There are symmetry relations

$$\Theta(klpq)=\Theta(lkpq)=\Theta(pqkl)^* \tag{B23}$$

and

$$\Omega(klpq)=\Omega(lkpq)=\Omega(pqkl)^*. \tag{B24}$$

Equation (B23) follows from Eq. (B9) and Eq. (B24) is imposed so that $G_2G_2^{-1}=G_2^{-1}G_2$.

Equations (B18) and (B20) show that the expansion coefficients $I^{\gamma\delta}$ have connected and disconnected parts. The three-body contributions to $I^{\gamma\delta}$ have disconnected parts which begin to show an exponential cluster structure Eq. (A25). This remarkable circumstance prompts the conjecture that the complete "optimal" expansion of the unit operator obeys the exponential cluster structure exactly. If the conjecture is correct, the "optimal" two-body approximation can be improved upon by including some fragments of three-body states so that the cluster properties are maintained exactly. More generally, in the "improved" m-body approximation, the relation $\mathcal{I}=\exp\mathcal{I}_c$ is kept exactly and an m-body approximation to $\mathcal{I}_c$ is calculated. The viability of the conjecture and the validity of the "improved" approximation will be examined elsewhere.

PHYSICAL REVIEW VOLUME 164, NUMBER 5 25 DECEMBER 1967

Strong-Coupling Solution of the Lee Model*

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(Received 24 July 1967)

The Lee model isobar states are derived for large bare coupling ($Z\rightarrow 0$) by the methods of old-fashioned strong-coupling theory. There is one isobar in each sector with energy $\propto n(n-1)/g_0^2$, where n is the maximum number of mesons allowed in the sector. Renormalization constants are also presented for all sectors.

THE purpose of this note is to solve the simplest known static model[1] in the strong-coupling limit by a method similar to old-fashioned strong-coupling theory.[2] Though the Lee model is extremely crude, relatively little progress has been made in solving it in general. Amado's solution[3] of the V-θ sector caused a revival of interest, but the higher sectors remain essentially unexplored.

It is remarkable that the Lee model even has a strong-coupling solution[4] similar to richer models, since the mesons involved do not have antiparticles (crossing symmetry). Also, the mesons belong to a "singlet" representation; therefore, no "rotational" states can emerge. Finally, the renormalized coupling constant remains finite as the bare coupling is taken to infinity.[5]

We present here a simple intuitive derivation of the isobar states which satisfy the Schrödinger equation. As in other strong-coupling derivations, it is necessary to make a few assumptions whose validity is checked at the end. We use a notation analogous to charged scalar theory; i.e., n, p, π^- are the particles.

The Hamiltonian[6] for the nonrelativistic Lee model may be written:

$$H=\tfrac{1}{2}(1-\tau_3)\epsilon_0+\int d^3r\left[\frac{\nabla\phi^\dagger(\mathbf{r})\cdot\nabla\phi(\mathbf{r})}{2m}+m\phi^\dagger(\mathbf{r})\phi(\mathbf{r})\right.$$
$$\left.+g_0\int d^3r\, u(r)[\phi(\mathbf{r})\tau_-+\phi^\dagger(\mathbf{r})\tau_+]\right], \tag{1}$$

where

$\phi(\mathbf{r})=\pi^-$-meson field,

$u(r)=$ source function,

$g_0=$ unrenormalized coupling constant,

$m=$ meson mass,

$$\tau_3=\begin{pmatrix}1&0\\0&-1\end{pmatrix},\quad \tau_+=\begin{pmatrix}0&1\\0&0\end{pmatrix},\quad \tau_-=\begin{pmatrix}0&0\\1&0\end{pmatrix},$$

$\epsilon_0=$ level shift for the neutron.

The τ matrices operate on bare proton and neutron states:

$$|p)=\begin{pmatrix}1\\0\end{pmatrix},\quad |n)=\begin{pmatrix}0\\1\end{pmatrix}.$$

Since we work throughout in the Schrödinger picture,

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* Supported by U. S. Atomic Energy Commission Contract No. AT-30-1-2171.

[1] T. D. Lee, Phys. Rev. **95**, 1329 (1954).

[2] G. Wentzel, Helv. Phys. Acta **13**, 169 (1940).

[3] R. Amado, Phys. Rev. **122**, 696 (1961).

[4] The strong-coupling limit is discussed in the lowest (soluble) sectors by M. T. Vaughn, R. Aaron, and R. D. Amado, Phys. Rev. **124**, 1258 (1961).

[5] This excludes the use of Goebel's unitarity argument on the pole terms of the Chew-Low equation. Cf. C. J. Goebel, Midwest Research Conference, 1965 (unpublished); also T. Cook, C. Goebel, and B. Sakita, Phys. Rev. Letters **15**, 35 (1965).

[6] In this nonrelativistic version of the Lee model, the meson field has dimensions $L^{-3/2}$, the cutoff $[u(r)]=L^{-3}$, the coupling $[g_0]=L^{1/2}$.

the meson-field operators satisfy

$$[\phi(\mathbf{r}),\phi^\dagger(\mathbf{r}')]=\delta^3(\mathbf{r}-\mathbf{r}'),\quad [\phi(\mathbf{r}),\phi(\mathbf{r}')]=0. \tag{2}$$

Inserting Fourier transforms for the meson fields, Eq. (1) becomes

$$H=\tfrac{1}{2}(1-\tau_3)\epsilon_0+\textstyle\sum_{\mathbf{k}}[a_{\mathbf{k}}^\dagger(k^2/2m+m)a_{\mathbf{k}} + g_0u_{\mathbf{k}}a_{\mathbf{k}}\tau_- + g_0u_{\mathbf{k}}a_{\mathbf{k}}^\dagger\tau_+], \tag{3}$$

with

$$a_{\mathbf{k}}=\int d^3r\,\phi(\mathbf{r})e^{-i\mathbf{k}\cdot\mathbf{r}}.$$

The charge, which is a constant of the motion, is given by

$$Q=\tfrac{1}{2}(1+\tau_3)-\textstyle\sum_{\mathbf{k}} a_{\mathbf{k}}^\dagger a_{\mathbf{k}}. \tag{4}$$

In order to solve the Schrödinger equation corresponding to Eq. (3), we must make the "field splitting" assumption of the old strong-coupling theory. The assumption is that the field may be split into bound and quasifree parts:

$$a_{\mathbf{k}}=(u_{\mathbf{k}}/\lambda)a+B_{\mathbf{k}},$$

where

$$\lambda^2=\int\frac{d^3k}{8\pi^3}u_{\mathbf{k}}^2\equiv\sum_{\mathbf{k}}u_{\mathbf{k}}^2. \tag{5}$$

Now $u_{\mathbf{k}}/\lambda$ is the bound-state wave function for the bound field quantum, and $B_{\mathbf{k}}$ may be expanded

$$B_{\mathbf{k}}=\textstyle\sum_{\mathbf{q}} v_{\mathbf{kq}}b_{\mathbf{q}},$$

where $v_{\mathbf{kq}}$ is the $\mathbf{k}$ component of the Fourier transform of the scattering state of a meson with momentum $\mathbf{q}$. The functions $v_{\mathbf{kq}}$ have the property

$$\textstyle\sum_{\mathbf{k}} u_{\mathbf{k}}v_{\mathbf{kq}}=0. \tag{6}$$

Equation (6) gives immediately

$$a=\textstyle\sum_{\mathbf{k}} u_{\mathbf{k}}a_{\mathbf{k}}/\lambda. \tag{7}$$

Now since

$$[a_{\mathbf{k}},a_{\mathbf{k}}^\dagger]=\delta_{\mathbf{kk}'},$$

we have[7]

$$[a,a^\dagger]=1. \tag{8}$$

Other commutation relations are easily derived to be

$$[a,B_{\mathbf{k}}^\dagger]=0, \tag{9}$$

$$[B_{\mathbf{k}},B_{\mathbf{k}'}^\dagger]=\delta_{\mathbf{kk}'}-u_{\mathbf{k}}u_{\mathbf{k}'}/\lambda^2. \tag{10}$$

We shall return to discuss the validity of the field-splitting assumption after solving the problem assuming it holds.

Expressed in terms of the new variables, the Hamiltonian may be written

$$H=\tfrac{1}{2}(1-\tau_3)\epsilon_0+(\textstyle\sum_{\mathbf{k}}(E_{\mathbf{k}}/\lambda^2)u_{\mathbf{k}}^2)a^\dagger a+g_0\lambda[a\tau_-+a^\dagger\tau_+] +\sum_{\mathbf{k}}(k^2/2m\lambda)u_{\mathbf{k}}[B_{\mathbf{k}}^\dagger a+a^\dagger B_{\mathbf{k}}] +\sum_{\mathbf{k}}E_{\mathbf{k}}B_{\mathbf{k}}^\dagger B_{\mathbf{k}}. \tag{11}$$

The crossed terms (connecting isobar states) between bound and quasifree operators are small when the cutoff momentum of the source $u_{\mathbf{k}}$ is much less than the meson mass. Precise conditions for the validity of this assumption will be given later. Finally, on account of Eq. (9), we may disregard quasifree meson excitations altogether since the Hamiltonian separates into two commuting parts.

The bound part of the Hamiltonian simplies to

$$H=\begin{pmatrix}\omega a^\dagger a & fa^\dagger\\ fa & \omega a^\dagger a+\epsilon_0\end{pmatrix}, \tag{12}$$

with $\omega\equiv\sum_{\mathbf{k}}E_{\mathbf{k}}u_{\mathbf{k}}^2\lambda^{-2}$ and $f\equiv g_0\lambda$. The eigenvectors of Eq. (12) are easily seen to be

$$v_n=\beta_n\begin{pmatrix}\psi_n\\ \alpha_n\psi_{n-1}\end{pmatrix}, \tag{13}$$

where α_n and β_n are constant coefficients, and the ψ_n are harmonic oscillator wave functions with

$$a^\dagger a\psi_n=n\psi_n, \tag{14}$$

$$\left(\frac{(a^\dagger)^n}{\sqrt{(n!)}}\right)\psi_0=\psi_n. \tag{15}$$

We have

$$Hv_n=E_nv_n, \tag{16}$$

and

$$Qv_n=-(n-1)v_n. \tag{17}$$

The eigenvectors are normalized so that

$$\beta_n^2=(1+\alpha_n^2)^{-1}, \tag{18}$$

and the E_n and α_n are to be found from the two simultaneous equations:

$$\omega n+\alpha_nf\sqrt{n}=E_n,\quad f\sqrt{n}+\alpha_n(n-1)\omega=\alpha_n(E_n-\epsilon_0). \tag{19}$$

ϵ_0 is to be adjusted to make the physical neutron have zero energy, i.e., $E_1=0$. The result is $\epsilon_0=f^2/\omega$. The solutions of Eq. (19) are then

$$E_n^\pm/\omega=\tfrac{1}{2}[(f/\omega)^2+2n-1]\pm\tfrac{1}{2}\{[(f/\omega)^2+2n-1]^2-4(n^2-n)\}^{1/2}, \tag{20}$$

$$\alpha_n^\pm/\omega=\frac{E_n^\pm}{f\sqrt{n}}-\frac{\sqrt{n}}{f}. \tag{21}$$

Now for large f, the two solutions $E_n^\pm$ are widely separated. Only E_n^- is retained since E_n^+ is of order f^2;

[7] Equation (8) is to be contrasted with the "coupling condition" of Goebel's crossing-symmetric models where the bound fields commute. A consequence of Eq. (8) is that in the Lee model the scattering amplitude has a single pole in the strong-coupling limit, whereas in crossing-symmetric models one obtains a double pole; cf. Ref. (5).

and, therefore, its eigenvectors couple only weakly with low-lying states. For large f the results then are

$$E_n=\omega^3 n(n-1)/f^2, \quad (22)$$

$$\alpha_n=-(\sqrt{n})\omega/f, \quad (23)$$

$$\beta_n=1-n\omega^2/2f^2. \quad (24)$$

Now that the eigenvectors are known it is a simple matter to compute the renormalization constants for any sector (charge). First the wave function renormalization for the neutron is

$$Z_1=\beta_1^2\alpha_1^2=\omega^2/f^2. \quad (25)$$

All other renormalization constants are related to

$$(v_n,\tau_3 v_n)=\beta_n^2(1-\alpha_n^2) =1-2n\omega^2/f^2, \quad (26)$$

and

$$(v_{n-1},\tau_+ v_n)=\beta_n\alpha_n =-(\sqrt{n})\omega/f. \quad (27)$$

For example, from Eq. (27), the renormalized $(n-1, n, \pi^-)$ coupling constant is

$$g_{n-1,n}=-(\sqrt{n})\omega/\lambda. \quad (28)$$

One application of Eq. (28) is that in the n-meson sector we find that the scattering amplitude has a crossed and direct pole; the residue of the crossed pole is $(n-1)\omega^2/\lambda^2$ while the residue of the direct pole is $-n\omega^2/\lambda^2$. According to Eq. (22), the poles come together as $f\to\infty$; the result is a single pole with residue $-\omega^2/\lambda^2$. This confirms the conjecture[4] that as $Z_1\to 0$ the scattering in all sectors becomes physically equivalent to scattering in the $\pi^- p$ channel.

Now we must return to the assumptions made early in the derivation. The field splitting assumption is easily verified by looking at matrix elements between the physical states of the equation of motion for $a_{\mathbf{k}}$:

$$i\dot{a}_{\mathbf{k}}^{rs}=[a_{\mathbf{k}},H]^{rs} =E_{\mathbf{k}}a_{\mathbf{k}}^{rs}+g_0 u_{\mathbf{k}}\tau_+^{rs}. \quad (29)$$

Now since E_r-E_s is of order ω^3/f^2, we may conclude that the field is nearly static, i.e.,

$$a_{\mathbf{k}}^{rs}\approx g_0(u_{\mathbf{k}}/E_{\mathbf{k}})\tau_+^{rs}. \quad (30)$$

Now for strong enough cutoff, $E_{\mathbf{k}}\approx m$, and Eq. (30) has the same momentum dependence as Eq. (5).

We may estimate the effect of the crossed terms[8] in Eq. (11) by looking at them as a perturbation on the masses in Eq. (22). We use second-order perturbation theory to show that the effect on isobar n is

$$\Delta E_n=-\sum_{\mathbf{q}}\frac{|\langle n|V|n-1,\mathbf{q}\rangle|^2}{m+q^2/2m}, \quad (31)$$

where

$$V=\sum_{\mathbf{k}}\frac{k^2}{2m\lambda}u_{\mathbf{k}}a^\dagger B_{\mathbf{k}}. \quad (32)$$

For a rough estimate we may take the $v_{\mathbf{kk}'}$ of Eq. (6) to be plane waves, i.e.,

$$v_{\mathbf{kk}'}\sim\delta_{\mathbf{kk}'}. \quad (33)$$

Then the matrix element of V is

$$\langle n|V|n-1,\mathbf{q}\rangle=(\sqrt{n}/2m\lambda)q^2u_{\mathbf{q}}, \quad (34)$$

and Eq. (31) becomes

$$\Delta E_n=-\frac{n}{4\lambda^2m^2}\sum_{\mathbf{q}}\frac{u_{\mathbf{q}}^2q^4}{m+q^2/2m}. \quad (35)$$

For a cutoff such that $\sum_{\mathbf{q}}\to\int_0^{R^{-1}}4\pi q^2dq$ and $R^{-1}\ll m$ (nonrelativistic mesons only), we have $\Delta E_n\sim+4\pi nm^4/7(mR)^7\lambda^2$. For the crude cutoff above, $\lambda^2\sim 4\pi/3R^3$. Then

$$\Delta E_n\sim-3nm/7(Rm)^4. \quad (36)$$

Now for our results to be valid, ΔE_n must be much less than E_n [Eq. (22)], which in terms of the cutoff radius is

$$E_n\sim 3n^2(Rm)^3/4\pi g_0^2. \quad (37)$$

The requirements then are two: that the isobars be tightly bound,

$$3n^2(Rm)^3/4\pi\ll mg_0^2; \quad (38)$$

and that the quasifree mesons only perturb the isobar masses slightly,

$$\frac{\Delta E_n}{E_n}\sim\left(\frac{4\pi}{7n}\right)\left(\frac{mg_0^2}{(Rm)^3}\right)\left(\frac{1}{Rm}\right)^4\ll 1. \quad (39)$$

Inequalities (38) and (39) are easily satisfied by taking

$$1\ll(mg_0^2)/(mR)^3\ll(mR)^4. \quad (40)$$

ACKNOWLEDGMENT

I am grateful to Professor R. D. Amado for reading the manuscript.

[8] In principle this limitation could be eliminated by using the strong-coupling prescription of Pauli. The present method is considerably simpler, however, so we retain it. Cf. W. Pauli and S. M. Dancoff, Phys. Rev. 62, 85 (1942).