$ is the He$^3$ effective mass; $W(\theta,\varphi)$ is the quasiparticle-quasiparticle transition rate per unit time from an initial state where $\theta$ is the angle between the momenta of the particles (lying on the Fermi surface) to a final state where the plane of the two momenta is rotated by angle $\varphi$. The brackets denote an average over the $4\pi$ solid angle swept by $\theta$ and $\varphi$.

The change from the predominantly $T$ behavior of $\kappa_3$, from boundary scattering, to the $T^{-1}$ Fermi-liquid behavior occurs when $\kappa_{3\mathrm{b}}\approx\kappa_{3-3}$. For the 5% solution in a tube 2 mm in diameter investigated in Ref. 4, the temperature at which this occurs is ~0.2 m°K; a temperature in the decade beyond the range of current research. Thus for temperatures in the millidegree region $\kappa_3$ is proportional to $T^{-1}$. The phonon contribution is of the form $\kappa_{\mathrm{ph,b}}\sim T^3$ and it rapidly becomes comparable in magnitude to $\kappa_3$. For the 5% solution in Ref. 4, the crossover occurs for $T\sim 20$m°K. For higher temperatures $\kappa_3$ is negligible and the phonon contributions to heat transport dominate.

As we shall see, the phonon thermal conductivity has the following structure. The first interaction effect that enters as $T$ is raised is the absorption of phonons via the He$^3$ viscosity; this is the mechanism that attenuates first sound. The phonon mean free path $l$ for this process is proportional at low temperatures to $T^{-2}$ and is independent of the phonon wavelength $\lambda$. Thus $\kappa_{\mathrm{ph}}=\frac{1}{3}C_{\mathrm{ph}}sl$ changes from the roughly $T^3$ from characteristic of boundary scattering, to a form linear in $T$. At still higher temperatures, Rayleigh scattering of phonons by quasiparticles comes into play; this mechanism leads to a mean free path $\sim\lambda^4$. If it were the only mechanism limiting the phonon thermal current, one might expect the effective mean free path to be $\sim T^{-4}$. However, for very large $\lambda$, Rayleigh scattering clearly becomes very weak, and so long-wavelength phonons will continue to be attenuated primarily by the He$^3$ viscosity mechanism described above. As a result, temperature behavior somewhere between $T^{-2}$ and $T^{-4}$ emerges for the average phonon mean free path. This explains the observations of Fairbank and Sandiford[14] and Niels-Hakkenberg *et al.*,[15] in studying the propagation of temperature pulses in dilute solutions, of a $T^{-3}$ to $T^{-3.5}$ behavior for the phonon mean free path.

Above 0.6°K, scattering of phonons by phonons and rotons begins to become important; these mechanisms, as well as conduction due to diffusion of thermal excitations, important above 1°K, have been analyzed by Khalatnikov and Zharkov, and we do not reconsider them here.

We turn now to a detailed description of the interaction of phonons with He$^3$ quasiparticles.

## II. SCATTERING OF PHONONS BY He$^3$ QUASIPARTICLES

In this section we calculate the cross section for the scattering of a long-wavelength phonon by a slowly moving He$^3$ quasiparticle. The exact form of the coupling between these excitations is determined by thermodynamic and Galilean-invariance arguments. The derivation of this effective interaction and the calculation of the cross section are done in the spirit of Landau and Khalatnikov.[16] The details of our interaction, however, differ from theirs; in addition, we are able now to relate all the parameters of the interaction to independent macroscopically observed quantities. In the following derivation, we shall neglect the concentration dependence of the He$^3$-phonon interaction.

According to Landau and Pomeranchuk[17] the energy of a slowly moving quasiparticle of momentum $\mathbf{p}$ in superfluid He$^4$ at rest is simply

$$\epsilon_p=\epsilon_0+p^2/2m; \tag{7}$$

in the limit of zero concentration, $\epsilon_0$ is the chemical potential $\mu_3$ of the He$^3$ in He$^4$, and the effective mass $m$ is approximately $2.34m_3$, where $m_3$ is the bare mass of a He$^3$ atom.[6]

The long-wavelength low-frequency deviations of the He$^4$ from equilibrium may be described by a density variation $\rho_4(\mathbf{r},t)$, which is the local particle density minus $n_4$, the mean density, and by a superfluid velocity $\mathbf{v}_s(\mathbf{r},t)$, given by the gradient of the phase of the condensate wave function. To find the interaction between a long-wavelength phonon and a quasiparticle, we must determine how the quasiparticle energy (7) depends on the local He$^4$ density and superfluid velocity.

The dependence of $\epsilon_p$ on $\mathbf{v}_s$ can be determined by the requirements of Galilean invariance. Suppose that there is a uniform superfluid velocity in the liquid. Then in the frame in which $\mathbf{v}_s=0$, the energy of a quasiparticle of momentum $\mathbf{p}$ is given by (7). In the laboratory frame

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[12] A. A. Abrikosov and I. M. Khalatnikov, Rept. Progr. Phys. **22**, 329 (1959).

[13] The average in Eq. (6) may be evaluated by using the effective interaction between He$^3$ quasiparticles, as proposed by BBP; the results are 30 to 50% smaller than the measurements reported recently in Ref. 4. The discrepancy, we suspect, is due to inaccuracies in the solution of the He$^3$ kinetic equation that leads to (6); a detailed investigation of this point is in progress and the results will be published shortly.

[14] D. J. Sandiford and H. A. Fairbank, in *Proceedings of the Seventh International Conference on Low Temperature Physics* (University of Toronto Press, Toronto, 1961), p. 641.

[15] C. G. Niels-Hakkenberg, L. Meermans, and H. C. Kramers, in *Proceedings of the Eighth International Conference on Low Temperature Physics* (Butterworths Scientific Publishers Inc., New York, 1963), p. 45.

[16] L. D. Landau and I. M. Khalatnikov, Zh. Eksperim. i Teor. Fiz. **19**, 637 (1949); **19**, 709 (1949); I. M. Khalatnikov, *ibid.* **23**, 8 (1952).

[17] L. D. Landau and I. Pomeranchuk, Dokl. Akad. Nauk. SSSR **59**, 668 (1948).

the momentum required to create this quasiparticle is $\mathbf{p}+m_3\mathbf{v}_s$, while the energy required to create it is

$$\epsilon_p+\mathbf{p}\cdot\mathbf{v}_s+m_3v_s^2/2. \tag{8}$$

Thus the energy-momentum relation in the laboratory frame is

$$\epsilon_{\mathbf{p}+m_3\mathbf{v}_s}[\mathbf{v}_s]=\epsilon_0+p^2/2m+\mathbf{p}\cdot\mathbf{v}_s+m_3v_s^2/2$$

or

$$\epsilon_p[\mathbf{v}_s]=\epsilon_0+p^2/2m+(\delta m/m)\mathbf{p}\cdot\mathbf{v}_s-(\delta m/m)(m_3v_s^2/2), \tag{9}$$

where $\delta m=m-m_3$; the $[\mathbf{v}_s]$ denotes the $\mathbf{v}_s$ dependence of the quasiparticle energy in the laboratory frame.

The quasiparticle energy depends on $\rho_4$ through the dependence of $\epsilon_0$ and $m$ on the He$^4$ density. The dependence of $m$ on $\rho_4$ can be neglected since it leads to corrections of order $(v_f/s)^2\sim 0.01$ in the cross section for phonon-He$^3$ scattering. The dependence of $\epsilon_0$ on $\rho_4$ can be written to second order as

$$\epsilon_0[\rho_4]=\epsilon_0+(\partial\epsilon_0/\partial n_4)\rho_4+\tfrac{1}{2}(\partial^2\epsilon_0/\partial n_4^2)\rho_4^2. \tag{10}$$

The partial derivatives are at constant He$^3$ density.

The point now is that to a quasiparticle, a long-wavelength phonon looks like a uniform motion of the He$^4$ with a local velocity $\mathbf{v}_s$ and a local density $n_4+\rho_4$. Thus the change in the energy of a quasiparticle of momentum $\mathbf{p}$, due to the presence of a long-wavelength phonon, is, from (9) and (10),

$$\delta\epsilon_p(\mathbf{r},t)=(\partial\epsilon_0/\partial n_4)\rho_4(\mathbf{r},t)+(\delta m/m)\mathbf{p}\cdot\mathbf{v}_s(\mathbf{r},t)+\tfrac{1}{2}(\partial^2\epsilon_0/\partial n_4^2)\rho_4(\mathbf{r},t)^2-\tfrac{1}{2}(\delta m/m)m_3v_s(\mathbf{r},t)^2, \tag{11}$$

where $\mathbf{r}$ is the position of the quasiparticle. This form of the coupling is exact up to terms of second order in the small quantities $\rho_4$, $\mathbf{v}_s$, and $\mathbf{p}$.

Since $\epsilon_0$ is the He$^3$ chemical potential at zero concentration, we have[7]

$$(\partial\epsilon_0/\partial n_4)_{n_3}=(m_4s^2/n_4)(1+\alpha), \tag{12}$$

where $\alpha\approx 0.28$ is the fractional excess molar volume of He$^3$ in He$^4$. Thus

$$\left(\frac{\partial^2\epsilon_0}{\partial n_4^2}\right)=\frac{m_4s^2}{n_4^2}\left[(1+\alpha)\left(2\frac{n_4}{s}\frac{\partial s}{\partial n_4}-1\right)+n_4\frac{\partial\alpha}{\partial n_4}\right]. \tag{13}$$

The derivatives in this expression, experimentally, are[18]

$$n_4\partial\alpha/\partial n_4\approx -1.4,\quad (n_4/s)(\partial s/\partial n_4)\approx 2.7, \tag{14}$$

and so the square bracket in (13) is approximately 4.2.

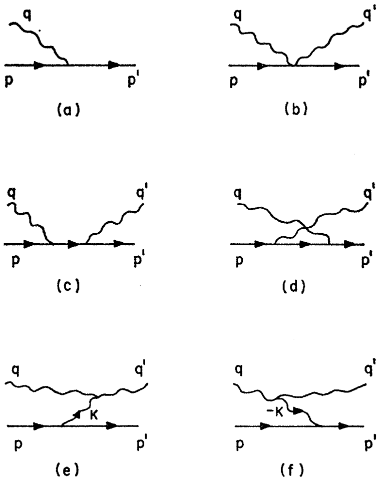

FIG. 1. Interactions of phonons (wavy lines) with quasiparticles (solid lines). (a) Phonon absorption; (b), (c), and (d); three processes leading to scattering of phonons by quasiparticles; (e) and (f); "phonon-induced" scattering of phonons by quasiparticles.

In order to calculate the amplitude for phonon-quasiparticle scattering we quantize the interaction (11) by replacing $\mathbf{p}$ and $\mathbf{r}$ by the quasiparticle momentum and position operators, and by expanding $\rho_4(\mathbf{r})$ and $\mathbf{v}_s(\mathbf{r})$ in terms of phonon annihilation and creation operators, $b_{\mathbf{q}}$ and $b_{\mathbf{q}}^\dagger$. In a box of volume $\Omega$, these expansions are[19]

$$\rho_4(\mathbf{r})=\sum_q\left(\frac{q^2n_4}{2m_4\omega_q\Omega}\right)^{1/2}(b_{\mathbf{q}}e^{i\mathbf{q}\cdot\mathbf{r}}+b_{\mathbf{q}}^\dagger e^{-i\mathbf{q}\cdot\mathbf{r}}) \tag{15}$$

and

$$\mathbf{v}_s(\mathbf{r})=\sum_q\left(\frac{\omega_q}{2m_4n_4\Omega}\right)^{1/2}\hat{q}(b_{\mathbf{q}}e^{i\mathbf{q}\cdot\mathbf{r}}+b_{\mathbf{q}}^\dagger e^{-i\mathbf{q}\cdot\mathbf{r}}). \tag{16}$$

$\omega_q$ is the frequency of the He$^4$ elementary excitation of wave number $q$; at long wavelengths $\omega_q=sq$.

Substituting (15) and (16) into (11) we find that the first two terms in (11), denoted by $V_1$, correspond to absorption or emission of a phonon by a quasiparticle. In quantizing (11), the $\mathbf{p}\cdot\mathbf{v}_s(\mathbf{r})$ term must be symmetrized as $\frac{1}{2}[\mathbf{p}\cdot\mathbf{v}_s(\mathbf{r})+\mathbf{v}_s(\mathbf{r})\cdot\mathbf{p}]$, since the position and momentum of the quasiparticle fail to commute. The matrix element for a quasiparticle of momentum $\mathbf{p}$ to absorb a phonon of momentum $\mathbf{q}$, and end up with momentum $\mathbf{p}''$, as in Fig. 1(a), is then

$$\langle\mathbf{p}''|V_1|\mathbf{p},\mathbf{q}\rangle=\delta_{\mathbf{p}'',\mathbf{p}+\mathbf{q}}\left(\frac{qn_4}{2m_4s\Omega}\right)^{1/2}\times\left[\frac{\partial\epsilon_0}{\partial n_4}+\frac{\delta ms}{n_4q}(\epsilon_{p''}-\epsilon_p)\right]. \tag{17}$$

Similarly, the final two terms in (11), denoted by $V_2$, correspond to the scattering of a phonon by a quasiparticle, as in Fig. 1(b); they lead, in addition, to two-phonon emission or absorption processes. The matrix element for the scattering process illustrated in

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[18] The derivative of $\alpha$ has been measured by C. Boghosian and H. Meyer, Phys. Letters, **25A**, 352 (1967); that of $s$ by K. R. Atkins and R. A. Stasior, Can. J. Phys. **31**, 1156 (1953).

[19] I. M. Khalatnikov, *An Introduction to the Theory of Superfluidity* (W. A. Benjamin, Inc., New York, 1965).

Fig. 1(b) is

$$\langle \mathbf{p}'\mathbf{q}'|V_2|\mathbf{pq}\rangle=\delta_{\mathbf{p}+\mathbf{q},\mathbf{p}'+\mathbf{q}'}\frac{(qq')^{\frac{1}{2}}n_4}{2m_4s\Omega}\times\left[\frac{\partial^2\epsilon_0}{\partial n_4{}^2}-\frac{\delta m}{m}\frac{m_3s^2}{n_4{}^2}\hat{q}\cdot\hat{q}'\right]. \tag{18}$$

The total amplitude $\langle \mathbf{p}'\mathbf{q}'|T|\mathbf{pq}\rangle$ for phonon-quasiparticle scattering is the sum of the matrix elements for the five processes in Figs. 1(b)–1(f). The first three processes 1(b), 1(c), and 1(d) are analogous to those in nonrelativistic scattering of photons by free electrons; their contribution, denoted by $\langle \mathbf{p}'\mathbf{q}'|T_d|\mathbf{pq}\rangle$, to the total scattering amplitude is calculated analogously. The resulting matrix element, evaluated on the energy shell $[\epsilon_p+sq=\epsilon_p{}'+sq']$, is

$$\langle \mathbf{p}'\mathbf{q}'|T_d|\mathbf{pq}\rangle=\delta_{\mathbf{p}+\mathbf{q},\mathbf{p}'+\mathbf{q}'}\frac{m_4s}{2n_4\Omega(qq')^{1/2}}\left[-\left(\frac{\delta m}{m_4}\right)^2\frac{\mathbf{q}\cdot\mathbf{q}'}{m}+\left(1+\alpha+\frac{\delta m}{m_4}\right)^2s^2qq'\left(\frac{1}{\epsilon_p+sq-\epsilon_{\mathbf{p}+\mathbf{q}}}+\frac{1}{\epsilon_{p'}-sq-\epsilon_{\mathbf{p}'-\mathbf{q}}}\right)\right]+\langle \mathbf{p}'\mathbf{q}'|V_2|\mathbf{pq}\rangle. \tag{19}$$

In deriving (19) we have neglected the exclusion principle in the intermediate states of Figs. 1(c) and 1(d); analysis shows that at the higher temperatures where this scattering process is important, the statistics of the $\mathrm{He}^3$ lead to a decrease in the scattering rate on the order of 10% for a 5% solution, and are negligible for a 1.3% solution.

The "direct" scattering amplitude (19) can be considerably simplified by noting that $v_f/s\ll1$; hence each of the energy denominators in (19) is dominated by the $sq$ and it is sufficient to expand them to terms proportional to $(sq)^{-2}$. Furthermore, in the temperature region where this scattering mechanism is important the relevant phonon energies are much smaller than $m_4s^2\approx27$°K. It is easy to verify then that, on the energy shell, quasiparticle-phonon scattering is essentially elastic, i.e., $q=q'$ to terms of relative order $KT/ms^2$ or $v_f/s$. Thus (19) reduces to

$$\langle \mathbf{p}'\mathbf{q}'|T_d|\mathbf{pq}\rangle\approx\delta_{\mathbf{p}+\mathbf{q},\mathbf{p}'+\mathbf{q}'}\frac{sq}{2n_4\Omega}\left\{\left[\left(1+\alpha+\frac{\delta m}{m_4}\right)^2\frac{m_4}{m}-\frac{\delta m}{m_4}\right]\times\cos\theta+\frac{n_4{}^2}{m_4s^2}\frac{\partial^2\epsilon_0}{\partial n_4{}^2}\right\}, \tag{20}$$

where $\theta$ is the angle between $\mathbf{q}$ and $\mathbf{q}'$.

The two processes in Figs. 1(e) and 1(f) are the scattering of a phonon by a quasiparticle through the exchange of a virtual phonon; they constitute the "phonon-induced" interaction between a phonon and a quasiparticle, analogous to that between two electrons in a metal. In order to calculate their contribution, denoted by $\langle \mathbf{p}'\mathbf{q}'|T_{ph}|\mathbf{pq}\rangle$, to the total scattering amplitude, we need to find the three-phonon vertex, as it occurs in Figs. 1(e) and 1(f). In the limit in which all three phonon wavelengths are long, the three-phonon interaction is given by[19]

$$V_3=\int d\mathbf{r}\left[\tfrac{1}{2}m_4v_s(\mathbf{r})\rho_4(r)v_s(\mathbf{r})+\frac{1}{6}\frac{\partial}{\partial n_4}\left(\frac{m_4s^2}{n_4}\right)\rho_4(\mathbf{r})^3\right], \tag{21}$$

where $\mathbf{v}_s$ and $\rho_4$ are given in terms of phonon annihilation and creation operators by (15) and (16). The three-phonon matrix element in Fig. 1(e) is then

$$\langle \mathbf{q}'|V_3|\mathbf{q},\mathbf{k}\rangle=\delta_{\mathbf{q}+\mathbf{k},\mathbf{q}'}\left(\frac{qq'ks}{8m_4n_4\Omega}\right)^{1/2}\times\left[2\frac{n_4}{s}\frac{\partial s}{\partial n_4}-1+\hat{q}\cdot\hat{q}'+\hat{q}\cdot\hat{k}+\hat{q}'\cdot\hat{k}\right], \tag{22}$$

while that in Fig. 1(f) is $\langle \mathbf{q}',-\mathbf{k}|V_3|\mathbf{q}\rangle=\langle \mathbf{q}|V_3|\mathbf{q}',-\mathbf{k}\rangle$.

The scattering amplitude due to phonon exchange, $\langle \mathbf{p}',\mathbf{q}'|T_{ph}|\mathbf{p},\mathbf{q}\rangle$, is simply evaluated on the energy shell when the scattering is elastic; then the $\delta m$ term in (17) vanishes as does $\hat{q}\cdot\hat{k}+\hat{q}'\cdot\hat{k}$ and the result is

$$\langle \mathbf{p}'\mathbf{q}'|T_{ph}|\mathbf{pq}\rangle=-\delta_{\mathbf{p}+\mathbf{q},\mathbf{p}'+\mathbf{q}'}\frac{sq}{2n_4\Omega}(1+\alpha)\times\left(2\frac{n_4}{s}\frac{\partial s}{\partial n_4}-1+\cos\theta\right). \tag{23}$$

Adding (20) and (23), and using (13), we find that the total phonon-quasiparticle scattering amplitude is[20]

$$\begin{aligned}\langle \mathbf{p}'\mathbf{q}'|T|\mathbf{pq}\rangle&=\langle \mathbf{p}'\mathbf{q}'|T_d|\mathbf{pq}\rangle+\langle \mathbf{p}'\mathbf{q}'|T_{ph}|\mathbf{pq}\rangle\\&=\delta_{\mathbf{p}+\mathbf{q},\mathbf{p}'+\mathbf{q}'}\frac{sq}{2n_4\Omega}\left[n_4\frac{\partial\alpha}{\partial n_4}+(1+\alpha+\delta m/m_4)\times\frac{m_4}{m}(1+\alpha-m_3/m_4)\cos\theta\right].\end{aligned} \tag{24}$$

The rate at which phonons are scattered from $\mathbf{q}$ to $\mathbf{q}'$ by the $\mathrm{He}^3$ is given by

$$2\sum_{p,p'}2\pi\delta(\epsilon_p+sq-\epsilon_{p'}-sq')|\langle \mathbf{p}'\mathbf{q}'|T|\mathbf{pq}\rangle|^2\times[1+n(\mathbf{q}')]f(\mathbf{p})[1-f(\mathbf{p}')]; \tag{25}$$

the initial factor of 2 accounts for $\mathrm{He}^3$ spin. $f(\mathbf{p})$ is the quasiparticle and $n(\mathbf{q})$ the phonon distribution function. For the purpose of calculating the phonon contribution to thermal conductivity, it is sufficient to assume the $\mathrm{He}^3$ to be in thermal equilibrium[1] and to neglect the exclusion principle for $\mathbf{p}'$. Then multiplying (25) by

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[20] Since the effective interaction (11) is in terms of completely renormalized vertices, the result (24) is exact in the limit that $s$ is much larger than typical $\mathrm{He}^3$ velocities.

$\delta(\cos\theta-\hat{q}.\hat{q}')$ and summing over all $\mathbf{q}'$ we find finally that the rate at which phonons of momentum $\mathbf{q}$ are scattered by angle $\theta$, divided by $Hn(q')$, is

$$\Gamma(q,\theta)=\frac{n_3 s q^4}{8\pi n_4^2}\left\{n_4\frac{\partial\alpha}{\partial n_4}+(1+\alpha+\delta m/m_4)(m_4/m)\right.$$
$$\left.\times(1+\alpha-m_3/m_4)\cos\theta\right\}^2$$
$$\approx\frac{xsq^4}{8\pi n_4}(-1.4+0.69\cos\theta)^2. \qquad (26)$$

This scattering rate has the $q^4$ dependence characteristic of Rayleigh scattering. Consequently its importance in limiting phonon-transport phenomena increases rapidly with temperature. For very long wavelengths, however, and consequently at very low temperatures, the absorption of phonons via ultrasonic attenuation, i.e., through the He$^3$ viscosity, plays a more dominant role in limiting the rates of phonon transport phenomena. We turn now to a discussion of this absorption mechanism.

## III. ABSORPTION OF PHONONS DUE TO He$^3$ VISCOSITY

At very low temperatures ($\lesssim 0.2$°K in a 5% solution) the attenuation of first sound is due to viscosity of the He$^3$, which are carried along in the sound wave. The theory of this attenuation has been given in Ref. 10. Since very long-wavelength phonons are simply first sound waves, ultrasonic attenuation is the dominant mechanism for limiting transport by such phonons. To make contact between the theory of Ref. 10 and the phonon-quasiparticle interaction derived in the previous section, we rederive the amplitude attenuation coefficient $\alpha_I$, this time from the interaction (11).

We ask then for the rate at which phonons of momentum $\mathbf{q}$ are absorbed by the He$^3$ quasiparticles. At very long wavelengths only $V_1$, the first two terms in (11), is significant for absorption. However, in calculating the absorption we cannot neglect the quasiparticle interaction and assume that the quasiparticles are simply in plane-wave states. The reason is that, since $v_f<s$, energy and momentum conservation forbid the absorption of a phonon by a noninteracting quasiparticle. It is convenient, therefore, to write the interaction (11) between the He$^3$ and the phonons in terms of the He$^3$ particle density operator $\rho_3(\mathbf{r})$ and the He$^3$ particle current operator $\mathbf{j}_3(\mathbf{r})$. In this language the interaction becomes

$$\int d\mathbf{r}\left\{\left[\frac{\partial\epsilon_0}{\partial n_4}\rho_4(\mathbf{r})+\frac{1}{2}\frac{\partial^2\epsilon_0}{\partial n_4^2}\rho_4(\mathbf{r})^2-\frac{1}{2}\frac{\delta m}{m}m_3 v_s(\mathbf{r})^2\right]\rho_3(r)\right.$$
$$\left.+\delta m\,\mathbf{j}_3(\mathbf{r})\cdot\mathbf{v}_s(\mathbf{r})\right\}. \qquad (27)$$

The matrix element for absorption of a phonon of momentum $\mathbf{q}$, with the He$^3$ going from initial state $|i\rangle$ to final state $|f\rangle$, is then

$$\langle f|V_1|i,\mathbf{q}\rangle=\left(\frac{qn_4}{2m_4 s\Omega}\right)^{1/2}\int d\mathbf{r}\,e^{i\mathbf{q}\cdot\mathbf{r}}$$
$$\times\left[\frac{\partial\epsilon_0}{\partial n_4}\langle f|\rho_3(\mathbf{r})|i\rangle+\frac{s\delta m}{n_4}\hat{q}\cdot\langle f|\,\mathbf{j}_3(\mathbf{r})|i\rangle\right]. \qquad (28)$$

From conservation of He$^3$ atoms, or quasiparticles, we can replace $\mathbf{q}\cdot\langle f|\,\mathbf{j}_3(\mathbf{r})|i\rangle$ in (28) by $(\omega_f-\omega_i)\langle f|\rho_3(\mathbf{r})|i\rangle$ $=sq\langle f|\rho_3(\mathbf{r})|i\rangle$, where $\omega_i$ and $\omega_f$ are the energies of the states $|i\rangle$ and $|f\rangle$.

The absorption rate is found by squaring (28) and summing over all $|f\rangle$ consistent with energy conservation. Using

$$\delta(\omega)=\int_{-\infty}^{\infty}e^{-i\omega t}d\omega/2\pi,$$

passing to the Heisenberg representation, and averaging over a thermal ensemble of initial states, we find

$$\Gamma_{\text{abs}}(q)=\frac{qm_4 s^3}{2n_4}(1+\alpha+\delta m/m_4)^2\int d\mathbf{r}\,e^{-i\mathbf{q}\cdot(\mathbf{r}-\mathbf{r}')}$$
$$\times\int_{-\infty}^{\infty}dt\,e^{isq(t-t')}\langle\rho_3(\mathbf{r}t)\rho_3(\mathbf{r}'t')\rangle. \qquad (29)$$

The Fourier transform of the density-density correlation function in (29) equals twice the imaginary part of $-\langle\rho_3\rho_3\rangle(q,\omega=sq)$, the Fourier transform of the retarded He$^3$ density-density response function. Thus

$$\Gamma_{\text{abs}}(q)=\frac{-qm_4 s^3}{n_4}(1+\alpha+\delta m/m_4)^2\,\text{Im}\langle\rho_3\rho_3\rangle(q,sq). \qquad (30)$$

Noting that the amplitude attenuation coefficient $\alpha_I$ equals $\Gamma_{\text{abs}}/2s$ we see that (30) is the general result of Ref. 10 for $\alpha_I$, to lowest order in $x$.

In the Appendix we evaluate $\langle\rho_3\rho_3\rangle(q,\omega)$ by solving the He$^3$ kinetic equation including He$^3$-He$^3$ collisions by means of a relaxation-time approximation that includes conservation of quasiparticle number, momentum, and energy. The result, for $\omega\gg v_f q$ and $\hbar\omega\ll KT$, is

$$\langle\rho_3\rho_3\rangle(q,\omega)=\frac{q^2 n_3}{m\omega^2+4iq^2\omega\eta/3n_3(1+\tau_\eta{}^2\omega^2)}; \qquad (31)$$

$\eta$, the He$^3$ first viscosity, is given in terms of the He$^3$ relaxation time $\tau_\eta$ for viscosity[21] by

$$\eta=P(T)\tau_\eta=\tfrac{1}{5}mv_f{}^2 n_3\tau_\eta Y(T), \qquad (32)$$

where $P(T)$ is the pressure, at temperature $T$, of an

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[21] It is not difficult to show that at low temperature $\tau_\eta$ is the same for longitudinal and transverse viscosity, i.e., the second viscosity of the He$^3$ is negligible.

ideal Fermi gas of the same effective mass and number density as the He$^3$ in the solution, and[22]

$$Y(T)=P(T)/P(0). \tag{33}$$

For $T\ll T_f$, the He$^3$ Fermi temperature,

$$Y(T)=1+\frac{5\pi^2}{12}\left(\frac{T}{T_f}\right)^2-\frac{\pi^4}{16}\left(\frac{T}{T_f}\right)^4+\cdots. \tag{34}$$

In Eqs. (31) and (32) we have omitted *all* terms of relative order $x$, such as the concentration dependence of the He$^3$ effective mass. Combining (30)–(32) we find

$$\Gamma_{\mathrm{abs}}(q)=2s\alpha_I=\frac{4}{15}\frac{v_f^2}{s^2}\frac{m_4}{m}x(1+\alpha+\delta m/m_4)^2 \times Y(T)\frac{\omega_q^2\tau_\eta}{1+\omega_q^2\tau_\eta^2}, \tag{35}$$

where $\omega_q=sq$. The only difference between this result and that of Ref. 10 is the corrected formula for $Y(T)$.

The lifetime $\tau_\eta$ is the one undetermined quantity in (35). Roach[23] has calculated this lifetime for $T\ll T_f$ from the empirical interaction between He$^3$ quasiparticles derived by BBP; the results for two representative concentrations are

$$\begin{aligned}\tau_\eta&=(17.8\times10^{-12}/T^2)\ \mathrm{sec}, & x&=5\%\\ \tau_\eta&=(15.0\times10^{-12}/T^2)\ \mathrm{sec}, & x&=1.3\%.\end{aligned} \tag{36}$$

A similar calculation can be performed in the high-temperature regime, $T\gtrsim 2T_f$, where the He$^3$ obey classical statistics, by using the extension[8] of the BBP interaction to higher momenta. In this regime the scattering time $\tau_\eta$ is determined by solving the classical Boltzmann equation for the viscosity $\eta$ to lowest order in the effective quasiparticle interaction, and using the relation (32). The calculation is standard[24]; the results for 5 and 1.3% solutions are drawn as dashed lines in Figs. 2 and 3. Unlike at very low temperatures, $\tau_\eta$ is here inversely proportional to the concentration.

Lacking a solution of the Boltzmann equation in the intermediate-temperature regime, we must interpolate between the very low and very high temperatures to obtain $\tau_\eta$ for all $T$. One can deduce the lifetime in a 5% solution for $T\ll T_f$ from the measurements of first sound attenuation in a 5% solution by Abraham *et al.*[5] Their data, shown in Fig. 4, have fairly well-defined low-temperature peaks for 20 Mc/sec at $T\approx0.055$°K and for 60 Mc/sec at $T\approx0.121$°K. The locations of the peaks are independent of the absolute normalization of the data. Theoretically one expects the corrections to the $T^{-2}$ dependence of $\tau_\eta$ to be even powers of $T/T_f$. We then write

$$\tau_\eta=\frac{A}{T^2}\left[1+B\left(\frac{T}{T_f}\right)^2\right], \tag{37}$$

and determine $A$ and $B$ empirically by requiring that the attenuation (35) peak at the two experimentally determined temperatures; the result of a numerical

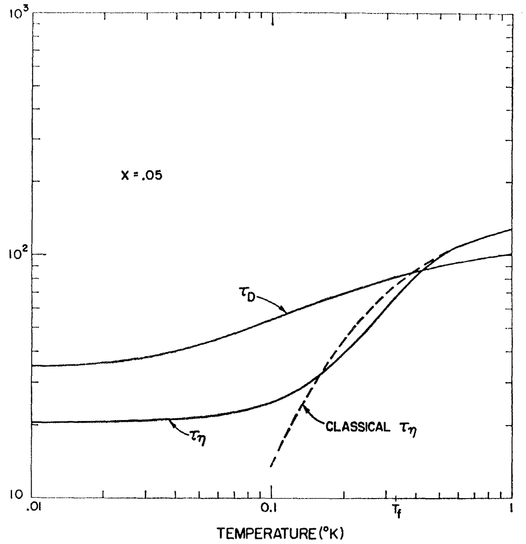

FIG. 2. Lifetimes for viscosity, $\tau_\eta$, and spin diffusion, $\tau_D$, in a 5% solution of He$^3$ in He$^4$. The dashed line is the form of $\tau_\eta$ for classical statistics.

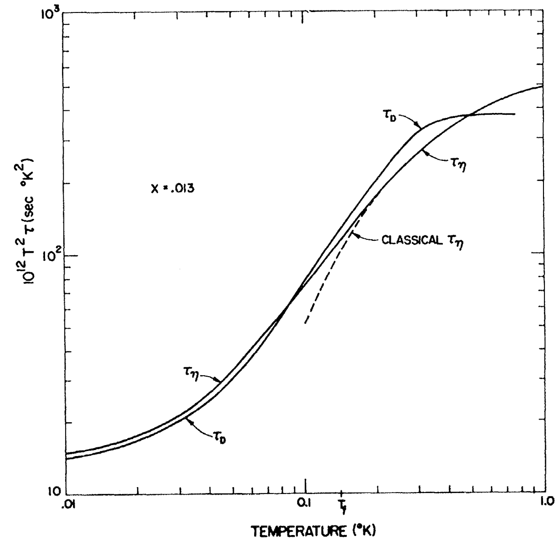

FIG. 3. Lifetimes for viscosity, $\tau_\eta$, and spin diffusion, $\tau_D$, in a 1.3% solution of He$^3$ in He$^4$. The dashed line is the form of $\tau_\eta$ for classical statistics.

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[22] We take this opportunity to point out that the formula for $Y(T)$ given in Ref. 10 is wrong. The error lay in solving the Boltzmann equation without including energy conservation in the collision term, and in an algebraic mistake. With the present formula (33) for $Y(T)$ all of the results of that paper are valid. Note that the present $\eta$ is, by definition, smaller by a factor of 4 than the $\eta$ of Ref. 10.

[23] W. R. Roach, dissertation, University of Illinois, 1966, p. 57 (unpublished).

[24] J. O. Hirschfelder, C. F. Curtiss and R. B. Bird, *Molecular Theory of Gases and Liquids* (John Wiley & Sons, Inc., New York, 1954), Chap. 8.

calculation is that

$$A=20.5\times10^{-12}\,\mathrm{sec}(°\mathrm{K})^2 \quad \text{and} \quad B=2.3. \tag{38}$$

Comparing this value of $A$ with the theoretical value (36), we find that the latter is 14% too small; this corresponds to an average inaccuracy in the BBP potential of 7%, which is entirely within reason.

The $\tau_\eta$ we shall use, drawn as a solid line in Fig. 2, is obtained by interpolating smoothly between the results (37) and (38) for $T\ll T_f$, and the calculated $\tau_\eta$ in the classical region.

In Fig. 4 we recalculate the attenuation for a 5% solution using Eq. (35) and the $\tau_\eta$ in Fig. 2; numerically

$$\alpha_I=(0.57\times10^{-8})Y(T)2\omega_q^2\tau_\eta/(1+\omega_q^2\tau_\eta^2). \tag{39}$$

The agreement with experiment is striking; the underestimate for the 20-Mc/sec data at higher temperature is due to residual attenuation from phonon-phonon interactions. [One should expect for a 5% solution no better than 10% agreement between Eq. (35) and experiment.] The lack of a systematic deviation of the theory from experiment confirms the absolute normalization of the experimental data.

For comparison with $\tau_\eta$, the lifetime $\tau_D$ for spin diffusion in a 5% solution is given in Fig. 2; we have extracted the lifetime from the measurements of the spin-diffusion coefficient $D$ by Anderson *et al.*,[3] by using the relation

$$D=\frac{1}{m^*}\left(\frac{\partial P(T)}{\partial n_3}\right)_T \tau_D, \tag{40}$$

which is derived by solving the He$^3$ kinetic equation in the relaxation-time approximation. In Fig. 3 we similarly plot $\tau_D$ for a 1.3% solution.

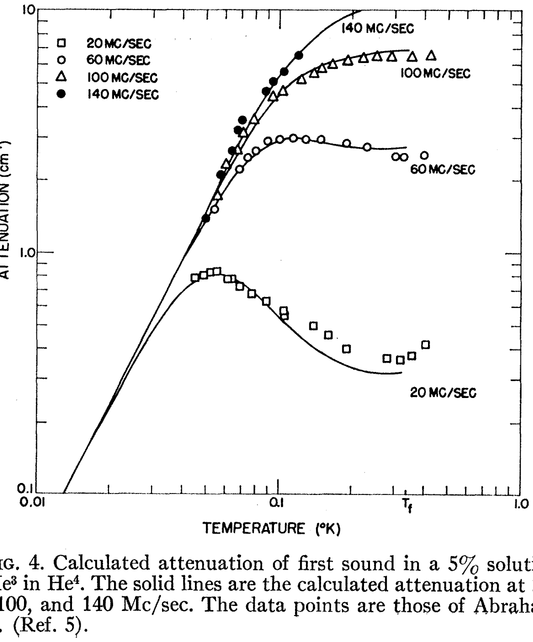

FIG. 4. Calculated attenuation of first sound in a 5% solution of He$^3$ in He$^4$. The solid lines are the calculated attenuation at 20, 60, 100, and 140 Mc/sec. The data points are those of Abraham *et al.* (Ref. 5).

For concentrations other than 5% no experimental information on $\tau_\eta$ is as yet available. We have therefore constructed a $\tau_\eta$ for use in calculations on a 1.3% solution as follows. A curve is drawn between the very low-temperature value (36) and the calculated classical $\tau_\eta$ assuming in this interpolation that $\tau_\eta$ is proportional to $\tau_D$ at low temperatures. $\tau_\eta$ and $\tau_D$ should be proportional at low temperatures and small concentrations to the extent that one can neglect the momentum dependence of the quasiparticle interaction. We expect the interpolation to be accurate to within 20%.

In Fig. 5 we present the theoretical attenuation $\alpha_I$ for a 1.3% solution, calculated using this $\tau_\eta$. Because of the sharp rise of $\tau_\eta T^2$ from its very low temperature to its classical values there is no peak in either the 20- or 60-Mc/sec attenuation. Measurements of these attenuations should provide detailed information on the behavior of $\tau_\eta$ for intermediate temperatures.

For $T\gtrsim 2T_f$ the lifetime $\tau_\eta$ becomes inversely proportional and $P(T)$ becomes proportional to the concentration. Thus at high temperatures, where $\omega\tau_\eta\ll1$, the attenuation due to He$^3$ viscosity becomes, for small $x$, independent of $x$. Furthermore $\alpha_I$ is then proportional to $\omega^2$ and decreases roughly as $T^{-1}$.

The lifetime used in the Boltzmann equation when calculating phonon attenuation is that appropriate to viscosity, $\tau_\eta$, only when $\kappa T\gg\hbar\omega_q$; this is the normal situation in ultrasonic-attenuation experiments. However, for thermal phonons $\hbar\omega_q\gtrsim\kappa T$, and thus, in order to use Eq. (35), $\tau_\eta$ should be replaced by the more

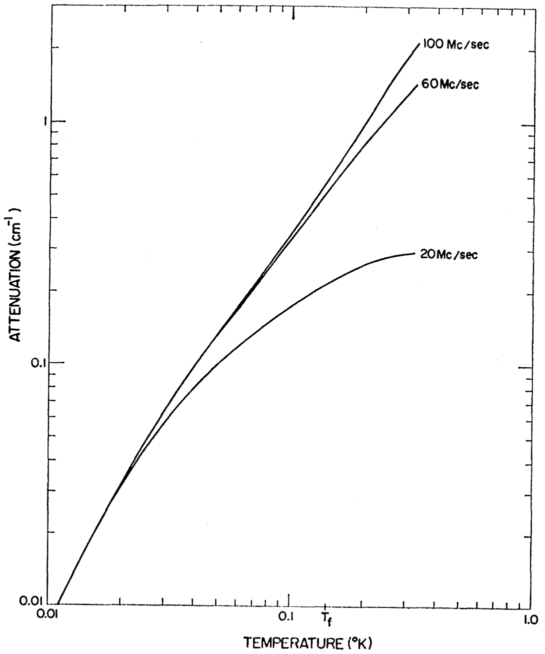

FIG. 5. Calculated attenuation of first sound in a 1.3% solution of He$^3$ in He$^4$.

general expression

$$\tau=\tau_\eta[1+(\hbar\omega/2\pi\kappa T)^2]^{-1}. \tag{41}$$

This can be shown by direct consideration of the collision integral; the details are given by Abrikosov and Khalatnikov.[12] Generally then

$$\Gamma_{\text{abs}}(q)=\frac{4}{3}\frac{m_4P(T)}{m^2n_4}(1+\alpha+\delta m/m_4)^2 \times\frac{q^2\tau_\eta[1+(\hbar sq/2\pi)T)^2]}{[1+(\hbar sq/2\pi\kappa T)^2]^2+s^2q^2\tau_\eta^2}. \tag{42}$$

For thermal phonons at very low temperatures, $\omega\tau_\eta$ is $\gg1$, in which case $\Gamma_{\text{abs}}$ is proportional to $\tau_\eta^{-1}\sim T^2$ and the phonon mean free path is proportional to $T^{-2}$.

## IV. THERMAL CONDUCTION BY PHONONS

The principal results of the last two sections are that phonons of momentum $q$ travelling through the solution are scattered by the $\text{He}^3$ at a rate given by (26) and are absorbed by the $\text{He}^3$ at a rate given by (42). In this section we combine these two results to calculate the thermal conductivity of the phonons, as limited by the[25] $\text{He}^3$. We ignore boundary scattering effects, relevant at very low $T$, and phonon-roton scattering, relevant for $T\gtrsim0.6$°K. Phonon-phonon scattering leads to no loss of heat current in the approximation that the phonon dispersion curve is linear; we do not include its effects.

An elastic scattering of a phonon by a very small angle does little to change the heat current of the phonon, whereas large-angle scattering reverses the heat current. The net scattering rate $\Gamma_{\text{sc}}(q)$ for thermal conductivity is thus an average of the rate (26) times 1-cos$\theta$:

$$\begin{aligned}\Gamma_{\text{sc}}(q)&=\int_{-1}^{1}d\cos\theta(1-\cos\theta)\Gamma(q,\theta)\\&=\frac{xsq^4}{12\pi n_4}\left[\left(\frac{m_4}{m}(1+\alpha+\delta m/m_4)(1+\alpha-m_3/m_4)-n_4\frac{\partial\alpha}{\partial n_4}\right)^2+2\left(n_4\frac{\partial\alpha}{\partial n_4}\right)^2\right]\\&\equiv\tau_1^{-1}\\&\simeq0.23xsq^4/n_4.\end{aligned} \tag{43}$$

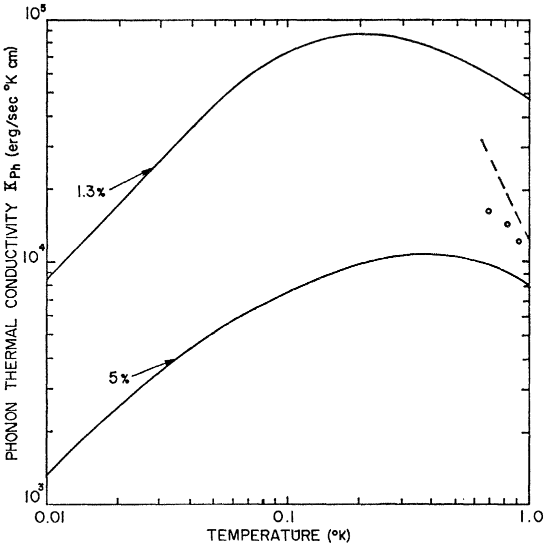

FIG. 6. Phonon thermal conductivity in 1.3% and 5% solutions of $\text{He}^3$ in $\text{He}^4$. The data points are those taken by Ptukha (Ref. 26) in a 1.36% solution, while the dashed line is the theory of Khalatnikov and Zharkov for a 1.36% solution.

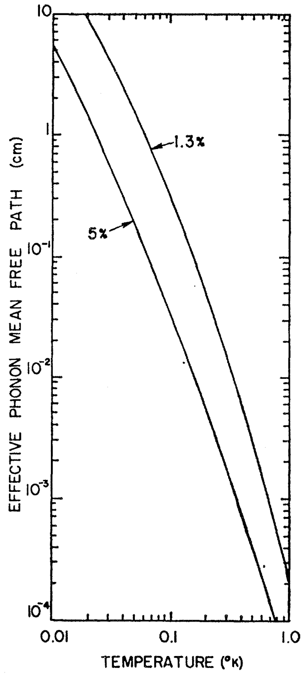

FIG. 7. Effective phonon mean free paths for thermal conductivity, in 1.3% and 5% solutions, neglecting scattering of phonons from boundaries and thermal excitations of the $\text{He}^4$.

To this scattering rate must be added the absorption rate (42) due to ultrasonic attenuation. The effective mean free path for a phonon of momentum $q$ is therefore

$$l(q)=\frac{s}{\Gamma_{\text{sc}}(q)+\Gamma_{\text{abs}}(q)}. \tag{44}$$

In the absence of phonon-phonon interactions the phonon thermal conductivity $\kappa_{\text{ph}}$ is simply a sum of the conductivities of each mode $q$; thus (as may be shown from the phonon Boltzmann equation),

$$\kappa_{\text{ph}}=\frac{1}{3}\int\frac{d\mathbf{q}}{(2\pi)^3}c(q)l(q), \tag{45}$$

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[25] The $\text{He}^3$ heat current that is produced in the collision of a phonon with a quasiparticle is negligible since $\text{He}^3$ velocities are much less than $s$.

where

$$c(q)=\hbar sq\frac{\partial}{\partial T}\frac{1}{e^{\hbar sq/\kappa T}-1} \tag{46}$$

is the contribution to the phonon specific heat of the mode **q**.

At very low temperatures, where thermal phonons have long wavelengths, the ultrasonic attenuation mechanism dominates the scattering mechanism; at higher temperatures ($\gtrsim 0.1$°K in a 5% solution) scattering is predominant. It should be noted, however, that the integral (45) diverges linearly at long wavelengths if scattering $\sim q^4$ is the only mechanism limiting transport; ultrasonic attenuation is still important at higher temperatures for long-wavelength phonons.

The integration of (45), using (42) and (43), reduces to a single complicated quadrature, which we have calculated numerically for 5 and 1.3% solutions between 0.01 and 1°K. The $\tau_\eta$'s used are those shown in Figs. 2 and 3. The results for $\kappa_{\mathrm{ph}}$ are shown in Fig. 6; they are characterized by a very broad maximum roughly in the neighborhood of the region where the effective scattering and attenuation rates are equal for a thermal phonon. We emphasize that, with the exception of the $\tau_\eta$, the calculated $\kappa_{\mathrm{ph}}$ contains only known parameters.

Ptukha[26] has measured the thermal conductivity of a 1.36% solution above 0.67°K; his data are indicated in Fig. 6. These fall below our calculated 1.3% curve, a discrepancy which is certainly due in part to the values of $\tau_\eta$ used in the high-temperature region. For comparison we indicate as a dashed line the fit below 1°K of Khalatnikov and Zharkov[1] to Ptukha's data. There are as yet no detailed measurements of $\kappa_{\mathrm{ph}}$ below 0.6°K. The present calculations appear, however, to be in reasonable agreement with preliminary, unpublished work of Abel, Johnson, Wheatley, and Zimmermann[27] on the thermal conductivity below 0.5°K of 1.3 and 5% solutions of $\mathrm{He}^3$ in $\mathrm{He}^4$.

It is illuminating to define an effective mean free path $l_{\mathrm{eff}}$ for thermal conductivity by writing

$$\kappa_{\mathrm{ph}}=\tfrac{1}{3}C_{\mathrm{ph}}sl_{\mathrm{eff}}, \tag{47}$$

where $C_{\mathrm{ph}}\sim T^3$ is the phonon specific heat per unit volume. Figure 7 shows $l_{\mathrm{eff}}$ as a function of $T$; numerical values of $l_{\mathrm{eff}}$ are given in Table I. Note that they do not scale with concentration.

One can expect boundary scattering effects to become important when $l_{\mathrm{eff}}$ becomes comparable with the sample size. We note that below 30 m°K the calculated phonon mean free paths are substantially larger than 2 mm; the data of Abel *et al.*[4] on the thermal conductivity below 30 m°K of 1.3 and 5% solutions in a 2-mm cylinder indicate effective mean free paths between 4 and 2 mm decreasing with $T$; thus the phonon thermal conductivity in their experiments is limited practically entirely by scattering of phonons by the walls.

TABLE I. Calculated values of the effective phonon mean free path in thermal conductivity, neglecting boundary scattering.

| $T$ (m°K) | $l_{\mathrm{eff}}(5\%)$ (cm) | $l_{\mathrm{eff}}(1.3\%)$ (cm) |
|---|---|---|
| 10 | 5.6 | 36. |
| 15 | 2.5 | 16. |
| 20 | 1.35 | 9.2 |
| 30 | 0.56 | 4.2 |
| 40 | 0.30 | 2.35 |
| 50 | 0.18 | 1.5 |
| 60 | 0.115 | 1.04 |
| 70 | 0.079 | 0.74 |
| 80 | 0.056 | 0.54 |
| 90 | 0.042 | 0.40 |
| 100 | 0.032 | 0.31 |
| 125 | 0.018 | 0.17 |
| 150 | 0.0110 | 0.105 |
| 175 | 0.0073 | 0.067 |
| 200 | 0.0051 | 0.046 |
| 225 | 0.0037 | 0.032 |
| 250 | 0.0028 | 0.023 |
| 275 | 0.0021 | 0.017 |
| 300 | 0.0017 | 0.013 |
| 350 | 0.00108 | 0.0078 |
| 400 | 0.00072 | 0.0050 |
| 450 | 0.00050 | 0.0034 |
| 500 | 0.00036 | 0.0023 |
| 550 | 0.00026 | 0.0017 |
| 600 | 0.00020 | 0.00123 |
| 650 | 0.00015 | 0.00093 |
| 700 | 0.000117 | 0.00071 |
| 750 | 0.000092 | 0.00056 |
| 800 | 0.000074 | 0.00044 |
| 850 | 0.000060 | 0.00036 |
| 900 | 0.000049 | 0.00029 |
| 950 | 0.000040 | 0.00024 |
| 1000 | 0.000034 | 0.00020 |

## ACKNOWLEDGMENTS

This work grew out of several discussions with Professor David Pines, Professor John Wheatley, and Dr. William Abel. Dr. Bernard Abraham and Dr. Yakov Eckstein have kindly provided us with detailed information about their sound-attenuation experiments.

## APPENDIX: SOLUTION OF $\mathrm{He}^3$ BOLTZMANN EQUATION IN THE RELAXATION-TIME APPROXIMATION

We calculate the density-density response function which appears in Eq. (30) by solving the linearized $\mathrm{He}^3$ Boltzmann equation in the presence of a potential of the form $Ue^{i\mathbf{q}\cdot\mathbf{r}-i\omega t}$; in this case the induced $\mathrm{He}^3$ density fluctuations are given by

$$\delta\rho_3=\langle\rho_3\rho_3\rangle(q,\omega)U. \tag{A1}$$

The linearized $\mathrm{He}^3$ Boltzmann equation, neglecting

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[26] T. P. Ptukha, Zh. Eksperim. i Teor. Fiz. **40**, 1583 (1961) [English transl.: Soviet Phys.—JETP **13**, 1112 (1961)].

[27] J. Wheatley (private communication).

Fermi-liquid effects, is

$$(\omega-\mathbf{q}\cdot\mathbf{v})\delta f_{\mathbf{p}}+U\mathbf{q}\cdot\mathbf{v}\partial f_p{}^0/\partial\epsilon_p=-i(\delta f_{\mathbf{p}}-\delta\bar{f}_{\mathbf{p}})/\tau\,, \tag{A2}$$

where $\mathbf{v}=\mathbf{p}/m$ and $\delta f_{\mathbf{p}}$ is the deviation of the quasiparticle distribution function from equilibrium: $f_{\mathbf{p}}=f_p{}^0+\delta f_p$; $\tau$ is the time for the distribution to relax to a local equilibrium distribution $f_p{}^0+\delta\bar{f}_{\mathbf{p}}$ with the same number, momentum, and energy as $f_{\mathbf{p}}$. Thus

$$\delta\rho_3=\sum_{\mathbf{p}}\delta f_{\mathbf{p}}=\sum_{\mathbf{p}}\delta\bar{f}_{\mathbf{p}}\,, \tag{A3a}$$

$$m\delta\mathbf{j}_3=\sum_{\mathbf{p}}\mathbf{p}\delta f_{\mathbf{p}}=\sum_{\mathbf{p}}\mathbf{p}\delta\bar{f}_{\mathbf{p}}\,, \tag{A3b}$$

and from energy conservation in collisions

$$\sum_{\mathbf{p}}p^2\delta f_{\mathbf{p}}=\sum_{\mathbf{p}}p^2\delta\bar{f}_{\mathbf{p}}. \tag{A3c}$$

The function $f_p{}^0+\delta\bar{f}_{\mathbf{p}}$ is a spherical distribution slightly displaced from the origin, and therefore is of the form

$$\delta\bar{f}_{\mathbf{p}}=\delta\bar{f}_p{}^0-(\partial n_p{}^0/\partial\epsilon_p)gv_z\,, \tag{A4}$$

where $\delta\bar{f}_p{}^0$ is spherically symmetric about the origin, and where the $z$ axis has been taken along $\mathbf{q}$. The constant $g$ is determined from Eqs. (A3a) and (A3b), together with the continuity equation; one finds

$$g=m\omega\delta\rho_3/qn_3\,; \tag{A5}$$

$n_3$ is the equilibrium He$^3$ density.

The next step is to combine the continuity equation with the momentum-conservation equation which is derived by multiplying $\mathbf{p}$ into (A2) and then summing over all momenta; the result is

$$\omega^2\delta\rho_3-q^2\int\frac{d\mathbf{p}}{4\pi^3}v_z{}^2\delta f_{\mathbf{p}}=q^2n_3U/m. \tag{A6}$$

Thus to find $\langle\rho_3\rho_3\rangle$ we need to determine $\int v_z{}^2\delta f_p$ in terms of $\delta\rho_3$. Since $v\ll s$ we are concerned only with $\omega\gg vq$; then it suffices to determine $\int v_z{}^2\delta f_{\mathbf{p}}$ to lowest order in $qv/\omega$. We see from (A6) that to lowest order in $q$,

$$q^2U=m\omega^2\delta\rho_3/n_3\,, \tag{A7}$$

and from (A5) that $qg=m\omega\delta\rho_3/n_3$. Thus it is necessary to keep terms of order $q^0\delta\bar{f}_p{}^0$, $qg$, and $q^2U$. Taking moments of the expanded $\delta f_{\mathbf{p}}$ with respect to $v^2$ and $v_z{}^2$, and using (A5) and (A7), we discover

$$(1-i\tau\omega)\int\frac{d\mathbf{p}}{4\pi^3}v^2\delta f_p=\int\frac{d\mathbf{p}}{4\pi^3}v^2\delta\bar{f}_p{}^0-\frac{5i\tau\omega}{mn_3}P(T)\delta\rho_3 \tag{A8}$$

and

$$(1-i\tau\omega)\int\frac{d\mathbf{p}}{4\pi^3}v_z{}^2\delta f_{\mathbf{p}}=\frac{1}{3}\int\frac{d\mathbf{p}}{4\pi^3}v^2\delta\bar{f}_p{}^0-\frac{3i\tau\omega}{mn_3}P(T)\delta\rho_3\,, \tag{A9}$$

where

$$P(T)=\int\frac{d\mathbf{p}}{4\pi^3}\frac{mv^2}{3}=-\frac{m^2}{15}\int\frac{d\mathbf{p}}{4\pi^3}v^4\frac{\partial n^0}{\partial\epsilon_p}$$

is the pressure of an ideal Fermi gas of density $n_3$ and mass $m$ at temperature $T$.

From (A3c), $\int v^2\delta\bar{f}_p=\int v^2\delta f_p$; thus we may solve (A8) and (A9), to find

$$\int\frac{d\mathbf{p}}{4\pi^3}v_z{}^2\delta f_p=\left(1-\frac{4}{5}\frac{i\tau\omega}{1-i\tau\omega}\right)\frac{5}{3}\frac{P(T)}{mn_3}\delta\rho_3\,, \tag{A10}$$

$$\int\frac{d\mathbf{p}}{4\pi^3}v^2\delta f_p=\frac{5P(T)}{mn_3}\delta\rho_3. \tag{A11}$$

The latter equation is the statement that the energy variation is given by

$$\delta\int\frac{d\mathbf{p}}{4\pi^3}\frac{mv^2}{2}f_p=\left(\frac{\partial E}{\partial n_3}\right)_S\delta\rho_3=\frac{E+P}{n_3}\delta\rho_3\,, \tag{A12}$$

where $S$ is the entropy per particle, and $E$ the energy per unit volume.

Substituting (A10) into (A6) and comparing with (A1) we finally derive

$$\langle\rho_3\rho_3\rangle(q,\omega)$$
$$=\frac{q^2n}{m\omega^2-[5P(T)/3n_3]q^2\{1-\frac{4}{5}[i\tau\omega/(1-i\tau\omega)]\}}. \tag{A13}$$

Equation (31) is determined by dropping the $q^2$ corrections to the real part of the denominator, and using $P(0)=mv_f{}^2n_3/5$.

A word of caution: (A13) appears, for $\omega\tau\ll1$, to have a pole at

$$\omega^2=\frac{5}{3}\frac{P(T)}{mn_3}q^2(1-\tfrac{4}{5}i\tau\omega)\,, \tag{A14}$$

corresponding to a wave propagating at the adiabatic sound velocity. The attenuation of this wave seems to be a factor $\frac{2}{3}$ less than one normally finds[28] for such a wave in a weakly interacting monatomic gas. The reason is that Eq. (A13) is valid only for $\omega^2\gg v^2q^2$; in the region where the pole of (A13) occurs, the $q^2/\omega^2$ correction increase the attenuation by a factor $\frac{3}{2}$.[29]

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[28] L. D. Landau and E. M. Lifshitz, *Fluid Mechanics* (Addison-Wesley Publishing Company, Inc., Reading, Massachusetts, 1959), Eq. (77.6).

[29] L. P. Kadanoff and P. C. Martin, Ann. Phys. (N. Y.) **24**, 419 (1963), Eq. (54).