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Parametrically Excited Plasma Fluctuations

D. F. DuBois
Hughes Research Laboratories, Malibu, California
AND
M. V. Goldman
Department of Physics, University of California, Los Angeles, California
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The enhancement of the spectrum of density fluctuations in a parametrically excited plasma is studied using the linearized theory. In the particular case studied, the plasma is "pumped" by a monochromatic electromagnetic field of either transverse or longitudinal polarization with frequency $\omega_0 \simeq$ (electron plasma frequency) + (ion acoustic frequency). A strong enhancement of the fluctuation power at the two individual frequencies is predicted as the previously calculated threshold for instability or oscillation of the linearized theory is approached. The experiments of Stern and Tzoar and other recent theories are discussed. Approximate expressions are derived for the strength of the enhanced resonances and the range of $\mathbf{k}$ vectors which actively participate. The cross section for inelastic scattering of radiation from the excited plasma is derived and compared with a result due to Berk, who did not consider the important regenerative process of parametric amplification. The saturation of the noise level for pump powers above the linear threshold is treated.

1. INTRODUCTION

THE parametric excitation of longitudinal electron plasma waves and ion acoustic waves in a plasma by strong, externally controlled, monochromatic pump fields was recently proposed and studied theoretically by the present authors.[1,2] Stern and Tzoar[3] recently reported the experimental generation of strongly enhanced noise signals at the optical and acoustic wave frequencies from a discharge plasma excited by a pump signal near the plasma frequency. This enhancement occurred only for pump powers above a threshold, which is characteristic of parametric excitation.

The idea of parametric amplification is, of course, a very old one going back to Lord Rayleigh and has received many applications in electrical devices[4] and more recently in nonlinear optics.[5] The main issue here is the detailed application of these ideas to plasmas, including the calculation of the relevant coupling parameters.

In I and II we treated the problem of a transverse (pump) beam of frequency ω_0 whose amplitude was weak in the sense that the ratio of the pump energy per unit volume to the plasma thermal energy per unit volume was small. When $\omega_0 \simeq \omega_1 + \omega_2$, where ω_1 and ω_2 are frequencies of (approximate) normal modes of the system, it was shown that for sufficient pump power (to overcome losses in these modes) these modes could be driven simultaneously unstable. It was shown that the lowest threshold for instability occurred when ω_1 was near the electron plasma frequency and ω_2 was in the range of ion acoustic frequencies. Another possibility was for both ω_1 and ω_2 to be near the electron plasma frequency, but this was seen to be much weaker and to depend on the spatial variation of the pump field.[2] In addition to the negative damping of the coupled modes there was a small power-dependent shift of the resonant frequencies.

A rather large number of other theoretical studies of parametric excitation of plasma waves have appeared. Silin[6] based his original theory on the cold-plasma equations and thus did not include the coupling of electron plasma waves and acoustic waves, nor did he include the effect of wave damping. The present authors[1,2] showed the coupling of Langmuir waves to ion acoustic waves to be the strongest three-wave coupling. We also consistently took into account damping effects which are essential to estimating the threshold. Our conclusions were substantially verified by Jackson,[7] who based his theory on the linearized Vlasov equation, and by Lee and Su,[8] who worked with a hydrodynamic model. Jackson[7] also extended the kinetic theory to the strong-field case and obtained partial agreement with the cold-plasma results of Silin.[6] A number of other papers[9] have considered parametric excitation of transverse waves in plasmas, but these do not bear directly on the experiments of Stern and Tzoar or on the considerations of the present paper.

Of these papers, only the work of Goldman[2,10] has been concerned with the noise power spectrum of

[1] D. F. DuBois and M. V. Goldman, Phys. Rev. Letters **14**, 544 (1965); henceforth referred to as I.
[2] M. V. Goldman, Ann. Phys. (N. Y.), **38**, 95 (1966); Hughes Research Laboratories Research Report No. 343, 1965 (unpublished), henceforth referred to as II.
[3] R. A. Stern and N. Tzoar, Phys. Rev. Letters **17**, 903 (1966).
[4] W. H. Louisell, *Coupled Mode and Parametric Electronics* (John Wiley & Sons, Inc., New York, 1960).
[5] N. Bloembergen, *Nonlinear Optics* (W. A. Benjamin, Inc., New York, 1965).
[6] V. P. Silin, Zh. Eksperim. i Teor. Fiz. **47**, 1977 (1965) [English transl.: Soviet Phys.—JETP **21**, 1127 (1965)].
[7] E. A. Jackson, Phys. Rev. **153**, 235 (1967).
[8] Y. C. Lee and C. H. Su, Phys. Rev. **152**, 129 (1966).
[9] M. V. Goldman and D. F. DuBois, Phys. Fluids **8**, 1404 (1965); A. Yariv, in *Proceedings of the Seventh International Conference on Ionization Phenomena in Gases, Belgrade, 1965* (Belgrade, 1966), Paper No. 4.4.5(3); D. Montgomery and I. Alexeff, Phys. Fluids **9**, 1362 (1966).
[10] M. V. Goldman (to be published).

parametrically excited density fluctuations. Experiments of the type of Stern and Tzoar can measure this incoherent fluctuation spectrum. The other theories apply, strictly speaking, to amplification of coherent signals which are introduced into the plasma. Therefore, the main purpose of this paper is to analyze the fluctuation spectrum in detail for three-mode coupling in a two-temperature plasma.

We also wish to comment on the applicability of this theory to the experiments of Stern and Tzoar.[3] In these experiments a finite, inhomogeneous cylindrical discharge plasma was used in which the excited optical modes were the radial Tonks-Dattner resonances (the analog of the electron plasma waves in an infinite homogeneous plasma). In addition, the monochromatic pump wave used here was itself a longitudinal Tonks-Dattner resonance which was excited by the externally produced fields.

The simplest model, and the one used by all authors to date, is that of an infinite, homogeneous plasma. The pump field can be taken to have a longitudinal polarization, in contrast to the transverse polarization explicitly considered in I and II. It is physically obvious, however, that in *the limit of the infinite pump wavelength*, which was used in all theories[1–3,6–8] for the plasma-acoustic wave coupling, there can be no physical difference between a transverse pump and a longitudinal pump field in a uniform, isotropic plasma.[11] This was demonstrated explicitly by Lee and Su,[8] who obtained exactly the results of I and II by using a longitudinal pump. A more serious question, which was not observed by these authors, is that the infinite-wavelength limit is *not* kinematically consistent for the three-mode coupling with longitudinal pumping in an infinite, homogeneous plasma. Momentum conservation and the approximate frequency-matching conditions cannot be satisfied for $k_0\equiv 0$. However, we will show that even when the finite k_0 is taken into account, the three-mode coupling is unchanged provided $(k_0/k_D)\ll 1$ and the dispersion relation of I and II still applies.

The pump intensity threshold level predicted by this three-mode theory is higher than that observed by Stern and Tzoar. The differences might reasonably be expected to result from the oversimplified model of the plasma. However, a more serious qualitative difference still exists. The noise spectrum was observed[3] to have strong components at frequency Ω (acoustic frequency) and at $\omega_0-\Omega$ (plasma resonance), as predicted by the three-mode theory, plus an *equally* strong component at $\omega_0+\Omega$, *not* predicted by this theory. It is clear that this can be explained only by a *four*-wave parametric coupling. The nonlinear susceptibilities for this coupling and the appropriate dispersion relation were also calculated by Goldman[2] for the case of a transverse pump, but were not analyzed for parameters corresponding to these experiments. In another paper now in preparation we will present a detailed analysis of the predictions of this four-mode theory. The threshold intensity turns out to be equal to or greater than the three-mode threshold.

The present paper will be devoted to the analysis of the fluctuation spectrum in the *three-mode case*. There are clearly defined experimental conditions under which this coupling is more important than the four-mode coupling.[12] The analysis of the simpler three-mode case is essential as a first step in understanding and carrying out the more complicated four-mode case. The general analysis is presented in a form applicable to any three-mode parametric coupling.

In Sec. 2 a derivation of the basic equations of the theory is presented which should make the theory more accessible to experimentalists than the quantum derivations of I and II.

The linear and nonlinear electromagnetic susceptibilities for the plasma are derived from the collisionless Boltzman-Vlasov equation for longitudinal or transverse pump fields. The finite k_0 of the longitudinal pump field is taken into account. Coupled Maxwell equations for the high- and low-frequency plasma modes are solved, and expressions are derived for the fluctuation spectrum.

The behavior of the density-fluctuation spectrum $S_0(k,\omega)$ is dominated by the resonances in the response function of the linearized system $\epsilon^{NL}(k,\omega)^{-1}$. In Sec. 3 the zeros of the analytic continuation of $\epsilon^{NL}(k,\omega)$ in the complex ω plane and the residues at these poles are determined. This can be done analytically if the frequency arguments of the linear dielectric functions, which make up ϵ^{NL}, are near the complex zeros of these linear functions. This is seen to be valid for frequencies near the electron plasma frequency ω_p and, if the electron temperature is greater than the ion temperature, for frequencies near the ion acoustic frequency $\omega_i(k)$. For equal temperatures the formulas derived in this approximation are seen to be qualitatively accurate but not quantitatively so, by comparison with numerical work done in I and II.

In Sec. 4 we use the properties of $\epsilon^{NL}(k,\omega)^{-1}$ derived in Sec. 3 to examine the behavior of $S_0(k,\omega)$ in the vicinity of the resonances. We find that as the pump power approaches the threshold for which the most favorably matched mode **k** goes unstable, there is a great enhancement of the resonances in the spectrum. This enhancement occurs for a narrow range of wave vectors around **k** which receive comparable negative damping from the parametric coupling. For most cases of interest the resonance near ω_p is more strongly en-

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[11] The brief theoretical discussion by Stern and Tzoar (Ref. 3) implied that this was not the case. Their theory also seemed to imply that the threshold condition was not a symmetrical function of the optical and acoustic wave losses, in turn implying a different threshold for the two waves. This would violate the Manley-Rowe relations which must hold for such problems.

[12] Recent experiments by Wong on three-mode parametric excitation of drift waves in a plasma have been reported [A. Y. Wong and M. V. Goldman (to be published)].

hanced than that near ω_i. Near threshold, however, both resonances have the *same* width which goes to zero at resonance in the linearized theory. The spectral peaks are enhanced more strongly than the total area under the resonances (i.e., the total noise power), but the linearized theory predicts that the enhancement of both quantities goes to infinity at the point of linear threshold.

In Sec. 5 we discuss the cross section for inelastic scattering of another beam of higher frequency from the parametrically excited plasma. The differential cross section is known to be proportional to $S_0(k,\omega)$. The cross section is greatly enhanced in a narrow range of angles corresponding to the range of "active" **k** vectors which are parametrically excited. Because of k-dependent factors in the cross section, arising because the heavy ions do not respond directly to the scattering electromagnetic fields, the enhanced resonances in the scattering at ω_p and ω_i have comparable cross sections in many cases. The total scattering integrated over the resonances and over the narrow range of "active" **k** vectors is enhanced. Again all the enhancement factors diverge at linear threshold.

We compare our results with the work of Berk,[13] who did not include the regenerative parametric effect. Near threshold for instability this effect is extremely important, and Berk's result greatly underestimates the enhanced cross section.

In Sec. 6 we consider the important question of the nonlinear saturation of the level of fluctuations near and above the threshold for the linearized theory. The large longitudinal fluctuating currents induced in the system react back on the pump field, so that the steady-state pump amplitude is not that predicted by the linear theory. Conservation of energy arguments show that the self-consistent pump field adjusts intself so that the effective pump amplitude is always below but near the linear threshold. The enhancement factors then become exponentially increasing *but finite* functions of the pump intensity for intensities above the linear threshold.

In Sec. 7 we briefly comment on the effect of nonlinear longitudinal mode coupling, which has not been included in the present work.

2. BASIC THEORY

We begin by calculating the nonlinear susceptibility, which couples a monochromatic electromagnetic "pump" wave

$$\mathbf{E}(\mathbf{r},t)=\mathbf{E}_0\sin(\mathbf{k}_0\cdot\mathbf{r}-\omega_0 t)$$
$$\mathbf{B}(\mathbf{r},t)=(c\mathbf{k}_0/\omega_0)\times\mathbf{E}(\mathbf{r},t) \tag{2.1}$$

to the plasma and ion acoustic resonances of the self-consistent longitudinal field $U(\mathbf{r},t)$ in a two-temperature plasma. A simple perturbative approach based on the Vlasov equation is equally applicable to the case of transverse or longitudinal pump waves. Our result for the transverse pump agrees with previous calculations[1,2,7,8]; however, the result for the longitudinal pump disagrees with the hydrodynamic theory which Stern and Tzoar[3] used to interpret their experiment, in which a longitudinal pump parametrically excites the plasma and ion-acoustic waves.

The Vlasov equation for electrons in an electromagnetic field is

$$\partial f/\partial t+\mathbf{v}\cdot\partial f/\partial\mathbf{r}=-\mathbf{a}(\mathbf{v},\mathbf{r},t)\cdot\partial f/\partial\mathbf{v}\,, \tag{2.2}$$

where

$$\mathbf{a}(\mathbf{v},\mathbf{r},t)=(q/m)\{-\nabla U(\mathbf{r},t)+F(\mathbf{r},t)/q\}\,, \tag{2.3a}$$
$$F(\mathbf{r},t)/q=E(\mathbf{r},t)+(\mathbf{v}/c)\times\mathbf{B}(\mathbf{r},t)\,. \tag{2.3b}$$

a is the acceleration of an electron as a result of the self-consistent longitudinal field ∇U and the Lorentz force F of the total field $(\mathbf{E},\mathbf{B})$. In Fourier space, Eq. (2.2) may be rewritten as

$$f(\mathbf{v},\mathbf{k},\omega)=-i\int\frac{d^3k'd\omega'}{(2\pi)^4}\times\frac{\mathbf{a}(\mathbf{v},\mathbf{k}-\mathbf{k}',\omega-\omega')}{(\omega-\mathbf{k}\cdot\mathbf{v})}\cdot\frac{\partial f}{\partial\mathbf{v}}(\mathbf{v},\mathbf{k}',\omega')\,. \tag{2.4}$$

The singularities in ω are to be handled in the Landau sense. Let

$$f=(2\pi)^4\delta^3(\mathbf{k})\delta(\omega)f_0(\mathbf{v})+f_1(\mathbf{v},\mathbf{k},\omega)+f_2(\mathbf{v},\mathbf{k},\omega)+\cdots,$$

where $f_0(v)$ is the equilibrium distribution functions, f_1 is first order in **a**, f_2 is second order in **a**, etc. Thus,

$$f_1(\mathbf{v},\mathbf{k},\omega)=-\frac{i\mathbf{a}(\mathbf{v},\mathbf{k},\omega)}{\omega-\mathbf{k}\cdot\mathbf{v}}\cdot\frac{\partial f_0(\mathbf{v})}{\partial\mathbf{v}} \tag{2.5}$$

$$f_2(\mathbf{v},\mathbf{k},\omega)=(-i)^2\int\frac{d^3k'd\omega'}{(2\pi)^4}\frac{\mathbf{a}(\mathbf{v},\mathbf{k}-\mathbf{k}',\omega-\omega')}{(\omega-\mathbf{k}\cdot\mathbf{v})}\cdot\frac{\partial}{\partial\mathbf{v}}\times\left(\frac{\mathbf{a}(\mathbf{v},\mathbf{k}',\omega')}{\omega'-\mathbf{k}'\cdot\mathbf{v}}\cdot\frac{\partial f_0(v)}{\partial\mathbf{v}}\right), \tag{2.6}$$

etc. The average electron charge density is

$$\rho(k,\omega)=q\int d^3\mathbf{v}f(\mathbf{v},\mathbf{k},\omega)=(2\pi)^4\delta^3(\mathbf{k})\delta(\omega)qn_e+\rho_1(\mathbf{k},\omega)+\rho_2(\mathbf{k},\omega)\,, \tag{2.7}$$

where ρ_1 and ρ_2 are, respectively, first and second order in **a**. The average charge qn_e in equilibrium is exactly canceled by the positive ion background, and so may be neglected. We are interested only in that part of the charge density which is proportional to the self-consistent field U. Terms which go as U^2 (e.g., in ρ_2) are assumed shall compared with terms which are linear in U, this is an effective limitation on how hard we excite

[13] H. Berk, Phys. Fluids 7, 917 (1964).

the system, and terms independent of U are proportional to δ functions such as $\delta(\omega)$, $\delta(\omega\pm\omega_0)$, $\delta(\omega\pm 2\omega_0)$, etc., since the external field is monochromatic. Four our purposes $\omega\neq n\omega_0$, $n=0$, ± 1, $\cdots$; therefore, such terms vanish. A more detailed treatment of those neglected terms is found in I and II.

The equilibrium longitudinal linear susceptibility χ_e is defined by

$$4\pi\rho_1(\mathbf{r},t)=\nabla^2\int d^3r'dt'\,\chi_e(\mathbf{r}-\mathbf{r}',t-t')U(\mathbf{r}',t'). \quad (2.8)$$

From (2.5), integrated over velocity space, we retain

$$4\pi\rho_1(k,\omega)=-k^2\chi_e(\mathbf{k},\omega)U(\mathbf{k},\omega), \quad (2.9)$$

where

$$\chi_e(\mathbf{k},\omega)=\frac{4\pi e^2}{mk^2}\int d^3\mathbf{v}\frac{\mathbf{k}\cdot[\partial f_0(\mathbf{v})/\partial\mathbf{v}]}{\omega-\mathbf{k}\cdot\mathbf{v}+i\epsilon} \quad (2.10)$$

is the usual[14] linear susceptibility. It is convenient to define a quantity q (the "proper polarization part"), defined by

$$q(k,\omega)=\frac{1}{mn_e\beta}\int d^3v\frac{\mathbf{k}\cdot[\partial f_0(\mathbf{v})/\partial\mathbf{v}]}{\omega-\mathbf{k}\cdot\mathbf{v}+i\epsilon}, \quad (2.11)$$

where β is the inverse thermal energy. In terms of q,

$$\chi_e(k,\omega)=(k_D{}^2/k^2)q(k,\omega), \quad (2.12)$$

where $k_D{}^2=4\pi n_e{}^2\beta$ is the square inverse Debye length.

The integral of (2.6) over velocity space gives the Fourier component of the electron charge density to second order in a. Only the crossterms linear in U are to be retained in this expression, which (after a trivial change of variables in one term) may be written as

$$4\pi\rho_2(\mathbf{k},\omega)=-\int\frac{d^3k'd\omega'}{(2\pi)^4}\tilde{Q}_1(\mathbf{k},\omega;\mathbf{k}'-\mathbf{k},\omega'-\omega)\times U(\mathbf{k}-\mathbf{k}',\omega-\omega') \quad (2.13)$$

$$\tilde{Q}_1(\mathbf{k},\omega;\mathbf{k}'-\mathbf{k},\omega'-\omega)=-4\pi\frac{e^2i}{m^2}\sum_j\int d^3v\left\{\frac{k_i-k_i'}{\omega-\mathbf{k}\cdot\mathbf{v}}\frac{\partial}{\partial v_i}\right.\times\left[\frac{F_j(\mathbf{v},\mathbf{k}',\omega')}{\omega'-\mathbf{k}'\cdot\mathbf{v}}\frac{\partial f_0(v)}{\partial v_j}\right]+\frac{k_j-k_j'}{\omega-\mathbf{k}\cdot\mathbf{v}}F_i(\mathbf{v},\mathbf{k}',\omega')\frac{\partial}{\partial v_i}\times\left.\left[\frac{1}{\omega-\omega'-[\mathbf{k}-\mathbf{k}']\cdot\mathbf{v}}\frac{\partial f_0}{\partial v_j}(v)\right]\right\}. \quad (2.14)$$

$\tilde{Q}_1(\mathbf{k},\omega;\mathbf{k}'-\mathbf{k},\omega'-\omega)$ may be regarded as the Fourier transform of a nonlinear susceptibility $\tilde{Q}_1(\mathbf{r},t;\mathbf{r}',t')$ which contains the first-order modulating effects of the pump field $(\mathbf{E},\mathbf{B})$ of external origin. Thus, in coordinate space, $4\pi\rho_2(\mathbf{r},t)=-\int d^3r'dt'\tilde{Q}_1(\mathbf{r},t;\mathbf{r}',t')U(\mathbf{r}',t')$ in analogy[15] with (2.8).

After carrying out the differentiation with respect to v_i in (2.14), it is convenient to express the term

$$\frac{k_i-k_i'}{(\omega-\omega'-[\mathbf{k}-\mathbf{k}']\cdot\mathbf{v})^2}\frac{F_i(k',\omega')}{(\omega-\mathbf{k}\cdot\mathbf{v})}\frac{\partial f_0(v)}{\partial v_j}$$

$$=\left[\frac{\partial}{\partial v_i}\frac{1}{\omega-\omega'-[\mathbf{k}-\mathbf{k}']\cdot\mathbf{v}}\right]\frac{F_i(k',\omega')}{\omega'-\mathbf{k}'\cdot\mathbf{v}}\frac{\partial f_0(v)}{\partial v_j}+\frac{k_i-k_i'}{(\omega'-\mathbf{k}'\cdot\mathbf{v})^2}\left[\frac{1}{\omega-\mathbf{k}\cdot\mathbf{v}}-\frac{1}{\omega-\omega'-[\mathbf{k}-\mathbf{k}']\cdot\mathbf{v}}\right]\times\frac{\partial f_0}{\partial v_j}E_i(k',\omega'). \quad (2.15)$$

The first term on the right-hand side of (2.15) may then be integrated by parts. When this is done, all terms proportional to $(\partial^2 f_0/\partial v_i\partial v_j)$ in the integrand of (2.14) are seen to cancel. We may further use the relation

$$\frac{\partial F_i(\mathbf{k}',\omega')}{\partial v_i}=\frac{k_i'F_i-k_i'F_j}{\omega'-\mathbf{k}'\cdot\mathbf{v}},$$

which is easily proved from (2.3b), the Maxwell equation $\mathbf{B}(\mathbf{k}',\omega')=(c\mathbf{k}'/\omega')\times\mathbf{E}(\mathbf{k}',\omega')$, and standard vector identities. One then obtains the following useful expression for $\tilde{Q}_1$:

$$\tilde{Q}_1(\mathbf{k},\omega;\mathbf{k}'-\mathbf{k},\omega'-\omega)=4\pi e^2\frac{i}{m^2}\int d^3\mathbf{v}\frac{1}{(\omega'-\mathbf{k}'\cdot\mathbf{v})^2}\times\left[\frac{(\mathbf{k}'-\mathbf{k})\cdot\mathbf{F}(\mathbf{v};\mathbf{k}',\omega')\mathbf{k}}{\omega-\mathbf{k}\cdot\mathbf{v}}+\frac{\mathbf{k}\cdot\mathbf{F}(\mathbf{v};\mathbf{k}',\omega')(\mathbf{k}'-\mathbf{k})}{\omega'-\omega-[\mathbf{k}'-\mathbf{k}]\cdot\mathbf{v}}\right]\times\frac{\partial f_0(v)}{\partial\mathbf{v}}. \quad (2.16)$$

The magnetic part of the Lorentz force $\mathbf{F}(\mathbf{v},\mathbf{k}',\omega')$ vanishes for a longitudinal pump field, and for transverse pump radiation it may be ignored for the following reason: From (2.1) and (2.3b) it is evident that B is of order v_ek_0/ω_0 times the electric field, where $v_e=(\beta m)^{-1/2}$ is the electron thermal velocity. The pump frequency ω_0 must be very near the plasma frequency ω_p, since the ion-acoustic frequency is $\ll\omega_p$. Thus, $v_ek_0/\omega_0\approx k_0/k_D$. The dispersion relation for transverse radiation near the plasma frequency is $\omega_0=\omega_p(1+c^2k_0{}^2/v_e{}^2k_D{}^2)^{1/2}$, so we require $(k_0/k_D)\ll v_e/c$. The B term may then be ignored since we need not calculate beyond zeroth order

[14] B. D. Fried and L. D. Conte, *The Plasma Dispersion Function* (Academic Press, Inc., New York, 1961).

[15] We have expressed this relation in terms of the polarization $\tilde{Q}_1$ rather than a nonlinear susceptibility χ_1 related to it by $\tilde{Q}_1=+\nabla^2\chi_1$ in order to take advantage of certain symmetry properties present only in $\tilde{Q}_1$.

in k_0 or v_e/c for a transverse pump. The Lorentz force $\mathbf{F}$ may therefore be taken simply as

$$\mathbf{F}(\mathbf{k}',\omega')=-|e|\mathbf{E}(k',\omega')$$
$$=[(2\pi)^4 m\omega_0^2/2i]\mathbf{d}$$
$$\times[\delta(\omega'+\omega_0)\delta^3(\mathbf{k}'+\mathbf{k}_0)-\delta(\omega'-\omega_0)\delta^3(\mathbf{k}'-\mathbf{k}_0)], \quad (2.17)$$

where

$$\mathbf{d}=-(|e|/m\omega_0^2)\mathbf{E}_0 \quad (2.18)$$

is the maximum excursion distance from the trajectory of an electron perturbed by the monochromatic pump field $\mathbf{E}(\mathbf{r},t)=\mathbf{E}_0\sin(\mathbf{k}_0\cdot\mathbf{r}-\omega_0 t)$. If (2.17) is inserted into (2.16), we have

$$\tilde{Q}_1(\mathbf{k},\omega;\,\mathbf{k}'-\mathbf{k},\omega'-\omega)=Q_1(\mathbf{k},\omega;\,\mathbf{k}'-\mathbf{k},\omega'-\omega)(2\pi)^4$$
$$\times[\delta(\omega'+\omega_0)\delta^3(\mathbf{k}'+\mathbf{k}_0)-\delta(\omega'-\omega_0)\delta^3(\mathbf{k}'-\mathbf{k}_0)], \quad (2.19)$$

where

$$Q_1(\mathbf{k},\omega;\,k'-k,\omega'-\omega)=\frac{4\pi e^2\omega_0^2}{m^2}\int d^3\mathbf{v}\frac{1}{(\omega'-\mathbf{k}'\cdot\mathbf{v})^2}$$
$$\times\left[\frac{(\mathbf{k}'-\mathbf{k})\cdot\mathbf{dk}}{\omega-\mathbf{k}\cdot\mathbf{v}}+\frac{\mathbf{k}\cdot\mathbf{d}(\mathbf{k}'-\mathbf{k})}{\omega'-\omega-[\mathbf{k}'-\mathbf{k}]\cdot\mathbf{v}}\right]\cdot\frac{\partial f_0(v)}{\partial\mathbf{v}}. \quad (2.20)$$

This expression is easily evaluated to zeroth order in $(k'v_e/\omega)\approx k_0/k_D$ simply by neglecting $\mathbf{k}'\cdot\mathbf{v}$ compared with ω' in the first denominator. Then, for example,

$$Q_1(\mathbf{k},\omega;\,\mathbf{k}_0-\mathbf{k},\omega_0-\omega)=\tfrac{1}{2}k_D^2[(\mathbf{k}_0-\mathbf{k})\cdot\mathbf{d}q(k,\omega)$$
$$+\mathbf{k}\cdot\mathbf{d}q^*(|\mathbf{k}_0-\mathbf{k}|,\omega_0-\omega)], \quad (2.21)$$

where q is defined in (2.11) and (2.12) to be proportional to the linear electronic susceptibility. This applies to a transverse or longitudinal pump field, the only difference being whether $\mathbf{d}$ points in the $\mathbf{k}_0$ direction or transverse to it. We note, in passing, an important symmetry property in (2.21): $Q_1(\mathbf{k},\omega;\,\mathbf{k}_0-\mathbf{k},\omega_0-\omega)=Q_1^*(\mathbf{k}_0-\mathbf{k},\omega_0-\omega;\,\mathbf{k},\omega)$.[16] As we shall soon see, ω and $\mathbf{k}$ may correspond to the frequency and wave number of a plasma wave, $\omega\approx\omega_L(k)=\omega_p(1+3k^2/k_D^2)^{1/2}$, and $\omega_0-\omega$ and $\mathbf{k}_0-\mathbf{k}$ then correspond to the frequency and wave number of an ion acoustic wave $(\omega_i\approx k_i v_e(m/M)^{1/2}$, where $M=$ ion mass). For a transverse pump field the frequency matching condition tells us that $\omega_p(1+c^2k_0^2/v_e^2k_D^2)^{1/2}\approx\omega_p(1+3k^2/k_D^2)^{1/2}$, or k_0 is of order $(v/c)k$. In this case k_0 may be totally neglected in (2.20) and (2.21) which reduces to the result previously found in I and II for this case.[16] In particular, the dominant term in (2.21) is the second, since $q(k,\omega_0-\omega)\cong1$, whereas $q(k,\omega)\cong k^2/k_D^2$.

For a longitudinal pump field, k_0 must still be $<k_D$ for an undamped plasma wave, but it generally cannot be neglected in comparison with k. However, the dominant term in (2.21) is again the second, so for either a transverse *or* longitudinal pump

$$Q_1(\mathbf{k},\omega;\,\mathbf{k}_0-\mathbf{k},\omega_0-\omega)\approx\tfrac{1}{2}k_D^2\mathbf{k}\cdot\mathbf{d}. \quad (2.22)$$

The correction terms arising from the first-order expansion of $(\omega'-\mathbf{k}'\cdot\mathbf{v})^{-2}$ in (2.20) lead to a correction to (2.22) which is of order (k_0/k_D) or smaller, and generally negligible. One may also determine the convergence of this entire perturbative procedure in the strength $\mathbf{d}$ of the pump field. Such convergence is in fact guaranteed if $dk_D\ll1$, or if electron excursion distances are less than the Debye distance. This is demonstrated in II.

There will also be an ionic contribution to the total charge density, in which linear and nonlinear ionic susceptibilities are defined simply by replacing the electron mass m and temperature Θ_e (or thermal velocity v_e) by the ion mass M and temperature Θ_i (or thermal velocity v_i). However, the nonlinear ionic susceptibilities may be neglected because they are of order m/M times the nonlinear electronic susceptibilities, thus exhibiting the preference of the external field for the lighter electrons. However, the linear ionic susceptibility must be retained as it contributes to the ion acoustic resonance. We may then write for the total charge density (electrons+ions) in Fourier space,

$$-4\pi\rho_{\rm tot}(k,\omega)=k^2[\chi_e(\mathbf{k},\omega)+\chi_i(\mathbf{k},\omega)]U(\mathbf{k},\omega)$$
$$+Q_1(\mathbf{k},\omega;\,-\mathbf{k}_0-\mathbf{k},-\omega_0-\omega)U(\mathbf{k}+\mathbf{k}_0,\omega+\omega_0)$$
$$-Q_1(\mathbf{k},\omega;\,\mathbf{k}_0-\mathbf{k},\omega_0-\omega)U(\mathbf{k}-\mathbf{k}_0,\omega-\omega_0), \quad (2.23)$$

where we have used (2.13) and (2.19). Continuation of our perturbative expansion in d to next order gives rise to terms of the following form:

$$Q_2(\mathbf{k},\omega;\,-2\mathbf{k}_0-\mathbf{k},-2\omega_0-\omega)U(\mathbf{k}+2\mathbf{k}_0,\omega+2\omega_0)$$
$$+Q_2(\mathbf{k},\omega;\,2\mathbf{k}_0-\mathbf{k},2\omega_0-\omega)U(\mathbf{k}-2\mathbf{k}_0,\omega-2\omega_0)$$
$$-2Q_2(\mathbf{k},\omega;\,-\mathbf{k},-\omega)U(\mathbf{k},\omega), \quad (2.24)$$

where each of the Q_2's is proportional to d^2.

We wish next to calculate the correlation of total charge density fluctuations in the nonequilibrium steady state prevailing when the pump radiation intensity is just below the threshold for instability of plasma and ion-acoustic modes. By envisioning a steady state we neglect the relatively slow secular heating of the plasma by the pump radiation. A complete treatment of all terms in the fluctuation spectrum for the case of a transverse pump and equal electron and ion temperatures was given in II, while a simpler, more intuitive presentation was described in Ref. 10; valid for lossless mode coupling (real nonlinear susceptibilities). Since the imaginary parts of our nonlinear susceptibilities are much smaller than the real parts, this latter format is adequate for our purposes, and may be used for a generalized treatment which includes the possibility of a longitudinal pump and unequal electron and ion temperatures. Consider the self-consistent field U and the total charge density $\rho_{\rm tot}$ to be fluctuating quantities whose thermal ensemble averages vanish but whose average correla-

[16] In the notation of II, $Q_1(\mathbf{k},\omega;\,\mathbf{k}_0-\mathbf{k},\omega_0-\omega)=k^2\mathbf{A}_0\cdot\chi^{NL}\times(\mathbf{k}_0,\omega_0;\,\mathbf{k}-\mathbf{k}_0,\omega-\omega_0)$, where $\mathbf{A}_0$ is the vector magnitude of the vector potential for $E_0(\mathbf{r},t)$.

tions do not. Such quantities may be thought of either as stochastic classical variables or as quantum-mechanical Heisenberg picture operators. U is related to its source by Poisson's equation

$$-k^2U(\mathbf{k},\omega)=-4\pi\rho_{\rm tot}(\mathbf{k},\omega)-4\pi\rho^0(\mathbf{k},\omega)\,, \quad (2.25)$$

where $\rho^0(\mathbf{k},\omega)$ is the Fourier transform of the fluctuating total free charge density in the absence of self-consistent or external fields. $\rho_{\rm tot}$, as before, is the total fluctuating polarization charge density, proportional to components of the self-consistent field U. Suppose first that we take the external fields equal to zero, so

$$-4\pi\rho_{\rm tot}(\mathbf{k},\omega)=k^2\chi(\mathbf{k},\omega)U(\mathbf{k},\omega)\,, \quad (2.26)$$

where

$$\chi(\mathbf{k},\omega)=\chi_e(\mathbf{k},\omega)+\chi_i(\mathbf{k},\omega)\,. \quad (2.27)$$

Thus,

$$-k^2\epsilon^L(\mathbf{k},\omega)U(\mathbf{k},\omega)=-4\pi\rho^0(\mathbf{k},\omega)\,, \quad (2.28)$$

where

$$\epsilon^L(\mathbf{k},\omega)=1+\chi(\mathbf{k},\omega) \quad (2.29)$$

is the longitudinal dielectric function. The thermal average of (2.28) gives zero on both sides, but the average absolute square relates the correlation of longitudinal self-consistent-field fluctuations to correlations of total charge density fluctuations in the noninteracting plasma:

$$k^4|\epsilon^L(\mathbf{k},\omega)|^2\langle|U(\mathbf{k},\omega)|^2\rangle=(4\pi)^2\langle|\rho^0(\mathbf{k},\omega)|^2\rangle\,. \quad (2.30)$$

In a two-temperature equilibrium these correlation functions may be determined either by direct evaluation[2,10] or by application of Nyquist's theorem.[17] (We assume the collisionless approximation.) In the classical limit,

$$\lim_{\Omega,T\to\infty}\frac{\langle|\rho^0(\mathbf{k},\omega)|^2\rangle}{\Omega T}=\frac{2k^2}{4\pi\omega}[\Theta_e\,{\rm Im}\chi_e(\mathbf{k},\omega)+\Theta_i\,{\rm Im}\chi_i(\mathbf{k},\omega)]\,, \quad (2.31)$$

where Ω and T are the volume and time of observation, respectively. This determines $\langle|U(\mathbf{k},\omega)|^2\rangle$ through (2.30). Of more direct interest to us will be the correlation of total charge density fluctuations, defined by

$$e^2S_0(\mathbf{k},\omega)=\lim_{\Omega,T\to\infty}\frac{\langle|\rho_{\rm tot}(\mathbf{k},\omega)+\rho^0(\mathbf{k},\omega)|^2\rangle}{\Omega T}\,. \quad (2.32)$$

Employing Poisson's equation [Eq. (2.25)], we see that S_0 is proportional to $\langle|U|^2\rangle$:

$$\frac{4\pi e^2}{k^2}S_0(\mathbf{k},\omega)=\frac{k^2}{4\pi}\lim_{\Omega,T\to\infty}\frac{\langle|U(\mathbf{k},\omega)|^2\rangle}{\Omega T}\,. \quad (2.33)$$

Using the above information for the equilibrium plasma, we now calculate $S_0(\mathbf{k},\omega)$ in the presence of an external pump field of intensity below threshold (to guarantee a steady state). From (2.23) and (2.25),

$$\begin{aligned}k^2\epsilon^L(\mathbf{k},\omega)U(\mathbf{k},\omega)=&-Q_1(\mathbf{k},\omega;-\mathbf{k}_0-\mathbf{k},-\omega_0-\omega)\\&\times U(\mathbf{k}+\mathbf{k}_0,\omega+\omega_0)+Q_1(\mathbf{k},\omega;\mathbf{k}_0-\mathbf{k},\omega_0-\omega)\\&\times U(\mathbf{k}-\mathbf{k}_0,\omega-\omega_0)+4\pi\rho^0(\mathbf{k},\omega)\,.\end{aligned} \quad (2.34)$$

This generates an infinite set of coupled equations, of which another [obtained by displacement of (k,ω) into $(\mathbf{k}-\mathbf{k}_0,\omega-\omega_0)$] is

$$\begin{aligned}|\mathbf{k}-\mathbf{k}_0|^2&\epsilon^L(\mathbf{k}-\mathbf{k}_0,\omega-\omega_0)U(\mathbf{k}-\mathbf{k}_0,\omega-\omega_0)\\&=-Q_1(\mathbf{k}-\mathbf{k}_0,\omega-\omega_0;-\mathbf{k},-\omega)U(\mathbf{k},\omega)\\&\quad+Q_1(\mathbf{k}-\mathbf{k}_0,\omega-\omega_0;2\mathbf{k}_0-\mathbf{k},2\omega_0-\omega)\\&\quad\times U(\mathbf{k}-2\mathbf{k}_0,\omega-2\omega_0)+4\pi\rho^0(\mathbf{k}-\mathbf{k}_0,\omega-\omega_0)\,.\end{aligned} \quad (2.35)$$

The chain may be broken by noting at which frequencies $U(\mathbf{k},\omega)$ is resonant in equilibrium, and assuming the real parts of the nonequilibrium eigenfrequencies are not shifted from these frequencies by very much (the shifts will in fact be of order $k_Dd\ll1$). The equilibrium resonances are at $\pm\omega_p$ and $\pm\omega_i$, so that if ω is in the neighborhood of $+\omega_p$, $\omega-\omega_0$ will be in the neighborhood of $-\omega_i$, and $\omega-2\omega_0$ will be in the neighborhood of $-\omega_p$. Thus, $\omega-n\omega_0$, for n any positive or negative integer other than 1 or 2, will be far from any of the equilibrium resonances, and the corresponding U may be ignored. Suppose $\omega-\omega_0=-\omega_i$, so that the ion-acoustic wave is excited at resonance. Then $\omega-2\omega_0=-\omega-2\omega_i$, and $U(\omega-2\omega_0)$ can be resonant at $-\omega_p$ only if $2\omega_i$ is much less than the linewidth γ_L of the longitudinal plasma mode. The significance of the case $\omega_i\ll\gamma_L$ will be discussed shortly. For the present, assume that $\omega_i\gg\gamma_L$, so that the mode $U(\omega-2\omega_0)$ may also be ignored, and (2.34) and (2.35) become a closed set of equations:

$$\begin{aligned}k^2\epsilon^L(\mathbf{k},\omega)&U(\mathbf{k},\omega)-Q_1(\mathbf{k},\omega;\mathbf{k}_i,\omega_0-\omega)U(-\mathbf{k}_i,\omega-\omega_0)\\&=4\pi\rho^0(\mathbf{k},\omega)k_i{}^2\epsilon^L(-\mathbf{k}_i,\omega-\omega_0)U(-\mathbf{k}_i,\omega-\omega_0)\\&\qquad+Q_1(-\mathbf{k}_i,\omega-\omega_0;-\mathbf{k},-\omega)U(\mathbf{k},\omega)\\&=4\pi\rho^0(-\mathbf{k}_i,\omega-\omega_0)\,,\end{aligned} \quad (2.36)$$

where $\mathbf{k}_i\equiv\mathbf{k}_0-\mathbf{k}$. Then,

$$U(\mathbf{k},\omega)=\frac{4\pi}{k^2\epsilon^{NL}(\mathbf{k},\omega)}\times\left[\rho^0(\mathbf{k},\omega)+\frac{Q_1(\mathbf{k},\omega;\mathbf{k}_i,\omega_0-\omega)}{\epsilon^L(-\mathbf{k}_i,\omega-\omega_0)k_i{}^2}\rho^0(-\mathbf{k}_i,\omega-\omega_0)\right], \quad (2.37)$$

where the nonlinear dielectric function $\epsilon^{NL}(\mathbf{k},\omega)$ is defined by

$$\epsilon^{NL}(\mathbf{k},\omega)=\epsilon^L(\mathbf{k},\omega)-\frac{[Q_1(\mathbf{k},\omega;\mathbf{k}_i,\omega_0-\omega)]^2}{k^2k_i{}^2\epsilon^L(-\mathbf{k}_i,\omega-\omega_0)}\,. \quad (2.38)$$

Here we have used the property

$$Q_1(\mathbf{k},\omega;\mathbf{k}',\omega')=-Q_1(-\mathbf{k}',-\omega';-\mathbf{k},-\omega)\,,$$

readily seen from (2.21), and $q(\omega)^*=q(-\omega)$.

[17] H. Nyquist, Phys. Rev. **32**, 110 (1932).

If we now form $\langle |U(\mathbf{k},\omega)|^2\rangle$ from this expression, and note that fluctuations in ρ^0 at different frequencies are uncorrelated ($\langle\rho^0(\omega)\rho^0(\omega-\omega_0^*)\rangle=0$), Eqs. (2.31) and (2.33) yield for the nonequilibrium density correlation function

$$\frac{4\pi e^2}{k^2}S_0(\mathbf{k},\omega)=\frac{2}{|\epsilon^{NL}(\mathbf{k},\omega)|^2}\left[\frac{\Theta_e\,\mathrm{Im}\chi_e(\mathbf{k},\omega)+\Theta_i\,\mathrm{Im}\chi_i(\mathbf{k},\omega)}{\omega}\frac{1}{k^2k_i^2}\frac{|Q_1(\mathbf{k},\omega;\,\mathbf{k}_i,\omega_0-\omega)|^2}{|\epsilon^L(k_i,\omega-\omega_0)|^2}\times\frac{\Theta_e\,\mathrm{Im}\chi_e(-\mathbf{k}_i,\omega-\omega_0)+\Theta_i\,\mathrm{Im}\chi_i(-\mathbf{k}_i,\omega-\omega_0)}{\omega-\omega_0}\right].\quad(2.39)$$

The use of the equilibrium (external field-free) expressions for $\langle|\rho^0(\omega)|^2\rangle$ and $\langle|\rho^0(\omega-\omega_0)|^2\rangle$ here is only permissible when the imaginary part of Q_1 is negligible, as in our case.

With ω close to ω_p and $\omega_0-\omega$ close to ω_i, we may use (2.22) to write

$$(1/k^2)[Q_1(\mathbf{k},\omega;\,\mathbf{k}_i,\omega_0-\omega)]^2\approx\Lambda^2\psi k\chi^2\qquad(2.40)$$

where

$$\Lambda^2=(dk_D/2)^2=I_0/nc\Theta(\omega_p/\omega_0)^2\qquad(2.41)$$

$$I_0=(E_0/2)^2c/4\pi\qquad(2.42)$$

$$\psi=(\hat{k}\cdot\hat{d})^2.\qquad(2.43)$$

Λ^2 is proportional to the pump intensity[18] I_0, and is always assumed $\ll1$. ψ is an angular factor which attains its maximum value of 1 when the electric field vector of the pump (be it transverse *or* longitudinal) lies coincident with the propagation direction of the plasma wave ω.

All of the arguments made above rested upon the assumption that ω is close to ω_p and $\omega_0-\omega$ is close to ω_i. As we shall see in the next section, there is a zero in $\epsilon^{NL}(k,\omega)$ with the real part of ω close to ω_p and the imaginary part corresponding to decay or growth, depending on the value of Λ^2. However, in so far as $\mathrm{Im}Q_1$ is negligible, there is a symmetry in the condition $\epsilon^{NL}(k,\omega)=0$ which tells us that if (k,ω) is a root, then $(\mathbf{k}_0-\mathbf{k},\omega_0-\omega^*)$ must also be a root. This corresponds to the well-known Manley-Rowe relations for real mode coupling, and may be seen as follows. The linear susceptibilities as a function of complex ω are known to have the symmetry property $\chi(k,\omega)^*=\chi(k-\omega^*)$ [which implies $\epsilon^L(k_1\omega)^*=\epsilon^L(k_1,-\omega^*)$]. If we now set the right-hand side of (2.38) for $\epsilon^{NL}(\mathbf{k},\omega)$ equal to zero and then set its complex conjugate equal to zero, comparisons shows that both $(\mathbf{k},\omega)$ and $(\mathbf{k}_0-\mathbf{k},\omega_0-\omega^*)$ satisfy the same formal equation and are both valid roots, provided the imaginary part of $(Q_1)^2$ is negligible compared with the real part.

[18] Note that the definitions of intensity in (2.42) and in Refs. 1–8 differ in that $(E_0/2)^2$ appears here, but E_0^2 there, because of the definition of the pump field as a sine here and as twice the sine in I and II. This has led Jackson to mistakenly claim the results of I and II differ from his by a factor 4.

Returning to (2.39) for the density correlation function $S_0(\mathbf{k},\omega)$, a great simplicity arises in the limit of the high electron-to-ion temperature ratio, $\Theta_e/\Theta_i\gg1$, provided k_1, $k_s\ll k_D$. Under these conditions the ratio of $\mathrm{Im}\chi_i(k,\omega)$ to $\mathrm{Im}\chi_e(k,\omega)$ is exponentially small, regardless of whether $\omega\approx\omega_p$ or $\omega\approx\omega_i$, so that terms in (2.39) proportional to Θ_i may be neglected, and $S_0(k,\omega)$ may be written as

$$S_0(\mathbf{k},\omega)=2n\frac{k^2}{k_D^2}\frac{1}{|\epsilon^{NL}(\mathbf{k},\omega)|^2}\left[\frac{\mathrm{Im}\epsilon_L(\mathbf{k}\cdot\omega)}{\omega}+\frac{1}{k^2k_i^2}\times\frac{|Q_1(\mathbf{k},\omega;\,\mathbf{k}_i,\omega_0-\omega)|^2}{\omega-\omega_0}\mathrm{Im}\left(\frac{1}{\epsilon^L(k_i,\omega-\omega_0)}\right)\right],\quad(2.44)$$

where we have used $\mathrm{Im}\epsilon^L(\mathbf{k},\omega)=\mathrm{Im}\chi_e(\mathbf{k},\omega)$, which is valid in these frequency ranges, and the definition $k_D^2=(4\pi ne^2/\Theta_e)$. It is notable that (2.44) also follows from (2.39) in the equal-temperature case $\Theta_e=\Theta_i$, since $\mathrm{Im}\chi_e(\omega_p)/\mathrm{Im}\chi_i(\omega_p)$ is still exponentially small as long as $M\gg m$.

Some final points remain to be discussed. We have omitted terms proportional to the susceptibilities Q_2 in (2.24) from the analysis. The neglect of $U(\omega\pm2\omega_0)$, provided $\omega_i\gg\gamma_L$, has been justified earlier [beneath Eq. (2.35)]. However, the term $-2Q_2(\mathbf{k},\omega;-\mathbf{k},-\omega)\times U(\mathbf{k},\omega)$ which is of order Λ^2, properly should have been included in our above remarks. As discussed in II, such a term is partially responsible for the slight shift in the real part of frequency ω obeying $\epsilon^{NL}(\mathbf{k},\omega)=0$ from $\omega_L(k)$ to $\omega_L(k)[1+O(\Lambda^2)]$. It has no effect on the imaginary part of ω, which is of main interest here, and so is neglected.

A case of great interest since it is realized in the experiment of Stern and Tzoar,[3] is when $\omega_i\ll\gamma_L$, so that with ω close to ω_p, and $\omega-\omega_0\approx-\omega_i$, we have $\omega-2\omega_0\approx-\omega-2\omega_i$ differing from $-\omega_p$ by less than the linewidth γ_L of the resonance of $U(\omega-2\omega_0)$ at $-\omega_p$. This case was first treated by one of the authors[2] for a transverse pump and equal electron and ion temperatures, and will be generalized to unequal temperatures and a longitudinal pump in a current work in progress. Briefly, when the terms involving $U(\omega-2\omega_0)$ arising from (2.24) and (2.35) are taken into account, instead of the two coupled equations (2.36), we obtain three coupled equations, of form

$$k^2\tilde{\epsilon}^L(\omega)U(\omega)-Q_1(\omega;\omega_0-\omega)U(\omega-\omega_0)+Q_2(\omega;2\omega_0-\omega)U(\omega-2\omega_0)=4\pi\rho^0(\omega)|\mathbf{k}_0-\mathbf{k}|^2\tilde{\epsilon}^L(\omega-\omega_0)U(\omega-\omega_0)$$
$$+Q_1(\omega-\omega_0;-\omega)U(\omega)-Q_1(\omega-\omega_0;2\omega_0-\omega),$$
$$U(\omega-2\omega_0)=4\pi\rho^0(\omega-\omega_0),$$
$$|2\mathbf{k}_0-\mathbf{k}|^2\tilde{\epsilon}^L(\omega-2\omega_0)U(\omega-2\omega_0)+Q_1(\omega-2\omega_0;\omega_0-\omega)U(\omega-\omega_0)+Q_2(\omega-2\omega_0;-\omega)U(\omega)=4\pi\rho^0(\omega-2\omega).\quad(2.45)$$

We have suppressed the obvious wave-vector dependence of the Q's and U's. $\tilde{\epsilon}^L$ includes the correction arising from Q_2:

$$\tilde{\epsilon}^L(\mathbf{k},\omega)\equiv\epsilon^L(\mathbf{k},\omega)-(2/k^2)Q_2(\mathbf{k},\omega;-\mathbf{k},-\omega).$$

The threshold condition $\epsilon^{NL}(\mathbf{k},\omega)=0$ is now equivalent to the vanishing of a 3×3 determinant:

$$\begin{vmatrix} k^2\tilde{\epsilon}^L(\omega), & -Q_1(\omega;\omega_0-\omega), & +Q_2(\omega;2\omega_0-\omega) \\ Q_1(\omega-\omega_0,-\omega), & |\mathbf{k}_0-\mathbf{k}|^2\tilde{\epsilon}^L(\omega-\omega_0), & -Q_1(\omega-\omega_0;2\omega_0-\omega) \\ Q_2(-\omega,\omega-2\omega_0), & Q_1(\omega-2\omega_0,\omega_0-\omega), & |2\mathbf{k}_0-\mathbf{k}|^2\tilde{\epsilon}^L(\omega-2\omega_0) \end{vmatrix}=0,\quad(2.46)$$

With ω close to ω_p, the condition on Λ^2 for threshold ($\mathrm{Im}\omega=0$) is now altered; more importantly, we can show that if ω is a growing solution ($\mathrm{Im}\omega>0$), then so is $2\omega_0-\omega^*$, by taking the complex conjugate of (2.46) and using the symmetry about the off-diagonal axis. This means that in addition to stimulating an ion acoustic wave at ω_i and a plasma wave at $k_e\omega\approx\omega_0-\omega_i$, the pump also stimulates a plasma wave at $2\omega_0-\mathrm{Re}\omega\approx\omega_0+\omega_i$; which has, in fact been observed by Stern and Tzoar.[3] This symmetry about the pump frequency is well known in stimulated Raman scattering.[5] In fact, we may draw an analogy between the Raman vibrational level and ω_i, the scattered Stokes line and $\omega_0-\omega_i$, and the scattered anti-Stokes line and $\omega_0+\omega_i$. Further details of the effect of this anti-Stokes parametric exexcitation will be reserved for another publication.[19]

3. RESONANCE APPROXIMATIONS

The analysis and understanding of the function $\epsilon^{NL}(k,\omega)$ is simplified if we can assume that the functions $\epsilon^L(k,\omega)$ and $\epsilon^L(k,\omega_0-\omega)$ which appear in (2.38) and (2.39) have their frequency arguments very close to one of their complex zeros.[20] That is, we assume that we can write in the neighborhood of a zero at $\omega=\omega_\nu-i\gamma_\nu$

$$\epsilon^L(k,\omega)\simeq\frac{(\omega-\omega_\nu(k)+i\gamma_\nu(k))}{\omega_\nu Z_\nu}+O\left(\frac{\omega-\omega_\nu+i\gamma_\nu}{\omega_\nu}\right)^2,\quad(3.1)$$

where

$$\epsilon^L(k,\omega_\nu-i\gamma_\nu)\equiv0\quad(3.2)$$

and

$$Z_\nu^{-1}=\omega_\nu\frac{\partial\epsilon^L(k,\omega)}{\partial\omega}\bigg|_{\omega=(\omega_\nu-i\gamma_\nu)}.\quad(3.3)$$

It follows from the property $\epsilon^L(k,\omega)^*=\epsilon^L(k,-\omega^*)$ that for every zero at $\omega_\nu-i\gamma_\nu$ there will be a mirror zero at $-\omega_\nu-i\gamma_\nu$, with a Z_ν of the opposite sign.

As a working example throughout this paper we consider a classical two-component plasma. The general analysis applies to any 3-mode parametric excitation with suitable reinterpretation of Λ^2 and reidentification of the modes. One of the two least-damped roots of interest to us corresponds to electron plasma waves where ($\nu=L$) for a particular $k=k_L$

$$\omega_L=-(\omega_p^2+3k_L^2v_e^2)^{1/2}$$
$$\gamma_L=\left(\frac{\pi}{2}\right)^{1/2}e^{-3/2}\frac{k_D^3}{k_L^3}e^{-(1/2)k_D^2/k_L^2}+\frac{\lambda}{6\sqrt{2}\pi^{3/2}}\ln\left(\frac{\Theta}{\hbar\omega_p}\right)$$
$$Z_L=\tfrac{1}{2}+O(k_L^2/k_D^2)\quad(3.4)$$

where $v_e^2=\omega_p/m_e$ is the rms thermal velocity, $\omega_p^2=4\pi e^2n\times(1/m_e+1/m_{i\nu})$, $k_D^2=4\pi e^2/\Theta_e$, $\lambda=k_D^3/n$. The first term in the expression for γ_L is the collisionless Landau damping term, which vanishes rapidly as $k\to0$. The second term arises from electron-ion collisions.[21] The ratio γ_L/ω_L is $\ll1$ if $k<k_D$ and $\lambda\ll1$.

The second root of interest corresponds to the ion acoustic waves where ($\nu=i$) for a particular $k=k_i$

$$\omega_i=k_i[\kappa(\Theta_e/m_i)]^{1/2},\quad(3.5)$$

where κ is a constant depending on the role of collisions in the plasma. For a collisionless plasma with equal

[19] E. A. Jackson (Ref. 7) has shown that when $\omega_0-\omega\ll\omega_1$, then a relatively weak parametric coupling of the two nearly degenerate anodes at ω and $\omega-2\omega_0$ can still occur at much higher threshold levels. He also shows that this effect is canceled by collisional damping if $(\gamma/\omega_p)>10^{-4}$. We will not consider the regimes for which this process can occur in the present paper.

[20] The function $[\epsilon^L(\omega)]^{-1}$ is, of course, analytic in the lower half ω plane as demanded by causality. The zeros which we are discussing are actually in the analytic continuation of $[\epsilon^L(\omega)]$ from the upper into the lower half plane.

[21] D. F. BuBois, V. Gilinsky, and M. G. Kivelson, Phys. Rev. **129**, 2376 (1963); see also M. G. Kivelson and D. F. DuBois, Phys. Fluids **7**, 1578 (1964).

electron and ion temperature Θ, it is found[22]

$$\omega_i \simeq 1.6 k_i(\Theta_e/m_i)^{1/2} = C_i k \tag{3.6}$$

and

$$\gamma_i \simeq 0.6 k_i(\Theta_e/m_i)^{1/2}. \tag{3.7}$$

In this case γ_i/ω_i is not particularly small, and the mode is not well defined.[23] On the other hand, if the electron temperature Θ_e is greater than the ion temperature Θ_i (e.g., >5), then it is easily shown in the collisionless case[2] that

$$\epsilon^L(k_i,\omega) = 1 + \frac{k_D^2}{k_i^2}\left[1 - \left(\frac{\alpha v_e k_i}{\omega}\right)^2\right] - i\left(\frac{\pi}{2}\right)^{1/2} \alpha \frac{k_D^2}{k_i^2}, \tag{3.8}$$

from which we find, for $k \ll k_D$,

$$\omega_i = \alpha(k_i/k_D)\omega_p = k_i(\Theta_e/m_i)^{1/2} = C_i k_i \tag{3.9}$$

$$\gamma_i = \alpha^2/2(\pi/2)^{1/2}(k_i/k_D)\omega_p \tag{3.10}$$

and

$$Z_i = \tfrac{1}{2}(k_i^2/k_D^2) \tag{3.11}$$

where $\alpha = (m/M)^{1/2}$. In this case we see

$$\frac{\gamma_i}{\omega_i} = \frac{\alpha}{2}\left(\frac{\pi}{2}\right)^{1/2} \ll 1.$$

Our assumption is that roots of $\epsilon^{NL}(k,\omega)$ at $\omega = \omega^{NL} - i\gamma^{NL}$ for a particular $k = k_L$ lie close to the root of $\epsilon^L(k,\omega)$ at $\omega_L - i\gamma_L$ and that ω_0 can be adjusted so $\omega - \omega_0$ lies near the ion acoustic root of $\epsilon^L(k_i, \omega - \omega_0)$ at $-\omega_i - i\gamma_i$ (where $\mathbf{k}_i = \mathbf{k}_0 - \mathbf{k}_L$). Then we can write for ω in this neighborhood of the complex plane, using (2.38), (2.41)–(2.43)

$$\epsilon^{NL}(k,\omega) \simeq \frac{(\omega - \omega_L + i\gamma_L)}{\omega_L Z_L} + \frac{Z_i\omega_i}{(\omega - \omega_0 + \omega_i + i\gamma_i)} \Lambda^2 \frac{k_D^2}{k_i^2}\psi. \tag{3.12}$$

The complex zeros of ϵ^{NL} are roots of a quadratic which occur at

$$\omega = \omega_L - \frac{\Delta\omega}{2} - i\frac{(\gamma_L + \gamma_i)}{2} \pm \frac{\Delta\omega - i(\gamma_L - \gamma_i)}{2} \times \left(1 - \frac{4\Gamma^2}{[\Delta\omega - i(\gamma_L - \gamma_i)]^2}\right)^{1/2} \tag{3.13}$$

$$\Gamma^2 = \Lambda^2(k_D^2/k_i^2)\psi Z_i Z_L \omega_i \omega_L, \tag{3.14}$$

and where $\Delta\omega$ is the frequency mismatch

$$\Delta\omega \equiv \omega_L + \omega_i - \omega_0. \tag{3.15}$$

In the nearly degenerate case when $\gamma_L \simeq \gamma_i$ and $\Delta\omega \gg |\gamma_L - \gamma_i|$, we can write the solution as

$$\omega = \omega_L - \tfrac{1}{2}\Delta\omega - i\tfrac{1}{2}(\gamma_L + \gamma_i) \pm \tfrac{1}{2}\Delta\omega(1 - 4\Gamma^2/[\Delta\omega]^2)^{1/2}. \tag{3.16}$$

We see that in this case one of the square roots will have a positive imaginary part corresponding to a negative-damping contribution if

$$|\gamma_L - \gamma_i| \ll \Delta\omega < 2\Gamma. \tag{3.17}$$

The threshold condition in this case is

$$\Gamma(1 - [\Delta\omega]^2/4\Gamma^2)^{1/2} \gtrsim 1(\gamma_L + \gamma_i).$$

If $\Gamma \gg \Delta\omega$, the growth rate will be proportional to Γ. In most interesting cases for plasmas, and in particular for the experimental parameters of Stern and Tzoar,[3] the damping of the original roots differs considerably. In these experiments $\Delta\omega \ll \gamma_L + \omega_i \sim \gamma_L$ and $\gamma_L \gg \gamma_i$. If $|\gamma_L - \gamma_i| \gg \Delta\omega$, we can neglect the mismatch. In this case we have, for example, if $\gamma_L \gg \gamma_i$ and if

$$\Gamma^2/\gamma_L^2 \ll 1, \tag{3.18}$$

the roots

$$\hat{\omega}_1 = \hat{\omega}^{NL} - i\hat{\gamma}^{NL} = \omega_L - \Delta\omega(\Gamma^2/\gamma_L^2) - i\gamma_L - i\Gamma^2/\gamma_L,$$

$$\omega_1 = \omega^{NL} - i\gamma^{NL} = \omega_L - \Delta\omega - i\gamma_i + i\Gamma^2/\gamma_L. \tag{3.19}$$

If $\gamma_i \gg \gamma_L$ the solution has the same form, with γ_L and γ_i interchanged throughout. The root $\hat{\omega}_1$ (in the case $\gamma_L \gg \gamma_i$) has a negative-damping contribution proportional to Λ^2, while the root ω_1 receives an additional positive damping. At a critical value of the pump intensity the net damping of the first root (for the value of k involved) becomes zero. This occurs at

$$\frac{\Gamma_c^2(\mathbf{k})}{\gamma_L\gamma_i} = \Lambda_c^2 \frac{k_D^2}{k_i^2} \frac{Z_i\omega_i Z_L \omega_L \psi}{\gamma_i\gamma_L} = 1, \quad \Delta\omega \ll \gamma_i. \tag{3.20}$$

Since all the factors in this expression depend on k_L, or $k_i = |\mathbf{k}_0 - \mathbf{k}_L|$, Λ_c is a function of $\mathbf{k}_L$.

As we remarked following Eqs. (2.41)–(2.43), $\psi = 1$ for a transverse pump with electric vector in the $\mathbf{k}_L = (-\mathbf{k}_i)$ direction, or for a longitudinal pump with $\mathbf{k}_0$ collinear with both $\mathbf{k}_L$ and $\mathbf{k}_i$. Thus, as Λ is increased, modes propagating in the appropriate directions reach the critical point of instability first. Note that (3.20) is symmetrical in (ω_i/γ_i) and (ω_L/γ_L).

The dependence of the negative damping on $\Delta\omega$ can be seen in the following way: With $\omega = \omega^{NL} - i\gamma^{NL}$ we take the real and imaginary parts of $\epsilon^{NL}(k, \omega^{NL} - i\gamma^{NL}) = 0$ using (3.12). After some rearranging we can obtain the simultaneous equations

$$\gamma^{NL} = \gamma_i - \frac{\Gamma^2(\gamma_L - \gamma^{NL})}{(\Delta\omega^{NL})^2 + (\gamma_L - \gamma^{NL})^2}, \tag{3.21a}$$

$$\omega^{NL} = \omega_0 - \omega_i + \frac{\Gamma^2(\Delta\omega^{NL})}{(\Delta\omega^{NL})^2 + (\gamma_L - \gamma^{NL})^2}, \tag{3.21b}$$

[22] B. D. Fried and R. W. Gould, Phys. Fluids 4, 139 (1961).

[23] In the collision-dominated case, even for equal temperatures, $\gamma_i/\omega_i \ll 1$. The theory leading to the expression above for ϵ^{NL} applies strictly only to the near-collisionless case. Preliminary considerations indicate that this expression is valid even if the low-frequency root $\omega_0 - \omega$ is in the collision-dominated regime, as long as ω remains above the electron-ion collision frequency.

where now

$$\Delta\omega^{NL}=\omega^{NL}+\omega_i-\omega_0. \tag{3.22}$$

These equations must be solved self-consistently to obtain ω^{NL} and γ^{NL}. When (3.18) holds, and $\Delta\omega\ll\gamma_L$, we have seen in (3.19) that $\omega^{NL}\simeq\omega^L$. The value of ω^{NL} depends on $(\Delta\omega)$. If $\gamma^{NL}\ll\gamma_i$ (see below), then iteration of (3.21b), starting with $\Delta\omega^{NL}=\Delta\omega$, yields

$$\omega^{NL}=\omega^L+\frac{\Gamma^2(\Delta\omega)}{(\Delta\omega)^2+\gamma_L{}^2}. \tag{3.23}$$

If $\Delta\omega\ll\gamma_L$ we see that

$$\omega^{NL}=\omega^L+(\Gamma^2/\gamma_L{}^2)(\Delta\omega). \tag{3.24}$$

We have a small power-dependent frequency shift away from ω^L proportional to $\Delta\omega$. Such effects have been considered in detail by Goldman[2] and Jackson.[7] To calculate these shifts correctly it is necessary to include nonresonant corrections[24] to $\epsilon^L(\omega)$ and $\epsilon^L(\omega-\omega_0)$ which depend on Λ^2, as mentioned earlier. If $\Lambda^2(k^2/k_D{}^2)\ll1$, these shifts are small and can be neglected for most of our considerations. We will not deal with these shifts explicitly in this paper, but we will keep their existence in mind.

If we assume that ω^{NL} is known *and* if we are near threshold so that $|\gamma^{NL}|\ll\gamma_L$, we see from (3.21b) that

$$\gamma^{NL}=\gamma_i-\frac{\Gamma^2\gamma_L}{(\Delta\omega^{NL})^2+\gamma_L{}^2}. \tag{3.25}$$

When $\Delta\omega\simeq\Delta\omega^{NL}\ll\gamma_L$ we see that we recover the value of γ^{NL} obtained from (3.19) with $\omega^{NL}-i\gamma^{NL}=\omega_1$. However, we now see from (3.25) that appreciable negative damping occurs only in a range of order γ_L about $\Delta\omega=0$. This condition determines the range of k values which receive appreciable negative damping. We can write this as

$$\delta(\Delta\omega^{NL})\simeq\delta(\Delta\omega)=\gamma_L, \tag{3.26}$$

where $\delta(\Delta\omega)$ is the increment of k values in $\Delta\omega$. Using (3.4) and (3.5), we can express this in terms of $\delta k_L=-\hat{k}_i\cdot\hat{k}_L\delta k_i$. $\delta[\omega_L(k_L)+\omega_i(k_i)]=\gamma_L(k_L)$; for $\delta k_L/k_L\ll1$ we have

$$\delta k_L=\frac{\gamma_L(k_L)}{[\partial\omega_L(k_L)/\partial k_L]-[\partial\omega_i(k_i)(\hat{k}_i\cdot\hat{k}_L)/\partial k_i]}=\frac{\gamma_L(k_L)}{3v_e(k/k_D)-(\hat{k}_i\cdot\hat{k}_L)C_i}. \tag{3.27}$$

The range δk_L of active k_L vectors thus depends only on k_L, which we can take to be determined by $\Delta\omega\equiv0$. In the case of a transverse pump we have already seen $k_0\sim O((v/c)k)$, so $\mathbf{k}=-\mathbf{k}_i$ is essentially *independent of* k_0. For the case of a longitudinal pump, on the other hand, we have the condition

$$(\omega_p{}^2+3k_L{}^2v^2)^{1/2}+C_i|\mathbf{k}_0-\mathbf{k}_L|=\omega_0=(\omega_p{}^2+3k_0{}^2v_e{}^2)^{1/2}.$$

[24] See Eq. (3.46).

From this we find

$$k_i=2k_0x-\frac{2}{3}\frac{C_i}{\omega_p}k_D{}^2$$
$$k_L{}^2=k_0{}^2-\frac{4}{3}\frac{C_i}{\omega_p}k_D{}^2k_0x+\frac{4}{9}\frac{C_i^2}{\omega_p{}^2}k_D{}^2, \tag{3.28}$$

where

$$x=\hat{k}_0\cdot\hat{k}_i \tag{3.29}$$

and $1>x>(C_i/3\omega_p)(k_D{}^2/k_0)$. If

$$\frac{C_i}{3\omega_p}\frac{k_D{}^2}{k_0}=\frac{C_i}{3v_e}\frac{k_D}{k_0}=\frac{\sqrt{\kappa}}{3}\frac{m_e}{m_i}\frac{k_D}{k_0}\ll1,$$

then these relations simplify to read

$$k_i\cong2k_0x,\quad(1>x>0)$$
$$k_L\cong k_0. \tag{3.30}$$

Again we note that in the case of a longitudinal pump field we cannot neglect k_0 with respect to k_i or k_L. The most efficient collinear configuration which makes $\psi=1$ in (3.20) corresponds to $x=1$ in (3.30).

When $\Theta_e=\Theta_i$ it was shown for a transverse pump in 2 that for $(k\sim0.2k_D,\ \alpha k_D/k\ll1)$,

$$\Delta k/k_D\simeq\tfrac{1}{3}\alpha. \tag{3.31}$$

[This corresponds formally to $\gamma_i/\omega_i\simeq(3\sqrt{\kappa})^{-1}$ in (3.29)]

For k's away from the frequency-matched conditions, the threshold condition reads

$$\Lambda_c{}^2(\mathbf{k}_L)\frac{k_D{}^2}{k_i{}^2}\psi\frac{Z_i\omega_i}{\psi_i}\frac{Z_L\omega_L\gamma_L}{[\Delta\omega(\mathbf{k}_L)]^2+\gamma_L{}^2}=1. \tag{3.32}$$

In terms of Λ_c, the condition (3.18) becomes

$$\frac{\Lambda^2}{\Lambda_c{}^2}\frac{\gamma_i}{\gamma_L}\ll1. \tag{3.33}$$

With this definition (3.20) of $\Lambda_c=\Lambda_c(\mathbf{k}_L)$ we can rewrite (3.19) for k's which satisfy $\Delta\omega=0$ in the form

$$\hat{\omega}_1=\omega_L-i\gamma_L(1+\gamma_i\Lambda^2/\gamma_L{}'\Lambda_c{}^2)\simeq\omega_L-{}^i\gamma_L,$$
$$\omega_1=\omega_L-i\gamma_i(1-\Lambda^2/\Lambda_c{}^2), \tag{3.34}$$

all quantities except Λ being functions of $\mathbf{k}_L$.

The arguments following (2.43) show that in addition to the roots ω_1 and $\hat{\omega}_1$ for $k=k_L$, there are also two roots for $k=k_i=|\mathbf{k}_0-\mathbf{k}_L|$ at

$$\hat{\omega}_2=\omega_i-i\gamma_L(1+\gamma_i\Lambda^2/\gamma_L\Lambda_c{}^2)\simeq\omega_i-i\gamma_L,$$
$$\omega_2=\omega_i-i\gamma_i(1-\Lambda^2/\Lambda_c{}^2), \tag{3.35}$$

for $\Delta\omega=0$. Thus the roots ω_1 and ω_2 both have the same imaginary part. Thus modes with frequencies near ω_L and near ω_i (for $\Delta\omega=0$) have *identical* thresholds and growth rates, as demanded by the Manley-Rowe relations.[2,4]

Clearly, as $\Lambda\to 0$, then $\hat{\omega}_1\to\omega_L-i\gamma_L$ and $\omega_2\to\omega_i-i\gamma_i$, which are the linear roots. The roots ω_1 and $\hat{\omega}_2$ arise, therefore, from the nonlinear coupling, and must disappear from the theory as $\Lambda\to 0$.

To see this we expand $\epsilon^{NL}(k,\omega)$ about one of its zeros ω_ν:

$$\epsilon^{NL}(k_\nu,\omega)\simeq\frac{1}{\omega_\nu Z_\nu^{NL}}(\omega-\omega_\nu)+O(\omega-\omega_\nu)^2. \quad (3.36)$$

The residues of $[\epsilon^{NL}(k_\nu,\omega)]^{-1}$ at the simple poles corresponding to the above zeros of $\epsilon^{NL}(\omega)$ are defined by

$$(Z_\nu^{NL}\omega_\nu)^{-1}=\left.\frac{\partial\epsilon^{NL}(k_\nu,\omega)}{\partial\omega}\right|_{\omega=\omega_\nu}, \quad (3.37)$$

where ω_ν's are the zeros discussed above. Carrying this out, we find [again assuming the inequality (3.21)]

$$\hat{Z}_1^{NL}=\frac{Z_L}{1+(\Lambda^2/\Lambda_c^2)(\gamma_i/\gamma_L)}\simeq Z_L,$$

$$Z_1^{NL}=Z_L\frac{\Lambda^2}{\Lambda_c^2}\frac{\gamma_i}{\gamma_L},$$

$$Z_2^{NL}=\frac{Z_i}{1+(\Lambda^2/\Lambda_c^2)(\gamma_i/\gamma_L)}\simeq Z_i,$$

$$\hat{Z}_2^{NL}=Z_i\frac{\Lambda^2}{\Lambda_c^2}\frac{\gamma_i}{\gamma_L}. \quad (3.38)$$

The residues Z_1^{NL} and $\hat{Z}_2^{NL}$ at the poles at ω_1 and $\hat{\omega}_2$ vanish as $\Lambda\to 0$ while $\hat{Z}_1^{NL}\to Z_2$ and $Z_2^{NL}\to Z_i$, making contact with the linear theory. Because of (3.21), $Z_1^{NL}\ll\hat{Z}_1^{NL}$ and $\hat{Z}_2^{NL}\ll Z_2^{NL}$.

Examination of the terms neglected in using the first terms in the expansions of $\epsilon_L(\omega)$ and $\epsilon_L(\omega-\omega_0)$ about their zeros shows that this approximation is valid only if

$$\gamma^{NL}<\gamma_L<\gamma_i,\qquad \omega^{NL}\gtrsim\omega_L, \quad (3.39)$$

and

$$(\gamma_i/\omega_i)^2\ll 1 \quad\text{and}\quad (\gamma_L/\gamma_L)^2\ll 1. \quad (3.40)$$

For a classical plasma with $\Theta_e=\Theta_i$, this last condition does not hold for the ion acoustic mode. The dispersion relation $\epsilon^{NL}(k,\omega)$ must then be solved by a careful numerical analysis of the function $[k^2/k_D^2\epsilon_L\times(k,\omega-\omega_0)]^{-1}$ using the tabulated[14] collisionless plasma screening functions. Goldman[2] has carried this out for the transverse pump. He finds that a maximum negative proportional to Λ^2 leads to a frequency-matching condition as in (2.13), but with

$$\omega_i=1.7\alpha(k/k_D)\omega_p=1.7k(\Theta/m_i)^{1/2} \quad (3.41)$$

corresponding to $\kappa^{1/2}=1.7$ in (3.5).

When this condition is met, $\mathrm{Im}[k^2/k_D^2\epsilon^L(\omega-\omega_0)]^{-1}$ attains it maximum value of 0.58. The threshold value for the associated instability in this case is

$$(0.58)\Lambda_c^2Z_L(\omega_L/\gamma_L)(\hat{k}\cdot\hat{e}_0)^2=1, \quad (3.42)$$

which is close to the value obtained from (3.20) with $(\gamma_i/\omega_i)_{\text{eff}}\cong\frac{5}{6}$. The results in this case will always turn out to be *qualitatively* in agreement with the formulas derived above with $\gamma_i/\omega_i=O(1)$. However, these formulas are quantitatively inaccurate.

4. BEHAVIOR OF THE FLUCTUATION SPECTRUM

We now can use the approximations of the last section to understand the behavior of $S_0(k,\omega)$.

The factor $|\epsilon^{NL}(k,\omega)|^{-2}$ peaks at the real frequencies ω^{NL} and $\omega_0-\omega^{NL}$. As discussed above, each of these real frequencies is near a *pair* of complex zeros of $\epsilon^{NL}(k,\omega)$. For example, using (3.12) to (3.19) we can write in the neighborhood of $\omega^{NL}\simeq\omega_L$ for $k=k_L$

$$\epsilon^{NL}(k_L,\omega)=\frac{(\omega-\omega_1)(\omega-\hat{\omega}_1)}{\omega_LZ_L(\omega-\omega_L+i\gamma_i)}, \quad (4.1)$$

again assuming $\omega_0=\omega_L+\omega_i$. Thus, using (3.19) and (3.22), we have

$$\frac{1}{|\epsilon^{NL}(k_L,\omega)|^2}=\frac{\omega_L^2Z_L^2}{(\omega-\omega_L)^2+\gamma_L^2((1+\gamma_i\Lambda^2/\gamma_L\Lambda_c^2))^2}\times\frac{(\omega-\omega_L)^2+\gamma_i^2}{(\omega-\omega_L)^2+\gamma_i^2(1-\Lambda^2/\Lambda_c^2)^2}. \quad (4.2)$$

The last factor in (4.2) dominates if $1\gg 1-\Lambda^2/\Lambda_c^2$, i.e., near threshold, and we can write

$$\frac{1}{|\epsilon^{NL}(k_L,\omega)|^2}=\frac{(Z_1^{NL})^2\omega_L^2}{(\omega-\omega_L)^2+\gamma_i^2((1-\Lambda^2/\Lambda_c^2))^2}, \quad (4.3)$$

where Z_1^{NL} is given by (3.26). In the limit $\Lambda\to 0$ the last factor in (4.2) approaches unity, and we obtain the usual linear result.

Near $\omega_0-\omega^{NL}$ and for a shifted value of $k_i=|\mathbf{k}_0-\mathbf{k}_L|$, on the other hand, we can write (using $\Delta\omega=0$)

$$\epsilon^{NL}(k_i,\omega)=\frac{(\omega-\omega_2)(\omega-\hat{\omega}_2)}{\omega_iZ_i(\omega-\omega_i+i\gamma_L)} \quad (4.4)$$

and

$$\frac{1}{|\epsilon^{NL}(k_i,\omega)|^2}=\frac{\omega_i^2Z_i^2}{(\omega-\omega_i)^2+\gamma_i^2((1-\Lambda^2/\Lambda_c^2))^2}\times\frac{(\omega-\omega_i)^2+\gamma_L^2}{(\omega-\omega_i)^2+\gamma_L^2((1+\gamma_i\Lambda^2/\gamma_L'\Lambda_c^2))^2}. \quad (4.5)$$

Since $\gamma_i[1-(\Lambda^2/\Lambda_c^2)]\ll\gamma_L$ the behavior of this function for $(\omega-\omega_i)^2\ll\gamma_L^2$ is dominated by the first factor and we can write

$$\frac{1}{|\epsilon^{NL}(k_i,\omega)|^2} \simeq \frac{[\omega_0 Z_2^{NL}]^2}{(\omega-\omega_i)^2+\gamma_i^2((1-\Lambda^2/\Lambda_c^2))^2}. \quad (4.6)$$

In the limit $\Lambda \to 0$ the last factor in (4.5) approaches a constant value of unity and the first factor dominates, giving the usual linear result near the ion acoustic resonance.

On collecting results, using (2.44), we see near each resonance (provided $1 \gg 1-\Lambda^2/\Lambda_c^2$) that $S_0(k_\nu,\omega)$ has the form

$$S_0(k_\nu,\omega) = \frac{[Z^{NL}\omega]_\nu^2}{(\omega-\omega_\nu)^2+\gamma_i^2((1-\Lambda^2/\Lambda_c^2))^2} \frac{2n\gamma_\nu}{Z_\nu\omega_\nu} \times\left[\frac{1}{\omega_\nu}+\frac{1}{(\omega_0-\omega_\nu)}\frac{\Lambda^2}{\Lambda_c^2}\right], \quad (4.7)$$

where $\nu = i$ or L and $[Z^{NL}\omega]_L = Z_1^{NL}\omega_L$ and $[Z^{NL}\omega]_i = Z_2^{NL}\omega_i$. In obtaining this expression we note that since $\gamma^{NL} = \gamma_i(1-\Lambda^2/\Lambda_c^2) \ll \gamma_i$. γ_L, we have replaced all factors except the resonant term by their values at resonance. In our analysis we have been assuming $\gamma_L > \gamma_i$ but now both resonances have the *same* reduced width $\gamma_i(1-\Lambda^2/\Lambda_c^2)$ for values of Λ approaching Λ_c (i.e., for which $1 \gg 1-\Lambda^2/\Lambda_c^2$).

The values of $S(k,\omega)$ at the peak resonances $\omega = \omega_\nu$ are given by

$$(k_D^2/k_\nu^2)S_0(k_\nu,\omega) = 2n(Z_\nu/\gamma_\nu)A_\nu K^2, \quad (4.8)$$

where

$$K = \frac{1}{1-\Lambda^2/\Lambda_c^2} \quad (4.9)$$

and

$$A_\nu = \left[\frac{Z_\nu^{NL}\omega_\nu^{NL}}{Z_\nu\omega_\nu}\right]^2 \frac{\gamma_\nu^2}{\gamma^{i2}}\left[1+\frac{\omega_\nu}{\omega_0-\omega_\nu}\frac{\Lambda^2}{\Lambda_c^2}\right]. \quad (4.10)$$

When $A_\nu K^2 = 1$ we have the familiar result of the linear equlibrium theory.[25]

The nonlinear effect of the regenerative parametric coupling is contained in the factor K, which appears here squared. As the pump intensity parameter Λ^2 approaches the critical value $\Lambda_c^2(k)$ for the instability, K diverges. Physically, of course, the nonlinear processes which we have neglected in the analysis up to this point lead to a saturation of the effect at some finite but large value. The steady state saturation level for K is discussed in Sec. 6.

For the case $\nu = i$, Eq. (4.10) in conjunction with (3.26) and (3.27) gives

$$A_i = \frac{1}{[1+(\Lambda^2/\Lambda_c^2)(\gamma_i/\gamma_L)]^2}\left[1+\frac{\omega_i}{\omega_0-\omega_i}\frac{\Lambda^2}{\Lambda_c^2}\right] \simeq \left[1+\frac{\omega_i}{\omega_L}\frac{\Lambda^2}{\Lambda_c^2}\right] \simeq 1. \quad (4.11)$$

[25] E. E. Salpeter, Phys. Rev. **120**, 1528 (1960); D. F. DuBois and V. Gilinsky, *ibid.* **135**, A995 (1964).

On the other hand, we also have for $\nu = L$

$$A_L = \left[\frac{\Lambda^2}{\Lambda_c^2}\frac{\gamma_i}{\gamma_L}\right]^2 \frac{\gamma_L^2}{\gamma_i^2}\left[1+\frac{\omega_L}{\omega_i}\frac{\Lambda^2}{\Lambda_c^2}\right] \simeq \left(\frac{\Lambda^2}{\Lambda_c^2}\right)^3 \frac{\omega_L}{\omega_i}. \quad (4.12)$$

Near threshold $\Lambda \lesssim \Lambda_c$, A_L is a large number since

$$\frac{\omega_L}{\omega_i} = \frac{1k_D}{\alpha k}\frac{1}{\sqrt{\kappa}} \gg 1. \quad (4.13)$$

For most cases of interest for near collisionless plasmas and $\Lambda \sim \Lambda_c$,

$$A_i/A_L = \omega_i/\omega_L \ll 1, \quad (4.14)$$

so that the resonance near ω_p is much more enhanced than the low frequency resonance at $\omega_i(k)$.

The integrated power in one of the resonance peaks is of interest. Integrating over the Lorentzian of (4.7), we obtain

$$\frac{k_D^2}{k^2}\int d\omega S_0(k,\omega) = \pi Z_\nu 2nA_\nu K\frac{\gamma_i}{\gamma_\nu}. \quad (4.15)$$

The integrated spectrum is thus enhanced by a factor $A_\nu K(\gamma_i/\gamma_\nu)$ over the equilibrium value.

From these formulas we find that the ratio of peak power at $\omega = \omega_i$, $k = k_i$ to peak power at $\omega = \omega_L$, $k = k_L$ is

$$\frac{S_0(k_i,\omega_i)}{S_0(k_L,\omega_L)} = \frac{k_D^2}{k_i^2}\frac{(\gamma_L/\omega_L)}{(\gamma_i/\omega_i)}, \quad (4.16)$$

which can be greater than or less than one. Since the resonance widths (for $\Lambda \sim \Lambda_c$) are equal the ratio of the integrated power in the two resonances is the same as (4.16).

5. INELASTIC SCATTERING OF RADIATION FROM THE PARAMETRICALLY EXCITED FLUCTUATIONS

It is well known that the asymptotic differential cross section for scattering an incident beam of radiation of frequency $\omega_1 \gg \omega_p$ from the parametrically excited plasma is[25]

$$\frac{d^2\sigma(k,\omega)}{d\omega_2 d\Omega_2} = \frac{nr_0^2}{\pi}|1+\chi_{0i}^+(k,\omega)|^2 S_0(\mathbf{k},\omega)(\hat{e}_1\cdot\hat{e}_2)^2 \quad (5.1)$$

if ω is near a resonance of $\epsilon^{NL}(k,\omega)^{-1}$. Here n is the mean electron density, $r_0 = e^2/mc^2$ is the classical electron radius, ω is the difference between the incident and scattered frequencies

$$\omega = \omega_2 - \omega_1, \quad (5.2)$$

$\mathbf{k}$ is the wave vector difference

$$\mathbf{k} = \mathbf{k}_2 - \mathbf{k}_1, \quad (5.3)$$

and $\hat{e}_1$ and $\hat{e}_2$ are the polarization vectors of the incident and scattered radiation.

The factor $|1+\chi_{0i}{}^{+}(k,\omega)|^2$ arises because the cross section is really proportional to the *electron* density fluctuation spectrum and not the total density fluctuation spectrum $S_0(k,\omega)$. This factor essentially removes the effect of ion fluctuations from $S_0(k,\omega)$.

We have seen in Secs. 3 and 4 that as the pump power is increased toward the threshold value, $S_0(\mathbf{k},\omega)$ is enhanced near the resonant frequencies $\omega_\nu(k)$ for a narrow range of $\mathbf{k}$ vectors which satisfy (3.28) to (3.30). The scattering angle θ is essentially determined by k (if $\omega_1 \gg \omega_p$)

$$k = 2k_1 \sin\tfrac{1}{2}\theta. \tag{5.4}$$

Thus, for a restricted range of scattering angles $\delta\theta$ determined by δk (see (3.28) and (3.31)), we expect an enhanced spectrum $S_0(\mathbf{k},\omega)$ given by (4.7). From (5.4) we find

$$\sin\theta\delta\theta = 2\frac{\delta k}{k}(1-\cos\theta) = \frac{\delta k k}{k_1{}^2}. \tag{5.5}$$

Thus if the possible "active" k vectors were isotropically distributed, the enhanced scattered radiation would lie in a cone with an angle θ [determined by (5.4) with k the value for perfect frequency matching] and a thickness $\delta\theta$ determined by (5.5). However, the strength of the negative damping terms (3.19) is proportional to $(\mathbf{k}\cdot\hat{e}_0)^2$ [see (3.20)]. Thus, the greatest enhancement occurs for $\mathbf{k}\|\hat{e}_0$. We can make the dependence on $\cos\phi = (\hat{k}\cdot\hat{e}_0)$ explicit by writing

$$A_\nu = \frac{Z_\nu{}^{NL}\omega_\nu{}^{NL2}}{Z_\nu\omega_\nu}\frac{\gamma_\nu{}^2}{\gamma_i{}^2}\left[1+\frac{\omega_\nu}{\omega_0-\omega_\nu}\cos^2\phi\frac{\Lambda^2}{\Lambda_c{}^2(0)}\right], \tag{5.6}$$

$$K = \frac{1}{1-\cos^2\phi(\Lambda^2/\Lambda_c{}^2(0))}, \tag{5.7}$$

where $\Lambda_c{}^2(0) = \Lambda_c{}^2(\cos\phi = 0)$.

The geometry is made clear in Fig. 1, where we have taken the case in which the incident scattering beam is perpendicular to $\hat{e}_0$ and k_0.

The solid-angle increment in which the enhanced scattering lies is from $\delta\Omega = \sin\theta\delta\theta d\phi$. To obtain the total enchanced scattering in the cone of thickness $\delta\theta$ integrated over all ϕ and integrated over frequencies in the neighborhood of a resonance at ω_ν, we have, from (4.14), (5.1), and (5.5).

$$\sigma_\nu = \int d\omega_2 \int_0^{2\pi} d\phi \sin\theta\delta\theta \frac{d^2\sigma(k,\omega)}{d\omega d\Omega}$$

$$\cong nr_0{}^2\left(\frac{k^2}{k_D{}^2}\right)|1+\chi_{0i}{}^{+}(k,\omega_\nu)|^2(\hat{e}_1\cdot\hat{e}_2)^2 2\frac{\delta k}{k}(1-\cos\theta)$$

$$\times Z_\nu \frac{\beta_e\hbar\omega_\nu}{e^{\beta_e\hbar\omega_\nu}-1}A_\nu(\phi=0)[K(\phi=0)^{1/2}-1]. \tag{5.8}$$

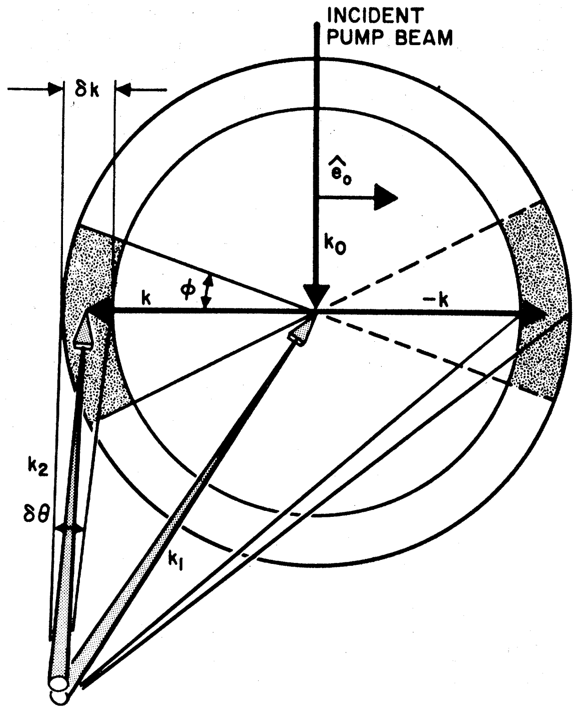

FIG. 1. Geometry for inelastic scattering experiment. Incident and scattered radiation $\mathbf{k}_1$ and $\mathbf{k}_2$ are essentially normal to the plane of the paper. The shaded region indicates the range of $\mathbf{k}$ vectors which receive appreciable negative damping.

Here we have evaluated the integral

$$\int_0^{2\pi} d\phi A_\nu(\phi)K(\phi) = 2\pi A_\nu(\phi=0)[K(\phi=0)^{1/2}-1] \tag{5.9}$$

using (5.6), (5.7), and well-known tabulated integrals.

The enhancement factor of the scattering cross section in this cone of angles is therefore

$$A_\nu(\phi=0)[K(\phi=0)^{1/2}-1] \tag{5.10}$$

where $A_\nu(\phi=0)$ and $K(\phi=0)$ are given by (5.6) and (5.7) with Λ_c for $\phi=0$. The singular behavior at threshold, though reduced by the angular integration, is still present in the enhanced scattering cross section.

Considering $\omega \sim \omega_p$ (so that $1+\chi_{0i}{}^{+}(\omega)\simeq 1$), using (5.7), and integrating over the resonance using (4.13), we obtain for the differential cross section per unit azimuthal angle

$$\frac{d\sigma}{d\phi}(k,\omega) = nr_0{}^2\left(\frac{k^2}{k_D{}^2}\right)(\hat{e}_1\cdot\hat{e}_2)^2 A_L(\phi)K(\phi)\frac{\delta k k}{k_1{}^2}, \tag{5.11}$$

where we have also used $Z_L \simeq 1$, $\beta_e\hbar\omega \ll 1$, and (5.5) for $\sin\theta d\theta$. Using only the dominant term in A_L from (4.11) and the expression (3.31) for δk, Eq. (3.41) for ω_i, and the definition (3.42) for $\Lambda_c{}^2$, we can write this as

$$\frac{d\sigma}{d\phi} = 0.009 nr_0{}^2\frac{(\mathbf{k}\cdot\mathbf{E}_0)^2}{k_1{}^2 n}\beta\frac{\omega_p}{\gamma_L}(\hat{e}_1\cdot\hat{e}_2)^2 K(\phi). \tag{5.12}$$

This expression, which is independent of the ion mass, is proportional to a result obtained by Berk[13,26] in treating a similar situation in which one radiation beam slightly above the plasma frequency enhances plasma fluctuations which subsequently scatter a second beam. However, Berk's model utilizes infinitely heavy ions. Therefore, there is no ion-acoustic mode in his theory and no possibility of regenerative parametric amplification, which requires a dynamic ion response. He does not, therefore, obtain the amplification factor $K(\phi)$. This would not be serious if his expression were restricted to powers considerably below threshold $\Lambda^2 \ll \Lambda_c^2$, $K \sim 1$. However, his numerical examples are considerably *above* threshold $\Lambda^2 \gg \Lambda_c^2$, and the results clearly are not valid. We can conclude from our results that near (but below) threshold the cross section will be considerably larger than Berk predicts because of the regenerative parametric process.

In a sense Berk's results represent the lowest order term of an expansion of our results in powers of E_0^2. However, the higher-order terms in this expansion have a secular character and are not small, even if E_0^2 is small, if the resonant frequency matching and threshold conditions are approximately satisfied.

6. NONLINEAR SATURATION CONDITIONS

The enhancement factor K which appears in an essential way in the above considerations diverges as Λ approaches the value Λ_c at which the most favorably matched $\mathbf{k}$ mode goes unstable in the linearized theory. This is clearly unphysical; we have neglected the reaction of the induced currents in the system back onto the pump field. When this is done self-consistently the *actual* steady-state pump amplitude never quite attains the threshold value, and K is large but finite. To solve this problem in general we must couple the equation for the pump field to the nonlinear current source provided by the enhanced longitudinal field fluctuations. This general nonlinear problem has not been solved in the case of modes with linear losses.

Fortunately, for the steady-state case which we are considering the self-consistent pump problem can be solved by simply invoking conservation of energy arguments. The basic simplifying assumption in the steady state case is that the pump field still has the form[27]

$$E_0(x,t) = \hat{e}^0/2\bar{E}_0 \exp[-i(\bar{\omega}_0 t - k_0 \cdot x)] + \text{c.c.} \quad (6.1)$$

where $\bar{E}_0$ is the *self-consistent* steady state pump amplitude resulting from the reaction of the induced fluctuating current. The amplitude $\bar{E}_0$ is *not* the value of the pump field calculated from the external sources and *linear* induced sources alone. It is determined by the steady state condition that the absorptive power P_0 supplied by the external sources equals the power dissipated by the self-consistent pump field *and* the fluctuating longitudinal fields. Clearly, the rms value of the longitudinal fields must remain finite if they are to dissipate a finite amount of power. Thus $\bar{E}_0$ must remain below the threshold value.

The power dissipated in the plasma is the average of the volume integral of $\mathbf{J}(1)\cdot\mathbf{E}(1)$ where $\mathbf{J}(1)$ is the microscopic plasma current density and $\mathbf{E}(1)$ is the total electric field. Thus

$$P_0 = \int d^3x_1 \langle\langle J_i(1) E_i(1)\rangle\rangle_t \quad (6.2)$$

where $\langle\ \rangle_t$ denotes a time average and $\langle\ \rangle$ denotes the ensemble average in steady state. The current density is related to the effective field via susceptibility functions in a way analogous to the charge density relation defined in Sec. 2:

$$J_i(1) = J_i^0(1) + \frac{\partial}{\partial t_1}\left\{\int d^4x_2 \chi_{ij}(1-2)E_j(2) + \int d^4x_2 \times \int d^4x_3 \chi_{ijk}(1-2,1-3)E_j(2)E_k(3) + \cdots\right\}. \quad (6.3)$$

Here $J_i^0(1)$ is the fluctuating "noise" current in the absence of an electric field E. Inserting this into (6.2) and noting that $J_i^0(1)$ and $E(1)$ are statistically-independent with zero average values, we can write

$$P_0 = \int d^3x_1 \int d^4x_2 \frac{\partial}{\partial t_1}\langle \chi_{ij}(1-2)\langle E_i(1)E_j(2)\rangle\rangle_t + \int d^3x_1 \int d^4x_2 \int d^4x_3 \frac{\partial}{\partial t_1} \times \langle \chi_{ijk}(1-2,1-3)\langle E_i(1)E_j(2)E_k(3)\rangle\rangle_t + \cdots. \quad (6.4)$$

Expressing χ's and E's in terms of Fourier components we can write this as

$$\frac{P_0}{\Omega} = \int \frac{d^3k}{(2\pi)^3}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi} \times \left\{-i\omega\chi_{ij}(k,\omega)\frac{\langle E_i(k,\omega)E_j(k,\omega)^*\rangle}{\Omega T}\right\} + \frac{P^{NL}}{\Omega}, \quad (6.5)$$

where

$$\frac{P^{NL}}{\Omega} \equiv \int \frac{d^3k}{(2\pi)^3}\int \frac{d^3k'}{(2\pi)^3}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi} \times \{-i(\omega+\omega')\chi_{ijk}(\mathbf{k},\omega;\mathbf{k},'\omega') \times \langle E_i(\mathbf{k},\omega)E_j^*(\mathbf{k}+\mathbf{k}',\omega+\omega')E_k(\mathbf{k}',\omega')\rangle\}. \quad (6.6)$$

[26] Aside from the factor K, which represents an important physical effect not included by Berk, there are several minor differences between our Eq. (5.12) and Beck's Eq. (7). He considers unpolarized pump radiation and therefore has $(1-(\hat{e}_1\cdot\hat{k}_2)^2)$ where we have $(\hat{e}_1\cdot\hat{e}_2)^2$ corresponding to an average over $\hat{e}_1$. In addition his factor k_1k_2 in the denominator is equivalent to our k_1^2 when $\omega_1 \gg \omega_p$. Finally, there is a numerical factor which is different and again appears to arise from our different handling of the dynamics of the ions.

[27] In general ω_0 will be a function of E_0 which can only be determined by a more detailed analysis of the coupled equations. This frequency shift affects only the frequency-matching conditions which we assume to be optimum.

Here Ω is the volume of the system and T is the time interval over which we are averaging. From the relations $E_i(\mathbf{k},\omega)=E_i(k,-\omega)^*$ and $J_i(\mathbf{k},\omega)=J_i(\mathbf{k},-\omega)^*$ it follows that $\chi_{ij}(\mathbf{k},\omega)=\chi_{ij}(k,-\omega)^*$; we can write

$$\frac{P_0}{\Omega}=\int\frac{d^3k}{(2\pi)^3}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\times\left\{\omega 2\,\mathrm{Im}\chi_{ij}(\mathbf{k},\omega)\frac{\langle E_i(\mathbf{k},\omega)E_j^*(\mathbf{k},\omega)\rangle}{\Omega T}\right\}+\frac{P^{NL}}{\Omega}. \quad (6.7)$$

It can be shown that the nonlinear dissipation P^{NL} vanishes if the dissipative part of the nonlinear susceptibility is small compared with its reactive part. This was shown to be the case in Sec. 2. The physical reason for this is clear. If the nonlinear susceptibility is nonlossy, then for every pump photon lost, one optical plasmon and one acoustic plasmon are gained. Thus the decrease in the plasmon loss rates due to the nonlinear parametric gain is exactly canceled by the increase in the pump photon loss rate. Ultimately, the heating or dissipation in the steady-state system arises solely from the *linear* losses. This assumes, of course, that *only* these three modes are appreciably coupled. In Sec. 7 we will briefly discuss the effect of further mode coupling.

The total electric field is made up of the self-consistent pump field plus the fluctuating longitudinal fields; thus

$$E_i(k,\omega)=\hat{e}_i^0\{\tfrac{1}{2}\bar{E}_0(2\pi)^4\delta(\omega-\omega_0)\delta^3(\mathbf{k}-\mathbf{k}_1)+\tfrac{1}{2}\bar{E}_0^*(2\pi)^4\delta(\omega+\omega_0)\delta^3(\mathbf{k}+\mathbf{k}_0)\}+E^L(\mathbf{k},\omega)k_i/|\mathbf{k}| \quad (6.8)$$

where $E^L(\mathbf{k},\omega)=-kU(\mathbf{k},\omega)$ is the fluctuating longitudinal field of Sec. 2. Since $\langle E^L(\mathbf{k},\omega)\rangle=0$, we can write (6.7) as[28]

$$\frac{P_0}{\Omega}=\tfrac{1}{2}|\bar{E}_0|^2\omega\,\mathrm{Im}\chi_0(\mathbf{k}_0,\omega_0)+\frac{d^3k}{(2\pi)^3}\int\frac{d\omega}{(2\pi)}\omega 2\,\mathrm{Im}\chi_L(\mathbf{k},\omega)\frac{\langle|E^L(\mathbf{k},\omega)|^2\rangle}{\Omega T}. \quad (6.9)$$

From (2.33) we have

$$\frac{\langle|E^L(\mathbf{k},\omega)|^2\rangle}{\Omega T}=\frac{(4\pi)^2e^2}{k^2}S_0(\mathbf{k},\omega;\bar{E}_0). \quad (6.10)$$

Here we have explicitly indicated that $S_0(\mathbf{k},\omega)$ is a function of $\bar{E}_0$ because of the parametric coupling. The expression (6.9) is not exactly correct since it implies that power is absorbed from external sources even when $\bar{E}_0=0$ and there are no parametrically excited plasma fluctuations. The thermal fluctuations which exist when $\bar{E}_0=0$ result from the balance between spontaneous emission of plasmons and the linear decay of plasmons. The energy for these fluctuations is provided by the heat bath which maintains the temperature. Therefore, in computing the power required of the external electrical sources we should subtract the effect of $S(\mathbf{k},\omega;\bar{E}_0=0)$.

$$\frac{P_0}{\Omega}=\tfrac{1}{2}|\bar{E}_0|^2\omega\,\mathrm{Im}\chi_0(\mathbf{k}_0,\omega_0)+\int\frac{d^3k}{(2\pi)^3}\int\frac{d\omega}{(2\pi)}\omega 2\,\mathrm{Im}\chi_L(\mathbf{k},\omega)\frac{(4\pi)^2e^2}{k^2}\times[S_0(\mathbf{k},\omega;\bar{E}_0)-S_0(\mathbf{k},\omega;0)]. \quad (6.11)$$

We assume again that $S_0(\mathbf{k},\omega;\bar{E}_0)$ is dominated by its resonance at $\omega_\nu-i\gamma_\nu$ and use (4.16) for the integral over ω

$$\frac{P_0}{\Omega}=\tfrac{1}{2}|\bar{E}_0|^2\gamma_0+2\Theta_e\int_{-\infty}^{\infty}\frac{d^3k}{(2\pi)^3}\sum_\nu Z_\nu\gamma_\nu\times[A_\nu(\phi)K(\phi)-1], \quad (6.12)$$

where we have used the relations

$$\gamma_0=\omega_0\,\mathrm{Im}\chi_0(\mathbf{k}_0,\omega_0)$$
$$\gamma_\nu=Z_\nu\omega_\nu\,\mathrm{Im}\chi_L(k_\nu,\omega_\nu). \quad (6.13)$$

By using (5.6) and (5.7) for $A_\nu(\phi)$ and $K(\phi)$, the integrations over ϕ and $\mathbf{k}$ can be approximated by the same arguments used to obtain (5.8):

$$\Lambda_0^2=\bar{\Lambda}^2+\frac{k^2\delta k}{n(2\pi)^2}\sum_\nu\frac{\gamma_\nu}{\gamma_0}\left\{\left[\frac{Z_\nu^{NL}}{Z_\nu}\frac{\omega_\nu^{NL}}{\omega_\nu}\right]^2\frac{\gamma_\nu^2}{\gamma_i^2}\times\left[\left(1+\frac{\omega_\nu}{\omega_0-\omega_\nu}\right)\frac{\Lambda_c}{\bar{\Lambda}}\ln\frac{\Lambda_c+\bar{\Lambda}}{\Lambda_c-\bar{\Lambda}}-\frac{2\omega_\nu}{\omega_0-\omega_\nu}\right]\right\}, \quad (6.14)$$

where

$$\Lambda_0^2=\frac{P_0}{\Omega n\Theta_e\gamma_0}(\theta_e=\beta_e^{-1}), \quad (6.15)$$

$$\bar{\Lambda}^2=|\bar{E}_0|^2/n\Theta_e. \quad (6.16)$$

This transcendental equation determines the self-consistent pump amplitude or $\bar{\Lambda}^2$ in terms of the power Λ^2 delivered by the external sources. Note that if $\bar{\Lambda}^2\ll\Lambda_c^2$, then $\bar{\Lambda}^2\approx\Lambda_0^2$ so

$$|\bar{E}_0|^2=P_0/\Omega\gamma_0=|E_0|^2, \quad (6.17)$$

i.e., in this case we can identify $\bar{E}_0$ as the field produced by the external sources plus the *linear* induced currents alone.

Now going to the explicit case considered in the text, $\gamma_L\gg\gamma_i$, $\omega_L\gg\omega_i$, we use (3.38), (3.34), and (3.35) to write (6.14) as

$$\Lambda_0^2=\bar{\Lambda}^2+F\left\{\frac{\Lambda_c}{\bar{\Lambda}}\ln\frac{\Lambda_c+\bar{\Lambda}}{\Lambda_c-\bar{\Lambda}}-2\right\}, \quad (6.18)$$

where

$$F\simeq\frac{k^2\delta k}{n(2\pi)^2}\left[\frac{\bar{\Lambda}^4}{\bar{\Lambda}_c^4}\frac{\omega_L}{\omega_i}\right]. \quad (6.19)$$

[28] Here $\chi_L\equiv\hat{k}_i\hat{k}_j$, χ_{ij}, $\chi_0\equiv\hat{e}_i^0\hat{e}_j^0\chi_{ij}$.

If we assume $\Lambda_0^2 > \Lambda_c^2$, then $\bar{\Lambda}$ must be near Λ_c and we have approximately

$$\Lambda^2 - \Lambda_c^2 \simeq F \ln\left(\frac{4}{1-\bar{\Lambda}^2/\Lambda_c^2}\right) = F \ln 4\bar{K} \quad (6.20)$$

or

$$\bar{K} = \tfrac{1}{4} \exp\left(\frac{\Lambda_0^2 - \Lambda_c^2}{F}\right). \quad (6.21)$$

Thus for power levels above threshold, i.e., $\Lambda_0^2 > \Lambda_c^2$, the amplification factor $\bar{K}$ increases exponentially with Λ_0^2 but always remains finite for finite Λ_0^2.

7. DISCUSSION

The inclusion of self-consistency requirements for the pump field is thus sufficient to make the three-mode parametric model finite. However, nonlinear longitudinal mode coupling effects which we have neglected will undoubtedly reduce (perhaps greatly) the steady state amplitude which can be experimentally attained. The principal mode coupling effects are probably further three-mode couplings of the type responsible for the parametric effect itself. The parametrically excited plasmons can themselves act as a pump which couples to another plasma mode and another ion acoustic mode. These secondary processes will not be as strong as the original parametric coupling since the pump energy is now spread over a number of spatial (**k**) modes and since the frequency-matching conditions for the secondary pump plasmons will not be optimum.

A complete treatment of the effect of longitudinal mode coupling on the saturation level has not yet been carried out.

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Collisionless Sound in Classical Fluids*

MARK NELKIN AND S. RANGANATHAN
Department of Applied Physics, Cornell University, Ithaca, New York
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The dynamic form factor $S(K,\omega)$ for a classical fluid is calculated from the linearized Vlasov equation. Following Percus or Zwanzig, the effective interatomic potential is taken as $-kTc(r)$, where $c(r)$ is the direct correlation function. The result for $S(K,\omega)$ is a simple closed expression with no free parameters except for the static structure factor $S(K)$. Using Ashcroft and Lekner's hard-sphere Percus-Yevick results for $S(K)$, we calculate the inelastic neutron scattering from liquid lead. The resulting scattering law shows a strong qualitative similarity with experiment. The narrow quasielastic peak observed experimentally is not, however, given by the calculation. The reasons for this discrepancy are discussed. An extension of the calculations to include a phenomenological collision term is also presented.

I. INTRODUCTION

RECENT slow-neutron inelastic-scattering experiments have shown a surprising persistence of phononlike excitations in the liquid state. The dynamic form factor $S(K,\omega)$ exhibits a structure associated with propagating sound waves in a variety of liquids including liquid helium above and below the lambda point[1] as well as classical liquids.[2] This naturally suggests that a mean field theory would provide a useful phenomenological description of such experiments.[3] In the present paper we present some simple calculations demonstrating that this is in fact the case for classical fluids.

The familiar classical limit of a mean field theory is the Vlasov equation. This equation has long been used[4] to calculate the dynamic form factor associated with electron density fluctuations in a plasma. To use the Vlasov equation in neutral fluids we must replace the actual interatomic potential by an appropriate effective potential. The desired substitution is $v(r) \to -kTc(r)$, where $c(r)$ is the direct correlation function. This replacement was first suggested by Percus and Yevick.[5] When used in a self-consistent way it leads to the well-known Percus-Yevick integral equation for the radial distribution function. The same replacement has been obtained by Zwanzig[6] from consideration of variational expressions for eigenfunctions of the Liouville equation. Within the context of the Vlasov equation, the replace-

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