Padé-Approximant Calculation of π-π Scattering*

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Encouraged by the results of potential scattering, the Padé method of summing the Born series is applied to $\lambda\phi^4$ theory. The results of the third-order calculation are compared with previous work, both theoretical and experimental.

I. INTRODUCTION

WE present here a calculation of π-π scattering based on a $4\pi\lambda(\phi\cdot\phi)^2$ interaction. The method is to improve the convergence of the perturbation expansion by the use of the Padé approximant.[1]

There have been many attempts at a one-parameter solution to the π-π problem.[2–8] Our approach is most similar in spirit to that of Ref. 7 where an algebraic rearrangement of the perturbation expansion is also used. We do not, however, feel that there is really any basic difference between our approach and the dispersion-theory approaches of Refs. 2–6. The main difference is in the method that one uses to rearrange the perturbation expansion. In Refs. 2–6, the methods are based on the analytic properties of the scattering amplitude in the energy variables, whereas we consider the Padé method to depend on analytic properties in the coupling constant.[9]

The present calculation has several motivations. The Padé method has shown itself to be a useful theoretical and numerical tool in potential scattering.[1,9,10] It was desirable to test the convergence of the method in field theory and to compare its predictions with dispersion calculations and experiment. The natural field-theory candidate is the $\lambda\phi^4$ theory, which is thought to have some connection with reality, and where the first few orders of perturbation theory have been explicitly calculated.[2,6,7] If the Padé method did demonstrate a reasonable degree of convergence, then one could hope to compare the predictions of $\lambda\phi^4$ theory with experiment. The experimental situation is such that there is now believed to be more structure than just a p-wave resonance. There have been suggestions of an ABC enhancement (300 MeV) and/or a σ particle (400 MeV), while an ϵ particle (750 MeV) has been used to explain the ρ-decay asymmetry parameter.[11,12] On the other hand, a recent experiment[11] has found no significant s-wave structure. One would thus like to answer the question, does $\lambda\phi^4$ theory predict any low-energy s- or p-wave structure?

In Sec. II we present the method, and in Sec. III the results are discussed. In summary, the s waves show no low-energy structure except for broad resonances for $\lambda \gtrsim 0.2$. The p wave resonates at the ρ mass for $\lambda=0.115$ and with a width $\Gamma\approx 20$ MeV. The p-wave result must not be taken too seriously until a fourth-order calculation is done in order to check convergence. In Sec. IV we conclude that the method should work well for intermediate-strength couplings but that the field-theory singularities associated with the Dyson "collapsed state"[13] may limit the method.

II. METHOD

In a previous paper it was shown that in potential scattering one can (for a suitable class of potentials) construct the scattering amplitude from its perturbation expansion. For momentum $q\geq 0$, one approximates $f(\lambda)=(1/q^{2l+1})\tan\delta_l$, where f has the power-series expansion

$$f(\lambda)=\lambda f_1(q)+\lambda^2 f_2(q)\cdots, \tag{1}$$

by a ratio of polynomials

$$[N,M]=\frac{p_0(q)+\lambda p_1(q)\cdots\lambda^M p_M(q)}{1+\lambda q_1(q)\cdots\lambda^N q_N(q)}. \tag{2}$$

The unique rational fraction $[N,M]$ is called the N,M Padé approximant[1] to f, and the coefficients $p_i(q)$ and $q_i(q)$ are determined by requiring that Eqs. (1) and (2) have the same power-series expansion up to and including the term λ^{N+M}.

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[1] G. A. Baker, Jr., in *Advances in Theoretical Physics*, edited by K. A. Brueckner (Academic Press Inc., New York, 1965), Vol. I, p. 1.

[2] M. Baker and F. Zachariasen, Phys. Rev. **118**, 1659 (1960).

[3] G. F. Chew and S. Mandelstam, Phys. Rev. **119**, 467 (1960).

[4] B. H. Bransden and J. W. Moffat, Nuovo Cimento **21**, 505 (1961); K. Kang, Phys. Rev. **139**, B126 (1965).

[5] V. V. Serebryakov and D. V. Shirkov, Fortschr. Physik **13**, 227 (1965).

[6] A. Saperstein and J. Uretsky, Phys. Rev. **140**, B352 (1965).

[7] M. Alexanian and M. Wellner, Phys. Rev. **140**, B1079 (1965).

[8] J. G. Cordes, Phys. Rev. **156**, 1707 (1967).

[9] D. Masson, J. Math. Phys. (to be published).

[10] J. L. Gammel and F. A. McDonald, Phys. Rev. **142**, 1245 (1966); S. Tani, *ibid.* **139**, B1011 (1965).

[11] C. J. S. Damerell, N. Middlemas, D. Newton, A. B. Clegg, W. S. C. Williams, and A. S. Carroll, Phys. Rev. **156**, 1451 (1967).

[12] D. Griffiths and R. J. Jabbur, Phys. Rev. **157**, 1371 (1967).

[13] F. J. Dyson, Phys. Rev. **85**, 631 (1952); W. M. Frank, J. Math. Phys. **5**, 363 (1964).

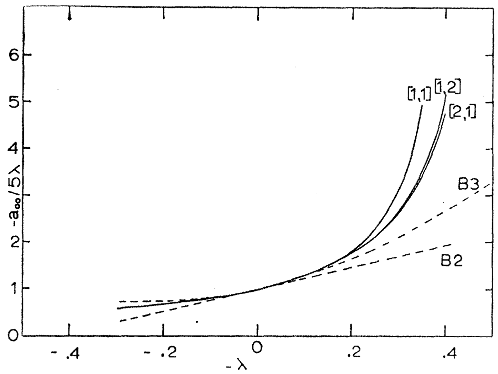

FIG. 1. The $I=0$, $J=0$ scattering length as a function of λ. The solid curves are the Padé approximants. The dashed curves are second- and third-order perturbation theory.

In Ref. 9, the following theorem is proved.

Theorem: If

$$\lim_{\lambda\to\infty} f(\lambda)/\lambda=0, \quad \arg\lambda\neq 0,\ \pi$$

and the potential $\lambda V(r)$ satisfies

$$V(r)>0 \quad \text{and} \quad \int_0^\infty [V(r)]^{1/2}dr<\infty\,,$$

then for q sufficiently small,

$$\lim_{N\to\infty}[N,\,N+j]=f(\lambda)\,.$$

For a general q, one may include only $j=0,\ \pm2,\ \pm4$.

Here, we optimistically apply the Padé method to $\lambda\phi^4$ theory up to third order in λ where we may calculate [1,1], [1,2], and [2,1].

We have used the perturbation calculations of Baker and Zachariasen[2] and cross-checked them with those of Alexanian and Wellner.[7] Some sign misprints in the former paper are corrected, and the results are (we use the Chew-Mandelstam definition of λ)

$$(\omega/q)\ \tan\delta_{JI}=\lambda C_1{}^{JI}+\lambda^2C_2{}^{JI}+\lambda^3C_3{}^{JI}\,. \tag{3}$$

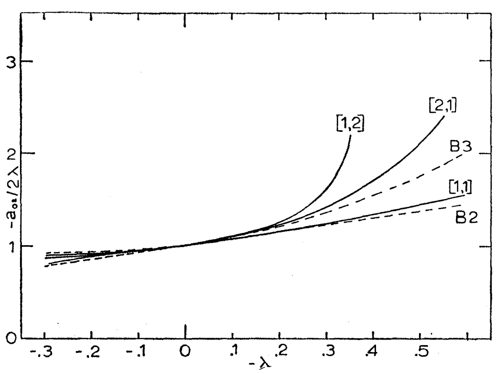

FIG. 2. The $I=2$, $J=0$ scattering length as a function of λ.

We define

$$C_i=\begin{pmatrix}C_i{}^{00}\\ C_i{}^{02}\end{pmatrix}, \tag{4}$$

and have

$$C_1=-\begin{pmatrix}5\\2\end{pmatrix}, \tag{5}$$

$$C_2=\begin{pmatrix}25\\4\end{pmatrix}F(s)+\begin{pmatrix}30\\18\end{pmatrix}[F(s)]_0\,, \tag{6}$$

$$C_3=-\begin{pmatrix}125\\8\end{pmatrix}F^2(s)-\begin{pmatrix}300\\72\end{pmatrix}G(s)-\begin{pmatrix}110\\86\end{pmatrix}[F^2(s)]_0-\begin{pmatrix}220\\112\end{pmatrix}[G(s)]_0\,, \tag{7}$$

$$C_1{}^{11}=0\,, \tag{8}$$

$$C_2{}^{11}=10[F(s)]_1\,, \tag{9}$$

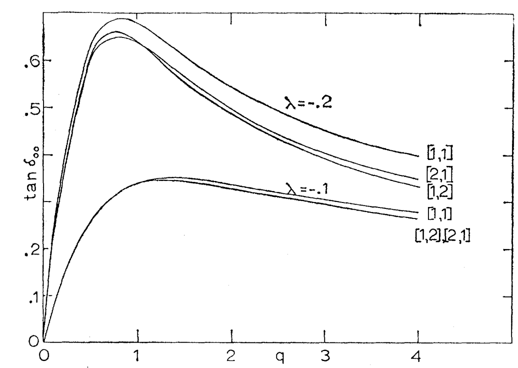

FIG. 3. $\tan\delta_{00}$ as a function of pion momentum q for negative λ.

$$C_3{}^{11}=-70[F^2(s)]_1-20[G(s)]_1\,, \tag{10}$$

where

$$F(s)=\frac{1}{\pi}(s-\tfrac{4}{3})P\int_4^\infty \frac{ds^1(s^1-4)^{1/2}}{(s^1-s)(s^1-\tfrac{4}{3})(s^1)^{1/2}}, \tag{11}$$

and

$$G(s)=\frac{(s-\frac{4}{3})}{\pi}P\int_4^\infty ds^1\frac{[F(s^1)]_0}{(s^1-s)(s^1-\frac{4}{3})}\left(\frac{s^1-4}{s^1}\right)^{1/2}. \tag{12}$$

Also, we have defined for any function $F(s)$

$$[F(s)]_l=\tfrac{1}{2}\int_{-1}^{1}F(t)P_l(\cos\theta)\quad d\cos\theta\,, \tag{13}$$

where $s=4\omega^2$ is the square of the center-of-mass energy $\omega^2=q^2+1$ and $t=-2q^2(1-\cos\theta)$.

III. RESULTS

The coefficients C_i^{JI} were calculated from the above equations using the IBM 7094 computer at Toronto. Once these were obtained, it was only a matter of algebra to calculate [1,1], [1,2], and [2,1]. The s-wave results are shown in Figs. 1–6.

In Figs. 1 and 2 we have plotted $-a_{00}/5\lambda$ and $-a_{02}/2\lambda$ as a function of $-\lambda$, where a_{JI} is the scattering length, together with the predictions of second- and third-order perturbation theory (B_2 and B_3). The convergence for $I=0$, $J=0$ is excellent, but for $I=2$ it is not as good. The [1,2] Padé predicts an $I=0$ s-wave bound state for $\lambda=-0.51$, in good agreement with Chew, Mandelstam, and Noyes[14] (−0.48), and Serebryakov and Shirkov[5] (−0.5).

In Figs. 3 and 4 we have plotted $\tan\delta_{00}$ and $\tan\delta_{02}$ for $\lambda=-0.1$ and -0.2 as a function of pion momentum q. There is observed to be no structure except an enhancement at low energy due to a large scattering length. The convergence is excellent even at large energies.

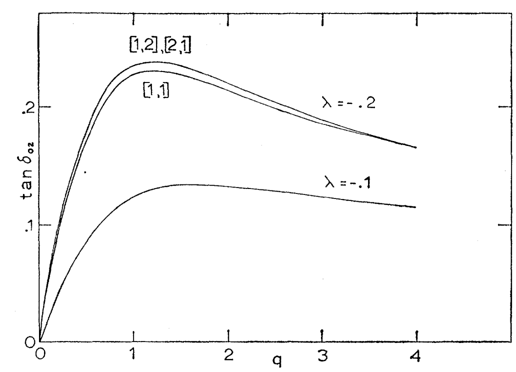

FIG. 4. tan δ_{02} as a function of pion momentum q for negative λ.

In Figs. 5 and 6 we have plotted δ_{00} and δ_{02} for the opposite sign of λ. Here we observe that for $\lambda=0.2$ and above the position of the ρ ($q=2.55$), both phase shifts pass through $-\pi/2$ causing broad resonances. The $I=2$ would yield the narrower width (≈500 MeV). One will note that this is a good test of the Padé method. The perturbation expansion is unable to predict these "resonances" while all approximants are in close agreement for their positions and widths. It should be mentioned that the coefficients C_2^{00} and C_2^{02} vanish at $q=0.93$ and $q=0.41$, respectively, and thus the [1,2] Padé approximant must be discarded near these values of q. It is precisely this situation that restricts the values of j in the theorem quoted in Sec. II.

For the p wave, since C_1^{11} vanishes, we are only able to use the [1,2] Padé approximant. We thus have no check on the convergence of the method and cannot rely on the results. For the scattering length, we get

$$a_{11}=(10/9\pi)\lambda^2/(1-\lambda/0.29). \tag{14}$$

For $\lambda=0.115$ one gets a p-wave resonance at the position of the ρ. The width Γ is given approximately by

$$\Gamma=\frac{4\lambda C_2^{11}q/\omega}{d(C_3^{11}/C_2^{11})/d\omega}, \tag{15}$$

where Eq. (15) is evaluated at $\lambda=0.115$, $q=2.55$. This gives $\Gamma=0.13$ (18 MeV). Note that in this lowest approximation one cannot help but get a ρ resonance, since the denominator of $\tan\delta_{11}$ is linear in λ and always has a zero for real λ. It will be interesting to look at fourth order and the [2,2] Padé approximant to see if the ρ resonance persists. We do feel that perhaps the sign of λ for which one obtains the ρ is relevant. That is to say, the p wave is most attractive for positive λ and negative s waves. This is in agreement with all other calculations, except those of the inverse-amplitude method[4] and Ref. 5. The former had difficulty with convergence for small positive λ, while the latter considered it inconsistent with their dynamical equations.

The results of Alexanian and Wellner[7] are in qualitative agreement with ours. The method is also similar in that they rearrange the perturbation expansion, and their lowest approximant is in fact the [1,1] Padé

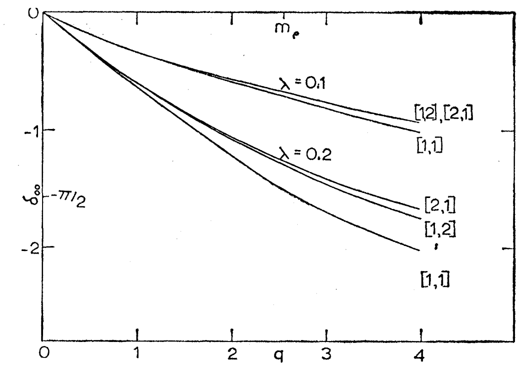

FIG. 5. δ_{00} as a function of q for positive λ.

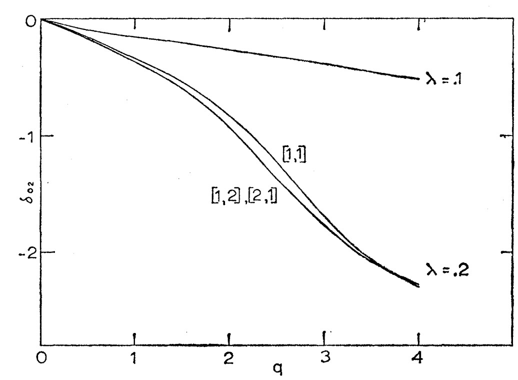

FIG. 6. δ_{02} as a function of q for positive λ.

[14] G. F. Chew, S. Mandelstam, and H. P. Noyes, Phys. Rev. 119, 478 (1960).

approximant. The methods differ in higher approximants. The Padé method is, however, much simpler since they must solve a nonlinear differential equation even in their second approximant. Also, their criterion for choosing a function to approximate is based on the smoothness of the function (thus they chose to use δ instead of $\tan\delta$). Our calculations are observed to converge more rapidly than do theirs.

The dispersion calculation which can be most easily compared with ours is that of Ref. 6. They calculated the left-hand cut in perturbation theory and then constructed the partial wave from the N/D method. They had difficulty in achieving convergence for $|\lambda|>0.1$. Although their method works in potential scattering, we remark that in field theory there is no reason to expect the left-hand cut to converge as a function of λ. We believe that had the Padé method been used to rearrange the perturbation expansion of the left-hand cut, their convergence would have improved.

Although for $\lambda=0.115$ we get a ρ resonance at the right position, its width is too small by a factor of 6. Also, for this value of λ, the $I=0$ s wave shows no resonance-type structure. This value of λ is, however, compatible with the $I=2$ data. A recent analysis[15] gave $|a_{02}|=0.204\pm0.028$. If we take the sign of δ_{02} from the analysis of the ρ-decay asymmetry parameter[12] and use Fig. 2, we get $\lambda=0.106\pm0.015$.

IV. CONCLUSIONS

We have seen that the Padé approximant can be successfully applied in field theory to yield converging results where the perturbation expansion fails. It is our belief that low-order approximants provide a good approximation for intermediate-strength couplings where only a small number of resonances dominate.

The present calculation yields a p-wave resonance at 760 MeV for $\lambda=0.115$ with a width of 18 MeV. For this value of λ, the s waves are small and negative. We do not, however, place much faith in the p-wave result since one must go to fourth order and the [2,2] Padé approximant in order to check convergence for the p wave.[16] For the s waves, convergence is already good. It will be interesting to see if the ρ resonance persists in fourth order.

As is mentioned in Ref. 9, the field-theory case is complicated by the presence of extra singularities associated with the Dyson "collapsed state." These extra singularities have a branch point at $\lambda=0$ which makes the perturbation expansion asymptotic. It has been shown that the Padé method can also cope with these extra singularities.[17] However, since the coefficients in the asymptotic expansion are believed to grow like $n!$, these extra singularities will dominate in higher orders and may limit the usefulness of the Padé method. A deeper understanding of this difficulty is desirable.

Note added in proof. After completing this work a CERN report by Bessis and Pusterla[18] was received. They do a similar calculation but include fourth-order perturbation theory and analytically continue in l. The real part of the Regge trajectories for the ρ and f_0 mesons are obtained correctly to within 15% for $\lambda\approx0.12$. We consider this to be very encouraging.

[15] T. S. Yoon, P. Berenyi, A. W. Key, J. D. Prentice, N. R. Steenberg, E. West, W. A. Cooper, W. Manner, L. Voyvodic, and W. D. Walker, Bull. Am. Phys. Soc. 12, 684 (1967).

[16] Fourth-order calculations are being done by M. A. Newton (private communication).

[17] G. A. Baker, Jr., and R. Chisholm, J. Math. Phys. 7, 1900 (1966).

[18] D. Bessis and M. Pusterla, Phys. Letters 25B, 279 (1967).

Errata

Low-Energy Theorem for the Weak Axial-Vector Vertex, S. L. Adler and Y. Dothan [Phys. Rev. **151**, 1267 (1966)]. In Eq. (80) for $M_{\lambda\alpha}{}^R$, the tensor multiplying $F_1{}^{V'}(0)$ should be $(q_\lambda\gamma_\alpha+k_\alpha\gamma_\lambda-\delta_{\lambda\alpha}\gamma\cdot k)$. In Eq. (82) for $O_{\lambda\alpha}$, the tensor multiplying $\bar{V}_2{}^{(0)}|_0$ should be $[(p_1+p_2)_\lambda k_\alpha-(p_1+p_2)\cdot k\delta_{\lambda\alpha}]$. Throughout Sec. III, $M^2{}_N$ should be read as $M_N{}^2$. We wish to thank Dr. J. Yellin for helpful discussions.

Towers as Sets of Composite Particles and Trouble with Infinite-Component Field Theories, M. B. Halpern [Phys. Rev. **159**, 1328 (1967)]. Strictly speaking, our discussion is only valid when one has crossing symmetry in the usual sense; that is, towers must be viewed as sets of composite particles if crossing is to be maintained. We thank Dr. C. Fronsdal for emphasizing to us that, in his theory, the external particles are also part of an infinite- dimensional representation—thus the usual crossing is lost.