PHYSICAL REVIEW VOLUME 164, NUMBER 5 25 DECEMBER 1967

Classification of Particle Multiplets*

BRUNO ZUMINO AND DANIEL ZWANZIGER
Department of Physics, New York University, New York, New York
(Received 9 August 1967)

In the present paper we observe that in considering how particle states transform under $\Theta = CPT$ one cannot neglect internal symmetries, since Θ changes the signs of all conserved internal additive quantum numbers; and that, furthermore, if a physical theory admits a parity symmetry $\mathcal{P}$, this symmetry may also, like Θ, effect an automorphism on the group of internal symmetries, as does, for example, CP, the correct parity operator in the presence of weak interactions. We find all types of particle multiplets for massive particles that are compatible with local field theory, particle multiplets being defined as the irreducible corepresentations of the group obtained by extending the internal symmetry group by $\mathcal{P}$ and Θ. Of the thirteen types that exist, only three occur in the various approximation schemes that are at present used to describe nature. The reason for the nonappearance of the other types is not to be found in the usual postulates of local field theory, because Lagrangian field-theoretical examples of all thirteen types do exist and are given.

I. INTRODUCTION

ONE of the most striking features of elementary particle physics is the existence of exact and approximate internal symmetry operations, such as those generated by baryon number and isotopic spin, which commute with all elements of the connected quantum mechanical Poincaré group. These internal symmetry operations also commute with the antilinear operator Θ, traditionally called CPT, which is guaranteed by a very general theorem[1] of elementary particle physics to be an exact symmetry. However, it is at present no longer possible to suppose that internal symmetries also commute with space and time reflections since the discovery[2] that it is CP, rather than P, which is (at least approximately[3]) conserved in weak interactions and which should properly be called the space reflection operator.

Michel[4] has proposed that, in general, the geometrical symmetry group may not be a subgroup of the quantum-mechanical symmetry group (the group of *all* transformations which leave the Lagrangian invariant), but is obtained instead as the factor group of the full symmetry group by an invariant subgroup. This idea finds its application in a recent work by Lee and Wick,[5] who classify all minimal extensions (to be described below) of the full internal symmetry group G by a parity operator $\mathcal{P}$.

In the present work we give a complete classification of the 13 types of particle multiplets, corresponding to irreducible corepresentations[6] of the group K, made by minimal extension of the group of internal symmetries G by a parity operator $\mathcal{P}$ and the CPT operator Θ. It is not necessary to adjoin an independent time-reversal operator $\mathcal{T}$ since it may be defined[5] by $\mathcal{T} = \mathcal{P}\Theta$. This classification is made in terms of the properties of the irreducible representations of the group G, assumed known, and the extension of G by $\mathcal{P}$ which is given. Starting from an irreducible representation of G of given dimensionality or particle multiplicity, it will be found that adjunction of $\mathcal{P}$ may or may not double the multiplicity and adjunction of Θ may or may not further double the multiplicity. Complete criteria for determining these alternatives, explicit formulas for the corepresentations, and illustrative examples are given. Our results also include *a fortiori* the case of theories, which one might wish to consider, that admit of no parity symmetry and, hence, no time reversal. The adjunction problem is then simpler since only Θ is adjoined, instead of both $\mathcal{P}$ and Θ.

As a simple example of the classification given here, consider some internal symmetry group G and adjoin only Θ. One of the three possible types of corepresentation of the resulting group is what may be called a self-conjugate particle multiplet. It occurs if and only if the irreducible representation $D^s(g)$ of G, which corresponds to the multiplet, is equivalent to a real representation [i.e., one in which the elements of the matrices $D^s(g)$ are real], in which case Θ acts within the multiplet; otherwise, the application of Θ to any state of the multiplet produces a state outside the multiplet. If the internal group is the isotopic spin

* This work was supported in part by the National Science Foundation.

[1] W. Pauli, *Niels Bohr and the Development of Physics* (Pergamon Press, Inc., London, 1955); J. Schwinger, Phys. Rev. **91**, 720 (1953); **94**, 1366 (1953); G. Lüders, Kgl. Danske Videnskab. Selskab, Mat.-Fys. Medd. **28**, No. 5 (1954); R. Jost, Helv. Phys. Acta. **30**, 409 (1957). See also R. F. Streater and A. S. Wightman, *PCT, Spin and Statistics and All That* (W. Benjamin, Inc., New York, 1964); and R. Jost, *The General Theory of Quantized Fields* (American Mathematical Society, Providence, Rhode Island, 1965).

[2] C. S. Wu *et al.*, Phys. Rev. **105**, 1413 (1957); T. D. Lee and C. N. Yang, ibid. **104**, 254 (1956); L. D. Landau, Nucl. Phys. **3**, 127 (1957).

[3] J. H. Christenson *et al.*, Phys. Rev. Letters **13**, 138 (1964).

[4] L. Michel, *Group Theoretical Concepts and Methods in Elementary Particle Physics*, edited by F. Gürsey (Gordon and Breach, Science Publishers, New York, 1964).

[5] T. D. Lee and G. C. Wick, Phys. Rev. **148**, 1385 (1966).

[6] Corepresentations are described by E. P. Wigner, *Group Theory and its Applications to the Quantum Mechanics of Atomic Spectra* (Academic Press Inc., New York, 1959), Ch. 26. They occur when one considers transformations of basis vectors in a vector space under a group that includes both linear and antilinear operators.

group $G=SU(2,C)$, the real representations correspond to integral isotopic spin, and there can be no self-conjugate multiplets of isotopic spin $\frac{1}{2}$. This particular result has been noted recently and thought to be a consequence of locality.[7] Actually, it is found here to depend on locality only to the extent that the CPT theorem is valid and thus holds also in a CPT-invariant S-matrix theory.[8] The first recognition of a degeneracy, which we now attribute to an invariance generated by an antilinear operator, dates back to Kramer's 1930 article,[9] and the classification of the three types of corepresentation obtained by adjoining an antilinear operator to a group of linear operators appears in Wigner's 1932 article on time inversion.[10] The new content of the present article is the classification of the 13 types of irreducible corepresentations obtained by adjoining both a parity operator of the type discussed by Lee and Wick[5] and an antilinear CPT operator Θ with $\Theta^2=(-1)^{2j}$.

Let us now turn our attention to the groups whose corepresentations we will find in the following sections. The complete quantum mechanical group of particle physics includes:

(1) Unitary operators representing the quantum-mechanical connected Poincaré group $P_+^\uparrow$. Its elements are given by (a,A), where a is a real 4 vector and $A\in SL(2,C)$ with the multiplication law $(a_2,A_2)(a_1,A_1)=(a_2+\Lambda(A_2)a_1,A_2A_1)$, where the Lorentz transformation $\Lambda(A)$ is specified by $a^{0\prime}+\boldsymbol{\sigma}\cdot\mathbf{a}'=A(a^0+\boldsymbol{\sigma}\cdot\mathbf{a})A^\dagger$ for $a'=\Lambda(A)a$. Its irreducible unitary representations have been found by Wigner.[11] We will be concerned here only with the irreducible unitary representations corresponding to positive mass. A similar analysis could also be effected for the case of zero mass, but this will not be done here.

(2) The antiunitary CPT operator Θ, whose multiplication law with (a,A) is given by $\Theta(a,A)\Theta^{-1}=(-a,A)$. Here and elsewhere in the article we will use the same symbol for the group element and the operator in Hilbert space corresponding to it. We consider here only the type occurring in field theory[5] where Θ is an antilinear operator satisfying[12]

$$\Theta^2=(-1)^{2j}. \tag{1.1}$$

We use $(-1)^{2j}$ as a shorthand notation for the element of $P_+^\uparrow$ specified by $(a,A)=(0,-1)$. It is also an element of the internal symmetry group corresponding to a rotation through 2π.

(3) Unitary operators g making up the *full* internal symmetry group G, assumed compact. Its irreducible representations will be labeled $D^s(g)$. The elements g commute with (a,A), and with Θ:

$$g\Theta=\Theta g. \tag{1.2}$$

This important property of Θ was stated and proven in Ref. 5. Alternatively, it may be proven, following Jost's proof of the CPT theorem in Ref. 1, by noting that because g commutes with the real Lorentz transformations it also commutes with their analytic continuation, the complex Lorentz transformations, and hence with Θ. (In general, $\mathcal{P}$ or $\mathcal{T}$ do not commute with g.[13]) If g is an element of a Lie group in the neighborhood of the identity and generated by the Hermitian operator F, so that $g=1+i\epsilon F$, we find from the antilinearity of Θ and Eq. (1.2)

$$\Theta F=-F\Theta.$$

Consequently Θ changes the sign of all internal additive quantum numbers, and hence carries every state into its antiparticle state.

(4) A parity operator $\mathcal{P}$, either as an exact or approximate symmetry. Its commutation relations with (a,A) and Θ are

$$\mathcal{P}(a,A)\mathcal{P}^{-1}=(a',\sigma_yA^*\sigma_y),$$

where $(a_0',\mathbf{a}')=(a_0,-\mathbf{a})$ and σ_y is the antisymmetric Pauli spin matrix, and[5,12]

$$\mathcal{P}\Theta\mathcal{P}^{-1}=(-1)^{2j}\Theta. \tag{1.3}$$

The commutation relations of $\mathcal{P}$ with g are specified only to the extent that

$$\mathcal{P}g\mathcal{P}^{-1}=F\cdot g\in G \tag{1.4}$$

is a given automorphism[14] F of G, and

$$\mathcal{P}^2=f\in G, \tag{1.5}$$

where f is a fixed point of the automorphism F,

$$\mathcal{P}f\mathcal{P}^{-1}=F\cdot f=f. \tag{1.6}$$

From $\mathcal{T}=\mathcal{P}\Theta$ and Eqs. (1.2)–(1.5) we find

$$\mathcal{T}g\mathcal{T}^{-1}=\mathcal{P}\Theta g\Theta^{-1}\mathcal{P}^{-1}=\mathcal{P}g\mathcal{P}^{-1}$$

or

$$\mathcal{T}g\mathcal{T}^{-1}=F\cdot g, \tag{1.7}$$

[7] P. Carruthers, Phys. Rev. Letters **18**, 353 (1967); Y. S. Jin, Phys. Letters **24B**, 411 (1967); G. N. Fleming and E. Kazes, Phys. Rev. Letters **18**, 764 (1967); Huan Lee, *ibid.* **18**, 1098 (1967); **19**, 57(E) (1967); M. B. Einhorn, (unpublished). However, more recently, O. Steinman [Phys. Letters **25B**, 234 (1967)] has independently recognized that this result depends only on the CPT theorem.

[8] H. Stapp, Phys. Rev. **125**, 2139 (1962) and *High-Energy Physics and Elementary Particles, Trieste Lectures* (IAEA, Vienna, 1965).

[9] H. A. Kramers, in his *Collected Scientific Papers* (North-Holland Publishing Company, Amsterdam), p. 525.

[10] E. P. Wigner, Göttinger Nachr., Math-Physik p. 546 (1932).

[11] E. P. Wigner, Ann. Math. **40**, 149 (1939).

[12] Other types may be considered (E. P. Wigner in Ref. 4) and analyzed along the lines given below. A classification of the representations of the Poincaré group including space and time reflections is given by Wigner in Refs. 4 and 11. The modifications which arise when the internal symmetries exist have not been considered, however.

[13] In the traditional formulation which has separate C, P, and T operators, whose product $CPT=\Theta$ commutes with g, P and T cannot both commute with all g since C does not.

[14] F^{-1} exists and $(F\cdot g_1)(F\cdot g_2)=F\cdot(g_1g_2)$.

and

$$\mathcal{T}^2 = \mathcal{P}\Theta\mathcal{P}\Theta = (-1)^{2j}\mathcal{P}^2\Theta^2 = \mathcal{P}^2$$

or

$$\mathcal{T}^2 = f. \tag{1.8}$$

In order to classify particle multiplets, it is sufficient to study the little group of the 4-vector $p_0 = (m,0,0,0)$, namely the group of all operators (apart from space-time translations) which leave the particle at rest. These include $A = u \in SU(2,C)$ (the quantum mechanical rotation group), $g \in G$, $\mathcal{P}$, Θ, and their products. In the rest frame, the rotations u are represented by

$$u|j\mu\rangle = \sum_\nu D^j{}_{\nu\mu}(u)|j\nu\rangle,$$

where the $D^j(u)$ are irreducible representations of $SU(2,C)$ of dimension $2j+1$. Now because g and $\mathcal{P}$ commute with u and $D^j(u)$ is irreducible, their representatives $D(g)$ and $D(\mathcal{P})$ are scalar matrices in spin space. However, even though Θ also commutes with u, it is not diagonal in spin space, but reverses the direction of all spins, as is known. This occurs because it is antilinear, so that from $\Theta u = u\Theta$ its representative $D(\Theta)$ satisfies

$$D(\Theta)D^{j*}(u) = D^j(u)D(\Theta).$$

It is convenient at this point to introduce instead of Θ the antilinear operator Θ_0 defined by

$$\Theta_0 = \Theta\exp(-i\pi J_y) = \exp(-i\pi J_y)\Theta, \tag{1.9}$$

which in the usual basis is a scalar in spin space. This is easily proven by verifying that the matrix $D(\Theta_0)$ commutes with all $D^j(u)$:

$$\begin{aligned} D^j(u)D(\Theta_0) &= D^j(u)D^j[\exp(-i\pi\sigma_y/2)]D(\Theta) \\ &= D^j[u(-i\sigma_y)]D(\Theta) = D(\Theta)D^{j*}[u(-i\sigma_y)] \\ &= D(\Theta)D^j[u^*(-i\sigma_y)^*] = D(\Theta)D^j[u^*(-i\sigma_y)] \\ &= D(\Theta)D^j[(-i\sigma_y)u] = D(\Theta)D^{j*}(-i\sigma_y)D^j(u) \\ &= D(\Theta_0)D^j(u). \end{aligned}$$

Consequently, in order to obtain the multiplet structure one may suppress all spin and momentum variables, because g $\mathcal{P}$, and Θ_0 leave the particle at rest, and their representatives are scalar matrices in spin space. Equations (1.1)–(1.3) for Θ are replaced by

$$\Theta_0^2 = 1, \tag{1.10}$$

$$g\Theta_0 = \Theta_0 g, \tag{1.11}$$

$$\mathcal{P}\Theta_0\mathcal{P}^{-1} = (-1)^{2j}\Theta_0. \tag{1.12}$$

The introduction of Θ_0 instead of Θ recalls the introduction, in the case of the isospin group, of isoparity or $\mathcal{G}$ parity, $\mathcal{G} = C\exp(-i\pi T_y)$, instead of the charge-conjugation operator C. Like Θ, Θ_0 changes the sign of all internal additive quantum numbers. One may, in fact, introduce for arbitrary momentum $\mathbf{p}$ a corresponding operator Θ_p which is a scalar in spin space.

As a further convenience, which will give the same form to representations for particles of integral and half-integral spin (hereafter called tensor and spinor particles, respectively) by eliminating factors of $(-1)^{2j}$, we introduce on the manifold of one-particle states the operator $\mathcal{P}_0$, instead of $\mathcal{P}$, defined by

$$\mathcal{P}_0 \equiv \sigma\mathcal{P}, \tag{1.13}$$

where $\sigma = 1$ for tensor particles and $\sigma = i$ for spinor particles. Using

$$f_0 \equiv (-1)^{2j} f \in G, \tag{1.14}$$

Eq. (1.3) or (1.12) is replaced by

$$\mathcal{P}_0\Theta_0 = \Theta_0\mathcal{P}_0, \tag{1.15}$$

by virtue of the antilinearity of Θ_0, and

$$\mathcal{P}_0^2 = f_0 \tag{1.16}$$

replaces Eq. (1.5). Equations (1.4) and (1.6) remain unchanged,

$$\mathcal{P}_0 g\mathcal{P}_0^{-1} = F\cdot g \in G, \tag{1.17}$$

$$\mathcal{P}_0 f_0\mathcal{P}_0^{-1} = F\cdot f_0 = f_0. \tag{1.18}$$

Instead of

$$\mathcal{T} = \mathcal{P}\Theta, \tag{1.19}$$

we also introduce for notational completeness

$$\mathcal{T}_0 = \mathcal{P}_0\Theta_0, \tag{1.20}$$

which satisfies

$$\mathcal{T}_0^2 = \mathcal{P}_0\Theta_0\mathcal{P}_0\Theta_0 = \mathcal{P}_0^2\Theta_0^2 = \mathcal{P}_0^2 = f_0, \tag{1.21a}$$

$$\mathcal{T}_0 g\mathcal{T}_0^{-1} = \mathcal{P}_0\Theta_0 g\Theta_0^{-1}\mathcal{P}_0^{-1} = \mathcal{P}_0 g\mathcal{P}_0^{-1} = F\cdot g. \tag{1.21b}$$

Equations (1.10), (1.11), (1.15)–(1.18), and (1.21) define the minimal extension of G by $\mathcal{P}_0$, Θ_0, and by $\mathcal{T}_0$.

The new operators $\mathcal{P}_0$ and Θ_0 have been defined to act on the manifold of one-particle states. If one were to apply them to the fields, they would be found to act nonlocally. However it will be found that when the fields are conjugated with $\mathcal{P}$ and Θ, and after the space-time transformation properties of the fields are taken into account (e.g., after multiplication by γ^0 or γ^5), the fields become multiplied by the representatives of $\mathcal{P}_0$ and Θ_0.

The main concern of the present article will be to find all irreducible corepresentations $D^u(k)$ of the group

$$K = \{k\} = \{G, G\mathcal{P}_0, G\Theta_0, G\mathcal{T}_0 = G\mathcal{P}_0\Theta_0\} \tag{1.22}$$

in terms of the irreducible representations $D^s(g)$ of the group G of all internal symmetry operators g. This will be done in two steps. In Sec. II we first find all irreducible representations $D^t(h)$ of the group

$$H = \{h\} = \{G, G\mathcal{P}_0\}, \tag{1.23}$$

which is the extension of G by $\mathcal{P}_0$. Secondly, in Sec. III, we similarly find all irreducible corepresentations[15]

[15] This second step is actually effected in Ref. 6. However, we repeat it here briefly for notational completeness and by a slightly simpler method.

$D^u(k)$ of the group K,

$$K=\{k\}=\{H,H\Theta_0\}, \tag{1.24}$$

viewed as the extension of H by the antilinear operator Θ_0. Then in Sec. IV the two steps are combined to yield the $D^u(k)$ in terms of $D^s(g)$. There turn out to be 13 types of irreducible corepresentations $D^u(k)$. They are displayed together in Table IV, which summarizes the principal results of the present investigation. In Sec. V our results are compared with previous analyses, and in particular a general discussion of the charge-conjugation operator C is given. The last section is devoted to illustrative examples from Lagrangian field theory. This is done by exhibiting Lagrangians which admit a certain internal symmetry group and possibly a parity operation. The group formed by these operators and the CPT operator transforms sets of multiple component fields in a manner which can be put in direct correspondence with the corepresentations studied in Secs. III and IV. Observe that the transformation laws of the fields, e.g. under parity, are not preassigned to them, but are instead determined by the Lagrangian. Consequently, in constructing field theoretic examples, our main task is to devise suitable interaction Lagrangians, which force particular transformation laws upon the fields.

All of our results are summarized in Tables I–VIII.

II. IRREDUCIBLE REPRESENTATIONS OF PARITY AND INTERNAL SYMMETRY GROUP

We seek the irreducible representations $D^t(h)$ of the group $H=\{h\}=\{G,G\mathcal{P}_0\}$. It will be found that they are expressible in terms of the irreducible representations $D^s(g)$ of G, assumed known. Since $h=g$ or $h=g\mathcal{P}_0$, we consequently must obtain $D^t(g)$ and $D^t(g\mathcal{P}_0)$. However, from the condition that $D^t(h)$ be a representation, we have $D^t(g\mathcal{P}_0)=D^t(g)D^t(1\cdot\mathcal{P}_0)$, for $1\in G$, and hence it is clearly sufficient to specify $D^t(g)$ and $D^t(1\cdot P_0)$, which we shall write as $D^t(\mathcal{P}_0)$. The equations that are required are (1.16)–(1.18), which we reproduce here:

$$\mathcal{P}_0 g\mathcal{P}_0^{-1}=F\cdot g, \tag{2.1}$$

$$\mathcal{P}_0^2=f_0, \tag{2.2}$$

$$\mathcal{P}_0 f_0\mathcal{P}_0^{-1}=F\cdot f_0=f_0. \tag{2.3}$$

Since G is a subgroup of H, every representation $D^t(h)$ is also a representation $D^t(g)$ of G. The group G is compact, by assumption, so a basis may be chosen such that $D^t(g)$ is completely reduced:

$$D^t(g)=\sum_s \oplus D^s(g),$$

where $D^s(g)$ is an irreducible unitary representation of G. Let us concentrate our attention on the set $|s\alpha\rangle$, which is the orthonormal basis for $D^s(g)$:

$$g|s\alpha\rangle=\sum_\beta D^s{}_{\beta\alpha}(g)|s\beta\rangle. \tag{2.4}$$

Multiply this equation on the left by $\mathcal{P}_0$:

$$\mathcal{P}_0 g|s\alpha\rangle=\mathcal{P}_0 g\mathcal{P}_0^{-1}\mathcal{P}_0|s\alpha\rangle =F\cdot g\mathcal{P}_0|s\alpha\rangle=\sum_\beta D^s{}_{\beta\alpha}(g)\mathcal{P}_0|s\beta\rangle,$$

and now replace g by $F^{-1}\cdot g$, where F^{-1} is the inverse automorphism to F. One finds

$$g\mathcal{P}_0|s\alpha\rangle=\sum_\beta D^s{}_{\beta\alpha}(F^{-1}\cdot g)\mathcal{P}_0|s\beta\rangle. \tag{2.5}$$

Consequently, the $\mathcal{P}_0|s\alpha\rangle$ form a basis for the representation $D^s(F^{-1}\cdot g)$, which is obviously unitary and irreducible because $D^s(g)$ is, and $F^{-1}G=G$. There are now two possibilities: Either $D^s(F^{-1}\cdot g)$ is equivalent to $D^s(g)$, which we write as $D^s(F^{-1}\cdot g)\sim D^s(g)$ and designate as case A_s, or $D^s(F^{-1}\cdot g)$ is inequivalent to $D^s(g)$, $D^s(F^{-1}\cdot g)$ not$\sim D^s(g)$, case B_s.

We consider first case A_s and let P_s^{-1} be the unitary[16] matrix which effects the equivalence,

$$D^s(F^{-1}\cdot g)=P_s^{-1}D^s(g)P_s. \tag{2.6}$$

This matrix, which depends on the automorphism F induced by $\mathcal{P}_0$ and on the representation s, is unique up to a phase factor since if P_s' also satisfies Eq. (2.6), then $P_s^{-1}P_s'$ commutes with all $D^s(g)$ and must be a constant times the identity. This constant is a phase factor because P_s and P_s' are unitary. We have further that

$$\begin{aligned} D^s(F^{-2}\cdot g)&=P_s^{-1}D^s(F^{-1}\cdot g)P_s=P_s^{-2}D^s(g)P_s^2\\ &=D^s(\mathcal{P}_0^{-2}g\mathcal{P}_0^2)=D^s(f_0^{-1}gf_0)\\ &=(D^s)^{-1}(f_0)D^s(g)D^s(f_0) \end{aligned} \tag{2.7}$$

by Eqs. (2.1) and (2.2). By the argument given above, P_s^2 and $D^s(f_0)$ differ at most by a phase factor. We now choose the arbitrary phase factor in P_s so that

$$P_s^2=D^s(f_0). \tag{2.8}$$

Only the sign of P_s remains undetermined. We will see that this sign may be regarded as the intrinsic parity of a multiplet.

Let us return to Eq. (2.5) and write

$$g\mathcal{P}_0|s\alpha\rangle=\sum_\beta [P_s^{-1}D^s(g)P_s]_{\beta\alpha}\mathcal{P}_0|s\beta\rangle$$

or

$$g[\mathcal{P}_0\sum_\gamma P_s^{-1}{}_{\gamma\alpha}|s\gamma\rangle] =\sum_\beta D^s{}_{\beta\alpha}(g)[\mathcal{P}_0\sum_\gamma P_s^{-1}{}_{\gamma\beta}|s\gamma\rangle]. \tag{2.9}$$

[16] Reference 6, Chap. 9. We will again and again make use of the simple theorems given in this chapter.

This shows that the orthonomal basis vectors

$$|s\alpha\rangle_1 \equiv \mathcal{P}_0 \sum_\gamma P_s^{-1}{}_{\gamma\alpha} |s\gamma\rangle \qquad (2.10)$$

also transform according to the irreducible representation $D^s(g)$ and hence[17]

$$\langle s\beta | s\alpha\rangle_1 = \lambda\delta_{\alpha\beta}, \qquad (2.11)$$

where λ is a constant independent of α, and $|\lambda|\leq 1$ since the vectors are normalized. We next show that with the choice of phase of Eq. (2.8), λ is real. From Eqs. (2.10) and (2.11) we have

$$\langle s\beta | \mathcal{P}_0 | s\alpha\rangle = \lambda P_{s\beta\alpha}.$$

The complex conjugate of this equation yields

$$\begin{aligned}
[\langle s\beta | \mathcal{P}_0 | s\alpha\rangle]^* &= \lambda^* P_s{}^*{}_{\beta\alpha} = \lambda^* P_s^{-1}{}_{\alpha\beta} \\
&= \langle s\alpha | \mathcal{P}_0{}^\dagger | s\beta\rangle \\
&= \langle s\alpha | \mathcal{P}_0{}^{-1} | s\beta\rangle \\
&= \langle s\alpha | \mathcal{P}_0\mathcal{P}_0{}^{-2} | s\beta\rangle = \langle s\alpha | \mathcal{P}_0 f_0{}^{-1} | s\beta\rangle \\
&= \sum_\gamma \langle s\alpha | \mathcal{P}_0 | s\gamma\rangle D^s{}_{\gamma\beta}(f_0{}^{-1}) \\
&= \sum_\gamma \langle s\alpha | \mathcal{P}_0 | s\gamma\rangle P_s^{-2}{}_{\gamma\beta}
\end{aligned}$$

by Eqs. (2.2) and (2.8). Hence

$$\langle s\alpha | \mathcal{P}_0 | s\beta\rangle = \lambda^* P_{s\alpha\beta},$$

and upon interchanging α and β we find

$$\langle s\beta | \mathcal{P}_0 | s\alpha\rangle = \lambda^* P_{s\beta\alpha}.$$

Consequently we find $\lambda=\lambda^*$, so λ is real and $-1\leq\lambda\leq 1$.

Consider the possibility $\lambda=\pm 1$. In this case, by Eqs. (2.10) and (2.11),

$$\mathcal{P}_0 \sum_\gamma P_s^{-1}{}_{\gamma\alpha} |s\gamma\rangle = \pm |s\alpha\rangle,$$

and hence,

$$\mathcal{P}_0 |s\alpha\rangle = \pm \sum_\beta P_{s\beta\alpha} |s\beta\rangle. \qquad (2.12)$$

The sign alternative $\pm$ could be absorbed into the indeterminacy of sign of P_s; however, we retain this notation as a reminder that either sign is possible. Equation (2.12) shows that if $\lambda=\pm 1$, the operator $\mathcal{P}_0$ acts within the multiplet $|s,\alpha\rangle$. Since the g already acts irreducibly there, we have obtained, if $\lambda=\pm 1$, an irreducible representation of H. It will be seen below that the case $-1<\lambda<1$ may be reduced, by a change of basis, to the present case and so in case A,

$$D^s(F^{-1}\cdot g) = P_s^{-1} D^s(g) P_s.$$

$D^t(h)$ is given by

$$D^t(g) = D^s(g), \quad D^t(\mathcal{P}) = \pm P_s. \qquad (2.13)$$

[17] Reference 6, p. 115. We will make frequent use of the theorem for irreducible representation s, s': $\langle s\alpha | s'\beta\rangle = \lambda\delta_{ss'}\delta_{\alpha\beta}$.

TABLE I. Irreducible representations $D^t(h)$ of the group $H=\{G,G\mathcal{P}_0\}$, where $\mathcal{P}_0{}^{-1}g\mathcal{P}_0=F^{-1}\cdot g$ and $\mathcal{P}_0{}^2=f_0\in G$. The elements g of G are linear operators as is $\mathcal{P}_0$.

Types 1 and 2 $D^s(F^{-1}\cdot g)=P_s^{-1}D^s(g)P_s$ $P_s{}^2=D^s(f_0)$	Type 3 $D^s(F^{-1}\cdot g)$ not$\sim D^s(g)$
$D^t(g)=D^s(g)$	$D^t(g)=\begin{pmatrix} D^s(g) & 0 \\ 0 & D^s(F^{-1}\cdot g)\end{pmatrix}$
$D^t(\mathcal{P}_0)=\pm P_s$	$D^t(\mathcal{P}_0)=\begin{pmatrix} 0 & D^s(f_0) \\ 1 & 0\end{pmatrix}$

The matrix P_s is fixed by Eqs. (2.6) and (2.8). This result is recorded in Table I, the two signs in Eq. (2.13) corresponding respectively to cases 1 and 2.

There remains to be examined the situation when $-1<\lambda<1$. Let us introduce a new set of basis vectors $|s\alpha\rangle_\perp$ defined by

$$|s\alpha\rangle_1 = \lambda|s\alpha\rangle + (1-\lambda^2)^{1/2}|s\alpha\rangle_\perp. \qquad (2.14)$$

Because $|s\alpha\rangle_1$ and $|s\alpha\rangle$ transform according to $D^s(g)$, so does $|s\alpha\rangle_\perp$. Using Eq. (2.11) one may easily verify that the set of vectors $\{|s\alpha\rangle, |s\alpha\rangle_\perp\}$ form an orthonormal set. From Eqs. (2.10) and (2.14) we have

$$\mathcal{P}_0|s\alpha\rangle = \sum_\gamma P_{s\gamma\alpha}[\lambda|s\gamma\rangle + (1-\lambda^2)^{1/2}|s\gamma\rangle_\perp]. \qquad (2.15)$$

Multiplying this equation by $\mathcal{P}_0$ and solving for $\mathcal{P}_0|s\alpha\rangle_\perp$ one obtains

$$\mathcal{P}_0|s\alpha\rangle_\perp = (1-\lambda^2)^{-1/2}\{\sum_\gamma P_s^{-1}{}_{\gamma\alpha}\mathcal{P}_0{}^2|s\alpha\rangle - \lambda\mathcal{P}_0|s\alpha\rangle\}.$$

By virtue of Eqs. (2.2) and (2.8), we have

$$\begin{aligned}
\mathcal{P}_0|s\alpha\rangle_\perp = (1-\lambda^2)^{-1/2}\{&\sum_\gamma P_{s\gamma\alpha}|s\alpha\rangle \\
&-\lambda\sum_\gamma P_{s\gamma\alpha}[\lambda|s\gamma\rangle + (1-\lambda^2)^{1/2}|s\alpha\rangle_\perp]\},
\end{aligned}$$

where Eq. (2.15) has been used for the last term, and hence,

$$\mathcal{P}_0|s\alpha\rangle_\perp = \sum_\gamma P_{s\gamma\alpha}[(1-\lambda^2)^{1/2}|s\alpha\rangle - \lambda|s\alpha\rangle_\perp]. \qquad (2.16)$$

Equations (2.15) and (2.16) constitute a representation of $\mathcal{P}_0$ in the orthonormal basis $\{|s\alpha\rangle, |s\alpha\rangle_\perp\}$,

$$D(\mathcal{P}_0) = P_s \otimes \begin{pmatrix} \lambda & (1-\lambda^2)^{1/2} \\ (1-\lambda^2)^{1/2} & -\lambda \end{pmatrix}. \qquad (2.17a)$$

On the same basis the g are represented by

$$D(g) = D^s(g) \otimes \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}. \qquad (2.17b)$$

This representation is easily reduced, the irreducible components being given by Eq. (2.13). This completes the proof that in case A_s all the irreducible representations are given by Eqs. (2.13).

The determination of the sign alternative in Eq. (2.13) is an intrinsic property of the multiplet corresponding to the irreducible representation $D^t(h)$ and is the generalization of the present case of the intrinsic parity of a particle. To see this one must verify that the different signs yield inequivalent representations. Let us assume the contrary. Then there exists a matrix U such that $D^s(g)=UD^s(g)U^\dagger$ and $UP_sU^\dagger=-P_s$. But, because $D^s(g)$ is irreducible, the first equation implies that U is a phase factor. The second equation then reads $P_s=-P_s$, which is a contradiction.

Let us turn to the second alternative, case B_s: $D^s(F^{-1}\cdot g)$ not $\sim D^s(g)$. By Eq. (2.5) the vectors $\mathcal{P}_0|s\alpha\rangle$ transform according to $D^s(F^{-1}\cdot g)$, which is inequivalent to $D^s(g)$, so,[17] $\langle s\alpha|\mathcal{P}_0|s\beta\rangle=0$, and hence the set $\{|s\alpha\rangle,\mathcal{P}_0|s\alpha\rangle\}$ forms an orthonormal basis. This basis yields the representation $D^t(k)$ for case B_s: $D^s(F^{-1}\cdot g)$, is not$\sim D^s(g)$,

$$D^t(g)=\begin{pmatrix} D^s(g) & 0 \\ 0 & D^s(F^{-1}\cdot g)\end{pmatrix};\qquad D^t(\mathcal{P}_0)=\begin{pmatrix} 0 & D^s(f_0) \\ 1 & 0\end{pmatrix}. \tag{2.18}$$

It must be verified that this representation is irreducible, for which it suffices to show that any matrix M which commutes with $D^t(h)$ is a constant matrix.[16] Let us write

$$M=\begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22}\end{pmatrix},$$

so from $MD^t(g)=D^t(g)M$, one has

$$\begin{pmatrix} M_{11}D^s(g) & M_{12}D^s(F^{-1}\cdot g) \\ M_{21}D^s(g) & M_{22}D^s(F^{-1}\cdot g)\end{pmatrix}=\begin{pmatrix} D^s(g)M_{11} & D^s(g)M_{12} \\ D^s(F^{-1}\cdot g)M_{21} & D^s(F^{-1}\cdot g)M_{22}\end{pmatrix}.$$

Hence, by Schur's lemma,[16] $M_{12}=M_{21}=0$, and $M_{11}=s_1$, $M_{22}=s_2$, where s_1 and s_2 are scalar matrices, or

$$M=\begin{pmatrix} s_1 & 0 \\ 0 & s_2\end{pmatrix}.$$

From the condition $MD^t(\mathcal{P}_0)=D^t(\mathcal{P}_0)M$, one easily finds $s_1=s_2$, so M is a scalar matrix and the representation (2.18) is irreducible, as asserted.

Lee and Wick, in Ref. 5, classify the automorphism induced in G by $\mathcal{P}_0$ according to whether it is inner (namely, there exists a $g_p\in G$ such that $F\cdot g=g_pgg_p^{-1}$), case A, or outer (there is no such g_p), case B. If case A holds, then every representation $D^s(g)$ of G will be of type A_s, $D^s(g)\sim D^s(F^{-1}\cdot g)$, and the equivalence transformation P_s that we have introduced will coincide with $D^s(g_p)$ to within a phase factor. If case B holds, then the representation $D^s(g)$ may be either in case A_s or case B_s. Equations (2.13) and (2.18) give all the irreducible representations of the group $H=\{G,G\mathcal{P}_0\}$. These are reproduced in Table I.

III. COREPRESENTATIONS WITH *CPT* CONJUGATION

This section follows the general method of the preceding section and, so as not to bore the reader, the justification of certain statements will now occasionally be omitted if it may be found after the corresponding statement there. We seek the irreducible corepresentations $D^u(k)$ of the group $K=\{k\}=\{H,H\Theta_0\}$ in a vector space where $h\in H$ acts unitarily and Θ_0 antiunitarily. Our immediate problem is not the most general problem of extension by an antilinear operator, because, by Eq. (1.10), (1.11), and (1.15),

$$\Theta_0 h\Theta_0^{-1}=h \tag{3.1}$$

and

$$\Theta_0^2=1. \tag{3.2}$$

However, it is just as simple to consider the general problem, and this will provide results that are useful later. So let us replace Θ_0 by A in Eqs. (3.1) and (3.2) and let

$$AhA^{-1}=E\cdot h, \tag{3.3}$$

$$A^2=e\in H, \tag{3.4}$$

where e is a fixed point of the automorphism E,

$$AeA^{-1}=E\cdot e=e. \tag{3.5}$$

We seek the irreducible corepresentations

$$D^u(k) \text{ of the group } K=\{k\}=\{H,HA\}.$$

Because A acts antiunitarily, $D^u(k)$ constitutes a set of unitary matrices which are a representation $D^u(h)$ of H and satisfy

$$D^u(hA)=D^u(h)D^u(A), \tag{3.6}$$

$$D^u(Ah)=D^u(A)D^{u*}(h), \tag{3.7}$$

$$D^u(A^2)=D^u(A)D^{u*}(A)=D^u(e). \tag{3.8}$$

We observe that it is sufficient to specify $D^u(h)$ and $D^u(A)$. Furthermore, if a unitary change of basis U is made such that

$$D^u(h)\to D'^u(h)=UD^u(h)U^\dagger, \tag{3.9a}$$

then, because A is antilinear, one easily verifies that

$$D^u(A)\to D'^u(A)=UD^u(A)U^T, \tag{3.9b}$$

so that the representative of an antilinear operator does not undergo a similarity transformation unless U is real. We call a corepresentation irreducible if it cannot be brought into diagonal block form by a transformation of the type (3.9).

Let the set $|t\alpha\rangle$ be the orthonormal basis for the irreducible representation $D^t(h)$ of H,

$$h|t\alpha\rangle=\sum_\beta D^t{}_{\beta\alpha}(h)|t\beta\rangle. \tag{3.10}$$

Multiplication on the left by A yields

$$Ah|t\alpha\rangle=(AhA^{-1})A|t\alpha\rangle=\sum_\beta D^{t*}{}_{\beta\alpha}(h)A|t\beta\rangle,$$

since A is antilinear. Upon replacing h by $A^{-1}hA=E^{-1}\cdot h\in H$, one finds

$$hA|t\alpha\rangle=\sum_\beta D^{t*}{}_{\beta\alpha}(E^{-1}\cdot h)A|t\beta\rangle. \tag{3.11}$$

Consequently, $A|t\alpha\rangle$ forms a basis for the unitary irreducible representation $D^{t*}(E^{-1}\cdot h)$ of H.

Let us now explore the possibility that $D^{t*}(E^{-1}\cdot h)\sim D^t(h)$. Let A_t be the unitary matrix, depending on the automorphism induced by A and the representation t, which effects the equivalence

$$D^{t*}(E^{-1}\cdot h)=A_t^{-1}D^t(h)A_t. \tag{3.12}$$

It is unique up to a phase factor. Upon writing $E^{-1}\cdot h$ for h and taking complex conjugates, we have

$$\begin{aligned}D^t(E^{-2}\cdot h)&=A_t^{*-1}D^{t*}(E^{-1}\cdot h)A_t^*\\&=A_t^{*-1}A_t^{-1}D^t(h)A_tA_t^*,\end{aligned}$$

and hence, using Eqs. (3.3) and (3.4),

$$\begin{aligned}D^t(e^{-1}he)&=(D^t)^{-1}(e)D^t(h)D^t(e)\\&=(A_tA_t^*)^{-1}D^t(h)(A_tA_t^*)\end{aligned}$$

or

$$D^t(h)D^t(e)(A_tA_t^*)^{-1}=D^t(e)(A_tA_t^*)^{-1}D^t(h)$$

for all h. Because $D^t(h)$ constitutes an irreducible representation, $D^t(e)(A_tA_t^*)^{-1}$ is a scalar matrix, and in fact a phase factor since the matrices are unitary:

$$D^t(e)(A_tA_t^*)^{-1}=\eta 1$$

or

$$D^t(e)=\eta A_tA_t^*;\quad |\eta|=1. \tag{3.13}$$

Let us now set $h=e$ in Eq. (3.12). By Eq. (3.5) we find

$$D^{t*}(e)=A_t^{-1}D^t(e)A_t,$$

which yields, when Eq. (3.13) is substituted into the left- and right-hand members,

$$\eta^*A_t^*A_t=A_t^{-1}(\eta A_tA_t^*)A_t=\eta A_t^*A_t,$$

and so η is real. Because it is also a phase factor, $\eta=\pm1$, and Eq. (3.13) becomes

$$A_tA_t^*=\pm D^t(e). \tag{3.14}$$

This equation is the analog of Eq. (2.8) for the unitary operator $\mathcal{P}_0$. The upper and lower signs in Eq. (3.14) will be designated as types 1 and 2, respectively.

Let us substitute Eq. (3.12) into Eq. (3.11):

$$hA|t\alpha\rangle=\sum_\beta [A_t^{-1}D^t(h)A_t]_{\beta\alpha}A|t\beta\rangle$$

or

$$h[\sum_\beta A_t^{-1}{}_{\beta\alpha}A|t\beta\rangle]=\sum_\gamma D^t{}_{\gamma\alpha}(h)[\sum_\beta A_t^{-1}{}_{\beta\gamma}A|t\beta\rangle]. \tag{3.15}$$

Consequently, the orthonormal basis vectors

$$|t\alpha\rangle_1\equiv\sum_\beta A_t^{-1}{}_{\beta\alpha}A|t\beta\rangle \tag{3.16}$$

also transform according to the irreducible representation $D^t(h)$ and hence[17]

$$\langle t\beta|t\alpha\rangle_1=\sum_\gamma \langle t\beta|A|t\gamma\rangle A_t^{-1}{}_{\gamma\alpha}=\lambda\delta_{\beta\alpha} \tag{3.17}$$

or

$$\langle t\beta|A|t\alpha\rangle=\lambda A_{t\beta\alpha}, \tag{3.18}$$

where λ is independent of α and $|\lambda|\leq1$. It will now be shown from the antiunitarity of A that λ vanishes for type 2.[18] We have

$$\langle t\beta|A|t\alpha\rangle=[\langle At\beta|A^2|t\alpha\rangle]^*$$

(where $\langle At\beta|$ is the bra corresponding to the ket$A|t\beta\rangle$),

$$=[\langle At\beta|e|t\alpha\rangle]^*=\sum_\gamma [\langle At\beta|t\gamma\rangle D^t{}_{\gamma\alpha}(e)]^*,$$

by Eq. (3.4), so

$$\begin{aligned}\lambda A_{t\beta\alpha}&=\sum_\gamma (D^t)^{-1}{}_{\alpha\gamma}(e)\langle t\gamma|A|t\beta\rangle\\&=\pm\sum_\gamma (A_tA_t^*)^{-1}{}_{\alpha\gamma}\lambda A_{t\gamma\beta}\end{aligned}$$

by Eqs. (3.14) and (3.18), or

$$\lambda A_{t\beta\alpha}=\pm\lambda(A_t^{*-1}A_t^{-1}A_t)_{\alpha\beta}=\pm\lambda A_t^{*-1}{}_{\alpha\beta}=\pm\lambda A_{t\beta\alpha}.$$

Consequently, λ vanishes when the lower sign holds, which is type 2.

We next concentrate our attention in detail on type 1, $A_tA_t^*=D^t(e)$. The phase of A_t has been arbitrary until now, and it is convenient to choose it such that λ in Eqs. (3.17) and (3.18) is a real non-negative number, $0\leq\lambda\leq1$. If $\lambda=1$, then by Eq. (3.17), $|t\alpha\rangle_1=|t\alpha\rangle$, and hence by Eq. (3.16),

$$A|t\alpha\rangle=\sum_\beta A_{t\beta\alpha}|t\beta\rangle, \tag{3.19}$$

and consequently A acts within the multiplet. Since the h acts irreducibly there, we have obtained, if $\lambda=1$, an irreducible representation of K. It will be seen below that the representation obtained when $\lambda<1$ may

[18] It is at this point that our method departs from Ref. 6.

TABLE II. Irreducible corepresentations $D^u(k)$ of the group $K=\{k\}=\{H,H\Theta_0\}$, where the "CPT" operator Θ_0 is an antilinear operator satisfying $\Theta_0^{-1}h\Theta_0=h$, $\Theta_0^2=1$, and $H=\{G,G\mathcal{P}_0\}$ is the group of linear operators obtained by extending the internal symmetry group G by the parity operator $\mathcal{P}_0$. This table also gives the irreducible corepresentations of the group $\{G,G\Theta_0\}$ if $D^t(h)$ and Θ_t are replaced by $D^s(g)$ and Θ_s.

Type 1 $D^{t*}(h)=\Theta_t^{-1}D^t(h)\Theta_t$ $\Theta_t\Theta_t^*=1$	Type 2 $D^{t*}(h)=\Theta_t^{-1}D^t(h)\Theta_t$ $\Theta_t\Theta_t^*=-1$	Type 3 $D^{t*}(h)$ not$\sim D^t(h)$
$D^u(h)=D^t(h)$	$D^u(h)=D^t(h)\otimes\begin{pmatrix}1&0\\0&1\end{pmatrix}$	$D^u(h)=\begin{pmatrix}D^t(h)&0\\0&D^{t*}(h)\end{pmatrix}$
$D^u(\Theta_0)=\Theta_t$	$D^u(\Theta_0)=\Theta_t\otimes\begin{pmatrix}0&-1\\1&0\end{pmatrix}$	$D^u(\Theta_0)=\begin{pmatrix}0&1\\1&0\end{pmatrix}$

be reduced to the one obtained here, and so for type 1,

$$D^{t*}(E^{-1}\cdot h)=A_t^{-1}D^t(h)A_t \quad\text{and}\quad A_tA_t^*=D^t(e)\,, \tag{3.20}$$

$D^u(k)$ is given by

$$D^u(k)=D^t(h)\,,\quad D^u(A)=A_t. \tag{3.21}$$

The phase of $D^u(A)=A_t$ may be changed by multiplying all basis vectors by a common phase factor.

To deal with the possibility $0\leq\lambda<1$, we introduce a new set of basis vectors $|t\alpha\rangle_1$ defined by

$$|t\alpha\rangle_1=\lambda|t\alpha\rangle+(1-\lambda^2)^{1/2}|t\alpha\rangle_\perp. \tag{3.22}$$

Because of Eq. (3.17), the set of vectors $\{|t\alpha\rangle,|t\alpha\rangle_\perp\}$ forms an orthonormal basis. By Eq. (3.16) we have

$$A|t\alpha\rangle=\sum_\beta A_{t\beta\alpha}[\lambda|t\beta\rangle+(1-\lambda^2)^{1/2}|t\beta\rangle_\perp]. \tag{3.23}$$

Multiplying this equation by A, and recalling that $A^2=e$ and that λ is real, one obtains, using Eq. (3.20),

$$\begin{aligned}e|t\alpha\rangle&=\sum_\beta D^t{}_{\beta\alpha}(e)|t\beta\rangle=\sum_\beta|t\beta\rangle(A_tA_t^*)_{\beta\alpha}\\&=\sum_\beta A_t{}^*{}_{\beta\alpha}[\lambda A|t\beta\rangle+(1-\lambda^2)^{1/2}A|t\beta\rangle_\perp],\end{aligned}$$

or

$$\begin{aligned}\sum_\beta|t\beta\rangle A_{t\beta\alpha}&=\lambda A|t\alpha\rangle+(1-\lambda^2)^{1/2}A|t\alpha\rangle_\perp\\&=\lambda\sum_\beta A_{t\beta\alpha}[\lambda|t\beta\rangle+(1-\lambda^2)^{1/2}|t\beta\rangle_\perp]\\&\qquad+(1-\lambda^2)^{1/2}A|t\alpha\rangle_\perp,\end{aligned}$$

in which Eq. (3.23) has been used. Solving for $A|t\alpha\rangle_\perp$, one obtains

$$A|t\alpha\rangle_\perp=\sum_\beta A_{t\beta\alpha}[(1-\lambda^2)^{1/2}|t\beta\rangle-\lambda|t\beta\rangle_\perp]. \tag{3.24}$$

Equations (3.23) and (3.24) yield the representative of A,

$$D(A)=A_t\otimes\begin{pmatrix}\lambda&(1-\lambda^2)^{1/2}\\(1-\lambda^2)^{1/2}&-\lambda\end{pmatrix}, \tag{3.25a}$$

in the basis $\{|t\alpha\rangle,|t\alpha\rangle_\perp\}$. In this basis the h are represented by

$$D(h)=D^t(h)\otimes\begin{pmatrix}1&0\\0&1\end{pmatrix}. \tag{3.25b}$$

This corepresentation is easily reduced by a *real* orthogonal transformation, whereby the representatives of the unitary and antiunitary operators transform in the same way, according to Eq. (3.9). The irreducible components of the corepresentation are then found to be of the form (3.21). This completes the discussion of type 1.

We now return to type 2, $A_tA_t^*=-D^t(e)$, for which, as we have seen, the right-hand side of Eq. (3.17) vanishes. Consequently the vectors

$$\{|t\alpha\rangle,\ |t\alpha\rangle_1=\sum_\beta A_t{}^{-1}{}_{\beta\alpha}A|t\beta\rangle\}$$

form an orthonormal set. We have

$$A|t\alpha\rangle=\sum_\beta A_{t\beta\alpha}|t\beta\rangle_1 \tag{3.26}$$

and also

$$\begin{aligned}A|t\alpha\rangle_1&=\sum_\beta A_t{}^{-1*}{}_{\beta\alpha}A^2|t\beta\rangle=\sum_\beta A_t{}^{-1*}{}_{\beta\alpha}e|t\beta\rangle\\&=\sum_{\beta\gamma}D^t{}_{\gamma\beta}(e)A_t{}^{-1*}{}_{\beta\alpha}|t\gamma\rangle=-\sum_\gamma(A_tA_t^*A_t^{*-1})_{\gamma\alpha}|t\gamma\rangle\end{aligned}$$

or

$$A|t\alpha\rangle_1=-\sum_\beta A_{t\beta\alpha}|t\beta\rangle. \tag{3.27}$$

Consequently, we find for type 2,

$$D^{t*}(E^{-1}\cdot h)=A_t^{-1}D^t(h)A_t \text{ and } A_tA_t^*=-D^t(e)\,, \tag{3.28}$$

that $D^u(k)$ is given by

$$D^u(h)=D^t(h)\otimes\begin{pmatrix}1&\\&1\end{pmatrix}, \tag{3.29a}$$

$$D^u(A)=A_t\otimes\begin{pmatrix}&-1\\1&\end{pmatrix}. \tag{3.29b}$$

It must be verified that this corepresentation is irreducible. The representation $D^u(h)$ [Eq. (3.29a)] of the subgroup H is already completely reduced. We may require that this representation of H remain invariant when attempting to reduce the full corepresentation, according to Eq. (3.9), by a transformation U. Using an argument similar to that which follows

TABLE III. Irreducible corepresentations $D^w(l)$ of the group $L=\{l\}=\{G,G\mathcal{T}_0\}$, with $\mathcal{T}_0^{-1}g\mathcal{T}_0=F^{-1}\cdot g$, $\mathcal{T}_0^2=f_0$. The elements g of G are linear operators; $\mathcal{T}_0$ is antilinear.

Type 1 $D^{s*}(F^{-1}\cdot g)=T_s^{-1}D^s(g)T_s$ $T_sT_s^*=D^s(f_0)$	Type 2 $D^{s*}(F^{-1}\cdot g)=T_s^{-1}D^s(g)T_s$ $T_sT_s^*=-D^s(f_0)$	Type 3 $D^{s*}(F^{-1}\cdot g)$ not$\sim D^s(g)$
$D^w(g)=D^s(g)$	$D^w(g)=D^s(g)\otimes\begin{pmatrix}1 & \\ & 1\end{pmatrix}$	$D^w(g)=\begin{pmatrix}D^s(g) & \\ & D^{s*}(F^{-1}\cdot g)\end{pmatrix}$
$D^w(\mathcal{T}_0)=T_s$	$D^w(\mathcal{T}_0)=T_s\otimes\begin{pmatrix} & -1\\ 1 & \end{pmatrix}$	$D^w(\mathcal{T}_0)=\begin{pmatrix} & D^s(f_0)\\ 1 & \end{pmatrix}$

Eq. (2.25), one finds that U must be of the form

$$U=1\otimes\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}.$$

It is unitary and may also be required to be unimodular, since if U is multiplied by a phase factor, this does not aid in the attempted reduction, but only multiplies $D^u(A)$ by a phase factor. Consequently it suffices to consider transformations of the form

$$U=1\otimes\begin{pmatrix}\alpha & \beta\\ -\beta^* & \alpha^*\end{pmatrix},\quad |\alpha|^2+|\beta^2|=1. \tag{3.30}$$

However, as one may easily verify, $D^u(A)$, given by Eq. (3.29b), remains invariant when transformed with this U according to Eq. (3.9b). Consequently the corepresentation (3.29) is irreducible. It is interesting to note the curious phenomenon that the basis vectors of the corepresentation (3.29) may be subjected to the transformation (3.30) and the corepresentation remains invariant. The particle multiplet which forms the representation space of $D^u(k)$ has twice the dimensionality of that of the multiplet corresponding to $D^t(h)$, even though there is no quantum number which distinguishes between $|t\alpha\rangle$ and $|t\alpha\rangle_1=\sum_\beta A_t^{-1}{}_{\beta\alpha}A\,|t\beta\rangle$.

Finally we consider the remaining possibility, $D^{t*}(E^{-1}\cdot h)$ not$\sim D^t(h)$, which we call type 3. In this case, because the vectors of the sets $\{|t\alpha\rangle\}$ and $\{A\,|t\alpha\rangle\}$ transform, by virtue of Eq. (3.11), according to inequivalent representations, they are orthogonal and consequently form an orthonormal set when taken together. From $A\,|t\alpha\rangle=(A\,|t\alpha\rangle)$ and

$$A\,(A\,|t\alpha\rangle)=A^2|t\alpha\rangle=e\,|t\alpha\rangle=\sum_\beta D^t{}_{\beta\alpha}(e)\,|t\beta\rangle,$$

in which Eq. (3.5) has been used, we find that the corepresentation $D^u(k)$ for type 3,

$$D^{t*}(E^{-1}\cdot h)\ \text{not}\sim D^t(h) \tag{3.31}$$

is given by

$$D^u(h)=\begin{pmatrix}D^t(h) & \\ & D^{t*}(E^{-1}\cdot h)\end{pmatrix}, \tag{3.32a}$$

$$D^u(A)=\begin{pmatrix} & D^t(e)\\ 1 & \end{pmatrix}. \tag{3.32b}$$

One may easily verify, using the arguments given below Eq. (3.29), that this corepresentation is irreducible.

The three types of irreducible corepresentations of the group obtained by extending any group of linear operators H by any antilinear operator A are given respectively by Eqs. (3.21), (3.29), and (3.32). We now apply this result by identifying the abstract group H and operator A with some physical groups and operators. Let H be the group of all internal symmetries extended by parity, $H=\{G,G\mathcal{P}_0\}$, and let A be the CPT operator Θ_0. Then the desired irreducible corepresentation $D^u(k)$ of $K=\{H,H\Theta_0\}$ in terms of the irreducible representations $D^t(h)$ of H are obtained from these equations by the substitutions

$$\begin{aligned}A\to\Theta_0,\quad A_t\to\Theta_t,\quad E^{-1}\cdot h\to h,\\ e\to 1,\quad D^t(e)\to 1,\end{aligned} \tag{3.33}$$

by virtue of Eqs. (3.1) and (3.2). Equations (3.12) and (3.14) become, respectively,

$$D^{t*}(h)=\Theta_t^{-1}D^t(h)\Theta_t \tag{3.34}$$

and

$$\Theta_t\Theta_t^*=\pm 1. \tag{3.35}$$

Because Θ_t is unitary, the upper or lower signs imply that Θ_t is symmetric or antisymmetric. The result of the substitutions (3.33) is recorded in Table II.

As a second application of the theory of corepresentations, Table II may also be used for the irreducible corepresentations of the group $\{G,G\Theta_0\}$ obtained by extending the internal symmetry group G by Θ_0. This group will be the full symmetry group (apart from $P_+{}^\uparrow$) in theories without parity invariance; otherwise it is a particular subgroup of $K=\{G,G\mathcal{P}_0,G\Theta_0,G\mathcal{P}_0\Theta_0\}$. To be consistent with the notation used elsewhere in the present paper, one should replace

$$D^t(h)\ \text{and}\ \Theta_t\ \text{by}\ D^s(g)\ \text{and}\ \Theta_s \tag{3.36}$$

for the corepresentations of $\{G,G\Theta_0\}$. Equations (3.34) and (3.35) then become

$$D^{s*}(g)=\Theta_s^{-1}D^s(g)\Theta_s, \tag{3.37}$$

$$\Theta_s\Theta_s^*=\pm 1, \tag{3.38}$$

and Θ_s is symmetric or antisymmetric. A canonical form for Θ_s is given in Sec. V. We note that the three types of corepresentation of Table II correspond respectively to $D^t(h)$ or $D^s(g)$ being potentially real, pseudoreal, or complex,[19] the matrix Θ_t or Θ_s being respectively symmetric, antisymmetric, or nonexistent in the three cases.

A final application is to let $H=G$, the group of internal symmetries, and let $A=\mathcal{T}_0$, the "time-reversal" operator. The irreducible corepresentations $D^v(l)$ of the group $L=\{G,G\mathcal{T}_0\}$ in terms of the irreducible representations $D^s(g)$ of G, are obtained from Eqs. (3.21), (3.29), and (3.32) by the substitutions, justified by Eqs. (1.21),

$$A\to\mathcal{T}_0,\quad A_t\to T_s,\quad h\to g,\quad D^t(h)\to D^s(g),\quad e\to f_0,\quad E^{-1}\cdot h\to F^{-1}\cdot g \tag{3.39}$$

as effected in Table III.

IV. EXTENSION OF INTERNAL SYMMETRY BY PARITY AND "CPT"

In the last section we found and classified the irreducible corepresentations $D^u(k)$, given in Table III, of the group $K=\{k\}=\{H,H\Theta_0\}$ obtained as the extension of H by Θ_0 satisfying

$$\Theta_0 h\Theta_0^{-1}=h\,, \tag{4.1}$$

$$\Theta_0^2=1\,. \tag{4.2}$$

The type of corepresentation $D^u(k)$ depends on the existence and symmetry or antisymmetry of a matrix Θ_t, satisfying, for each irreducible representation $D^t(h)$ of H,

$$D^{t*}(h)=\Theta_t^{-1}D^t(h)\Theta_t\,. \tag{4.3}$$

We now apply this criterion to the representations $D^t(h)$, given in Table I, of the group $H=\{G,G\mathcal{P}_0\}$. Equations (4.1) and (4.3) are respectively equivalent to

$$\Theta_0^{-1}g\Theta_0=g\,, \tag{4.4}$$

$$\Theta_0^{-1}\mathcal{P}_0\Theta_0=\mathcal{P}_0\,, \tag{4.5}$$

and to

$$D^{t*}(g)=\Theta_t^{-1}D^t(g)\Theta_t\,, \tag{4.6a}$$

$$D^{t*}(\mathcal{P}_0)=\Theta_t^{-1}D^t(\mathcal{P}_0)\Theta_t\,. \tag{4.6b}$$

We will express the corepresentations $D^u(k)$ in terms of the irreducible representations $D^s(g)$ of G.

We first consider representations $D^t(h)$ of types 1 and 2 for which

$$D^s(F^{-1}\cdot g)=P_s^{-1}D^s(g)P_s\,, \tag{4.7}$$

so that $D^t(h)$ is given by

$$D^t(g)=D^s(g)\,,\quad D^t(\mathcal{P}_0)=\pm P_s\,, \tag{4.8}$$

where

$$F^{-1}\cdot g=\mathcal{P}_0^{-1}g\mathcal{P}_0\,,\quad \mathcal{P}_0^2=f_0\in G\,, \tag{4.9}$$

and

$$P_s^2=D^s(f_0)\,. \tag{4.10}$$

[19] See, for example, Ref. 6, pp. 285–288. Usual convention designates our Θ_s by C^{-1}.

We must find out under what conditions Eq. (4.6) holds for these types of representation. From Eqs. (4.6a) and (4.8) we find that Θ_t must satisfy

$$D^{s*}(g)=\Theta_t^{-1}D^s(g)\Theta_t\,. \tag{4.11}$$

This is recognized as the criterion for the type of corepresentation of the group $\{G,G\Theta_0\}$, obtained by extending the internal symmetry group by Θ_0. These corepresentations are obtained from Table II by the substitution $H\to G$, $h\to g$, $D^t(h)\to D^s(g)$, $\Theta_t\to\Theta_s$. We observe that if the representation $D^s(g)$ is complex, then no Θ_s exists, and hence no Θ_t exists. The form of the representation $D^u(k)$ is then given by substituting Eq. (4.8) for $D^t(h)$ into the entry for case 3 in Table II:

$$D^u(g)=\begin{pmatrix}D^s(g) & \\ & D^{s*}(g)\end{pmatrix}, \tag{4.12a}$$

$$D^u(\mathcal{P}_0)=\pm\begin{pmatrix}P_s & \\ & P_s{}^*\end{pmatrix}, \tag{4.12b}$$

$$D^u(\Theta_0)=\begin{pmatrix} & 1\\ 1 & \end{pmatrix}, \tag{4.12c}$$

which also gives

$$D^u(\mathcal{T}_0)=D^u(\mathcal{P}_0)D^u(\Theta_0)=\pm\begin{pmatrix} & P_s\\ P_s{}^* & \end{pmatrix}. \tag{4.12d}$$

One may easily verify that the two possible sign determinations in Eq. (4.12b) correspond to inequivalent representations, by combining the argument which follows Eq. (2.17) with the proof that the representation (3.32) is irreducible. These two corepresentations appear as entries 9 and 10 in Table IV.

Let us now consider the alternative possibility, namely that $D^s(g)$ is potentially real or pseudoreal,[19] so that a unique solution (up to a phase factor) to Eq. (4.11) exists, and hence if there is a Θ_t, it is given by

$$\Theta_t=\Theta_s\,.$$

The remaining condition that Θ_t must satisfy is Eq. (4.6b), and so, by Eq. (4.8), Θ_t will exist, and be given by Θ_s, if Θ_s satisfies

$$P_s{}^*=\Theta_s^{-1}P_s\Theta_s\,. \tag{4.13}$$

This condition may be expressed in another way. Let us substitute $F^{-1}\cdot g$ for g in Eq. (4.11), then, with Θ_s written for Θ_t, we have

$$D^{s*}(F^{-1}\cdot g)=\Theta_s^{-1}D^s(F^{-1}\cdot g)\Theta_s=\Theta_s^{-1}P_s^{-1}D^s(g)P_s\Theta_s\,,$$

where use has been made of Eq. (4.7). If we set

$$T_s\equiv P_s\Theta_s\,, \tag{4.14}$$

TABLE IV. Irreducible corepresentations of the group $K=\{k\}=\{G,G\mathcal{P}_0,G\Theta_0,G\mathcal{T}_0\}$, where $\mathcal{T}_0=\mathcal{P}_0\Theta_0$, $\mathcal{P}_0^{-1}g\mathcal{P}_0=F^{-1}\cdot g$, $\mathcal{P}_0^2=f_0\in G$, $\Theta_0^{-1}g\Theta_0=g$, $\Theta_0^2=1$, $\Theta_0^{-1}\mathcal{P}_0\Theta_0=\mathcal{P}_0$; $g\in G$ and $\mathcal{P}_0$ are linear operators, Θ_0 is antilinear.

$D^s(F^{-1}\cdot g)=P_s^{-1}D^s(g)P_s$ $D^{s*}(g)=\Theta_s^{-1}D^s(g)\Theta_s$ $P_s^2=D^s(f_0)$ $T_s=P_s\Theta_s$		$D^{s*}(g)=\Theta_s^{-1}D^s(g)\Theta_s$ $D^s(F^{-1}\cdot g)$ not$\sim D^s(g)$
$\Theta_s\Theta_s^*=1$ $T_sT_s^*=D^s(f_0)$	$\Theta_s\Theta_s^*=1$ $T_sT_s^*=-D^s(f_0)$	$\Theta_s\Theta_s^*=1$
$D^u(g)=D^s(g)$	$D^u(g)=D^s(g)\otimes\begin{pmatrix}1&\\&1\end{pmatrix}$	$D^u(g)=\begin{pmatrix}D^s(g)&\\&D^s(F^{-1}\cdot g)\end{pmatrix}$
$D^u(\mathcal{P}_0)=\pm P_s$	$D^u(\mathcal{P}_0)=P_s\otimes\begin{pmatrix}1&\\&-1\end{pmatrix}$	$D^u(\mathcal{P}_0)=\begin{pmatrix}&D^s(f_0)\\1&\end{pmatrix}$
$D^u(\Theta_0)=\Theta_s$	$D^u(\Theta_0)=\Theta_s\otimes\begin{pmatrix}&1\\1&\end{pmatrix}$	$D^u(\Theta_0)=\begin{pmatrix}\Theta_s&\\&\Theta_s\end{pmatrix}$
Types 1, 2	Type 3	Type 4
$\Theta_s\Theta_s^*=-1$ $T_sT_s^*=-D^s(f_0)$	$\Theta_s\Theta_s^*=-1$ $T_sT_s^*=D^s(f_0)$	$\Theta_s\Theta_s^*=-1$
$D^u(g)=D^s(g)\otimes\begin{pmatrix}1&\\&1\end{pmatrix}$	$D^u(g)=D^s(g)\otimes\begin{pmatrix}1&\\&1\end{pmatrix}$	$D^u(g)=\begin{pmatrix}D^s(g)&\\&D^s(F^{-1}\cdot g)\end{pmatrix}\otimes\begin{pmatrix}1&\\&1\end{pmatrix}$
$D^u(\mathcal{P}_0)=\pm P_s\otimes\begin{pmatrix}1&\\&1\end{pmatrix}$	$D^u(\mathcal{P}_0)=P_s\otimes\begin{pmatrix}1&\\&-1\end{pmatrix}$	$D^u(\mathcal{P}_0)=\begin{pmatrix}&D^s(f_0)\\1&\end{pmatrix}\otimes\begin{pmatrix}1&\\&1\end{pmatrix}$
$D^u(\Theta_0)=\Theta_s\otimes\begin{pmatrix}&-1\\1&\end{pmatrix}$	$D^u(\Theta_0)=\Theta_s\otimes\begin{pmatrix}&-1\\1&\end{pmatrix}$	$D^u(\Theta_0)=\begin{pmatrix}\Theta_s&\\&\Theta_s\end{pmatrix}\otimes\begin{pmatrix}&-1\\1&\end{pmatrix}$
Types 5, 6	Type 7	Type 8

$D^s(F^{-1}\cdot g)=P_s^{-1}D^s(g)P_s$ $D^{s*}(g)$ not$\sim D^s(g)$ $P_s^2=D^s(f_0)$	$D^{s*}(F^{-1}\cdot g)=T_s^{-1}D^s(g)T_s$ $D^s(F^{-1}\cdot g)$ not$\sim D^s(g)$	
	$T_sT_s^*=D^s(f_0)$	$T_sT_s^*=-D^s(f_0)$
$D^u(g)=\begin{pmatrix}D^s(g)&\\&D^{s*}(g)\end{pmatrix}$	$D^u(g)=\begin{pmatrix}D^s(g)&\\&D^{s*}(g)\end{pmatrix}$	$D^u(g)=\begin{pmatrix}D^s(g)&\\&D^{s*}(g)\end{pmatrix}\otimes\begin{pmatrix}1&\\&1\end{pmatrix}$
$D^u(\mathcal{P}_0)=\pm\begin{pmatrix}P_s&\\&P_s^*\end{pmatrix}$	$D^u(\mathcal{P}_0)=\begin{pmatrix}&T_s\\T_s^*&\end{pmatrix}$	$D^u(\mathcal{P}_0)=\begin{pmatrix}&T_s\\-T_s^*&\end{pmatrix}\otimes\begin{pmatrix}1&\\&-1\end{pmatrix}$
$D^u(\Theta_0)=\begin{pmatrix}&1\\1&\end{pmatrix}$	$D^u(\Theta_0)=\begin{pmatrix}&1\\1&\end{pmatrix}$	$D^u(\Theta_0)=\begin{pmatrix}&1\\1&\end{pmatrix}\otimes\begin{pmatrix}&1\\1&\end{pmatrix}$
Types 9, 10	Type 11	Type 12

$D^s(F^{-1}\cdot g)$ not$\sim D^{s*}(g)$ not$\sim D^s(g)$
$D^u(g)=\begin{pmatrix}D^s(g)&&&\\&D^s(F^{-1}\cdot g)&&\\&&D^{s*}(g)&\\&&&D^{s*}(F^{-1}\cdot g)\end{pmatrix}$
$D^u(\mathcal{P}_0)=\begin{pmatrix}&D^s(f_0)&&\\1&&&\\&&&D^{s*}(f_0)\\&&1&\end{pmatrix}$
$D^u(\Theta_0)=\begin{pmatrix}&&1&\\&&&1\\1&&&\\&1&&\end{pmatrix}$
Type 13

then by reference to Table III we recognize that the representation $D^s(g)$ is of type 1 or 2 with respect to extension by $\mathcal{T}_0$. This could have been recognized by direct inspection of Eqs. (4.7) and (4.11) because $\mathcal{T}_0=\mathcal{P}_0\Theta_0$ and equivalence relations are transitive. Let us multiply Eq. (4.13) on the left by $P_s\Theta_s$ and on the right by Θ_s^*. This gives

$$P_s\Theta_sP_s^*\Theta_s^*=P_s^2\Theta_s\Theta_s^*,$$

and hence, by Eqs. (4.10) and (4.14), the condition

(4.13) for the existence of Θ_t is the same as the condition

$$T_sT_s^*=D^s(f_0)\Theta_s\Theta_s^*. \qquad (4.15)$$

Inspection of Table II [with $D^t(h)\to D^s(g)$ and $\Theta_t\to\Theta_s$] shows that $D^s(g)$ is of type 1 or 2 with respect to extension by Θ_0, according as $\Theta_s\Theta_s^*=1$ or $\Theta_s\Theta_s^*=-1$, and likewise, by Table III, it is of type 1 or 2 with respect to $\mathcal{T}_0$ as $T_sT_s^*=D^s(f_0)$ or $T_sT_s^*=-D^s(f_0)$. Consequently the condition (4.13) or (4.15) for the existence of a Θ_t is the condition that $D^s(g)$ be of the same type with respect to extension by Θ_0 and by $\mathcal{T}_0$. Let us now consider the various possibilities:

$$\text{(a)}\quad \Theta_s\Theta_s^*=1 \quad\text{and}\quad T_sT_s^*=D^s(f_0). \qquad (4.16)$$

In this case Eq. (4.15) is satisfied, so Θ_t exists and is given by Θ_s, which is symmetric. Hence, by Table II, type 1 and by Eq. (4.8), the representation $D^u(k)$ is given by

$$D^u(g)=D^s(g)\,, \qquad (4.17\text{a})$$

$$D^u(\mathcal{P}_0)=\pm P_s\,, \qquad (4.17\text{b})$$

$$D^u(\Theta_0)=\Theta_s\,, \qquad (4.17\text{c})$$

and hence,

$$D^u(\mathcal{T}_0)=D^u(\mathcal{P}_0)D^u(\Theta_0)=\pm P_s\Theta_s=\pm T_s. \qquad (4.17\text{d})$$

The opposite signs yield two inequivalent representations which are recorded as entries 1 and 2 in Table IV.

$$\text{(b)}\quad \Theta_s\Theta_s^*=-1 \quad\text{and}\quad T_sT_s^*=-D^s(f_0). \qquad (4.18)$$

Again Eq. (4.15) is satisfied and Θ_t is given by Θ_s, which is antisymmetric. Hence by Eqs. (4.3), (4.8), and Table II, type 2, the corepresentation $D^u(k)$ is given by

$$D^u(g)=D^s(g)\otimes\begin{pmatrix}1 & \\ & 1\end{pmatrix}, \qquad (4.19\text{a})$$

$$D^u(\mathcal{P}_0)=\pm P_s\otimes\begin{pmatrix}1 & \\ & 1\end{pmatrix}, \qquad (4.19\text{b})$$

$$D^u(\Theta_0)=\Theta_s\otimes\begin{pmatrix} & -1\\ 1 & \end{pmatrix}, \qquad (4.19\text{c})$$

$$D^u(\mathcal{T}_0)=D^u(\mathcal{P}_0)D^u(\Theta_0)=\pm T_s\otimes\begin{pmatrix} & -1\\ 1 & \end{pmatrix}; \qquad (4.19\text{d})$$

$$T_s=P_s\Theta_s.$$

Again the opposite signs yield inequivalent corepresentations which appear as entries 5 and 6 in Table IV.

$$\text{(c)}\quad \Theta_s\Theta_s^*=1 \quad\text{and}\quad T_sT_s^*=-D^s(f_0), \qquad (4.20)$$

or

$$\text{(d)}\quad \Theta_s\Theta_s^*=-1 \quad\text{and}\quad T_sT_s^*=D^s(f_0). \qquad (4.21)$$

Equation (4.15) is not satisfied and consequently no Θ_t exists so that the corepresentation $D^u(k)$ is of type 3 in Table II. From Eq. (4.8) we find for $D^u(k)$

$$D^u(g)=\begin{pmatrix}D^s(g) & \\ & D^{s*}(g)\end{pmatrix}, \qquad (4.22\text{a})$$

$$D^u(\mathcal{P}_0)=\pm\begin{pmatrix}P_s & \\ & P_s^*\end{pmatrix}, \qquad (4.22\text{b})$$

$$D^u(\Theta_0)=\begin{pmatrix} & 1\\ 1 & \end{pmatrix}. \qquad (4.22\text{c})$$

This corepresentation may be expressed in another form. From Eq. (4.20) we have instead of Eq. (4.15)

$$T_sT_s^*=-D^s(f_0)\Theta_s\Theta_s^*, \qquad (4.23)$$

or, by Eqs. (4.10) and (4.14),

$$P_s\Theta_sP_s^*\Theta_s^*=-P_s^2\Theta_s\Theta_s^*,$$

which reduces to

$$P_s^*=-\Theta_s^{-1}P_s\Theta_s. \qquad (4.24)$$

If this equation and Eq. (4.11) are substituted into Eqs. (4.22a) and (4.22b), they become

$$D^u(g)=\begin{pmatrix}D^s(g) & \\ & \Theta_s^{-1}D^s(g)\Theta_s\end{pmatrix},$$

$$D^u(\mathcal{P}_0)=\pm\begin{pmatrix}P_s & \\ & -\Theta_s^{-1}P_s\Theta_s\end{pmatrix}.$$

Now let the corepresentation $D^u(k)$ be transformed according to Eq. (3.9) with

$$U=\begin{pmatrix}1 & \\ & \Theta_s\end{pmatrix}. \qquad (4.25)$$

Then we find

$$D^u(g)\to D^{u\prime}(g)=D^s(g)\otimes\begin{pmatrix}1 & \\ & 1\end{pmatrix}, \qquad (4.26\text{a})$$

$$D^u(\mathcal{P}_0)\to D^{u\prime}(\mathcal{P}_0)=\pm P_s\otimes\begin{pmatrix}1 & \\ & -1\end{pmatrix} \qquad (4.26\text{b})$$

and

$$D^u(\Theta_0)\to D^{u\prime}(\Theta_0)=\begin{pmatrix}1 & \\ & \Theta_s\end{pmatrix}\begin{pmatrix} & 1\\ 1 & \end{pmatrix}\begin{pmatrix}1 & \\ & \theta_s{}^T\end{pmatrix}, \qquad (4.26\text{c})$$

$$D^{u\prime}(\Theta_0)=\begin{pmatrix} & \theta_s{}^T\\ \Theta_s & \end{pmatrix}=\Theta_s\otimes\begin{pmatrix} & \pm1\\ 1 & \end{pmatrix}. \qquad (4.26\text{d})$$

In the last line the upper or lower sign holds according as Θ_s is (c) symmetric or (d) antisymmetric, i.e., according as $D^s(g)$ is (c) real or (d) pseudoreal. In these two cases further transformation by

$$U=1\otimes\begin{pmatrix} & \pm 1 \\ 1 & \end{pmatrix} \tag{4.27}$$

interchanges the two signs in Eq. (4.26b) and leaves all other equations invariant. Consequently the two opposite signs in Eq. (4.26b) yield equivalent corepresentations. Since we are interested only in equivalence classes of corepresentations, we may suppress the lower sign in Eq. (4.26b). The two cases considered here, (c) and (d), appear as entries 3 and 7 respectively in Table IV. Our analysis of types 1 and 2 of $D^s(g)$, $D^s(F^{-1}\cdot g)\sim D^s(g)$, is now complete.

The analysis for type 3,

$$D^s(F^{-1}\cdot g)\ \text{not}\!\sim D^s(g)\,, \tag{4.28}$$

follows similar lines. The representation $D^t(h)$ is, by Table I,

$$D^t(g)=\begin{pmatrix} D^s(g) & \\ & D^s(F^{-1}\cdot g)\end{pmatrix}, \tag{4.29a}$$

$$D^t(\mathcal{P}_0)=\begin{pmatrix} & D^s(f_0) \\ 1 & \end{pmatrix}. \tag{4.29b}$$

The question to be settled is whether or not there exists a Θ_t satisfying Eqs. (4.6). It is convenient to decompose Θ_t into four square matrices

$$\Theta_{ij}\,,\quad i,\,j=1,\,2:$$

$$\Theta_t=\begin{pmatrix}\Theta_{11} & \Theta_{12}\\ \Theta_{21} & \Theta_{22}\end{pmatrix}. \tag{4.30}$$

We multiply Eq. (4.6a) on the left by Θ_t and substitute into it the expressions (4.29a) and (4.30):

$$\begin{pmatrix}\Theta_{11}D^{s*}(g) & \Theta_{12}D^{s*}(F^{-1}\cdot g)\\ \Theta_{21}D^{s*}(g) & \Theta_{22}D^{s*}(F^{-1}\cdot g)\end{pmatrix} =\begin{pmatrix} D^s(g)\Theta_{11} & D^s(g)\Theta_{12}\\ D^s(F^{-1}\cdot g)\Theta_{21} & D^s(F^{-1}\cdot g)\Theta_{22}\end{pmatrix}. \tag{4.31}$$

There are three possibilities to be considered:

$$D^{s*}(g)\sim D^s(g)\ \text{not}\!\sim D^s(F^{-1}\cdot g)\,, \tag{4.32}$$

$$D^{s*}(F^{-1}\cdot g)\sim D^s(g)\ \text{not}\!\sim D^s(F^{-1}\cdot g)\,, \tag{4.33a}$$

which as we shall see is equivalent to

$$D^{s*}(g)\sim D^s(F^{-1}\cdot g)\ \text{not}\!\sim D^s(g)\,, \tag{4.33b}$$

or

$$D^{s*}(g)\ \text{not}\!\sim D^s(g)\quad \text{and}\quad D^{s*}(g)\ \text{not}\!\sim D^s(F^{-1}\cdot g)\,. \tag{4.34}$$

If the first holds then[16] $\Theta_{12}=\Theta_{21}=0$, if the second holds $\Theta_{11}=\Theta_{22}=0$, and if the third holds then $\Theta_{12}=\Theta_{21}=\Theta_{11}=\Theta_{22}=0$. Consequently, if relation (4.34) holds $\Theta_t=0$ and Θ_t^{-1} does not exist, $D^u(k)$ is of type 3, Table II. In this latter case, if Eq. (4.29) is combined with entry 3 in Table II, one obtains

$$D^u(g)=\begin{bmatrix} D^s(g) & & & \\ & D^s(F^{-1}\cdot g) & & \\ & & D^{s*}(g) & \\ & & & D^{s*}(F^{-1}\cdot g)\end{bmatrix}, \tag{4.35a}$$

$$D^u(\mathcal{P}_0)=\begin{bmatrix} & D^s(f_0) & & \\ 1 & & & \\ & & & D^{s*}(f_0)\\ & & 1 & \end{bmatrix}, \tag{4.35b}$$

$$D^u(\Theta_0)=\begin{bmatrix} & & 1 & \\ & & & 1\\ 1 & & & \\ & 1 & & \end{bmatrix}, \tag{4.35c}$$

which is recorded in Table IV as entry 13.

Let us now suppose relation (4.32) holds, so that $D^s(g)$ is potentially real or pseudoreal,

$$D^{s*}(g)=\Theta_s^{-1}D^s(g)\Theta_s\,, \tag{4.36}$$

and $\Theta_{12}=\Theta_{21}=0$. We thus find that there exists a unitary Θ_t, satisfying Eq. (4.6a), and its diagonal blocks Θ_{11} and Θ_{22} differ at most by a phase from Θ_s. We may set $\Theta_{11}=\Theta_s$ since the phase of the latter is arbitrary, so Θ_t takes the form

$$\Theta_t=\begin{pmatrix}\Theta_s & \\ & \eta\Theta_s\end{pmatrix},$$

where η is a phase factor. If this expression and Eq. (4.29b) are substituted into Eq. (4.6b) one obtains $\eta=1$, and hence,

$$\Theta_t=\begin{pmatrix}\Theta_s & \\ & \Theta_s\end{pmatrix}. \tag{4.37}$$

We observe that Θ_t is symmetric or antisymmetric as Θ_s is, and so the representation $D^u(k)$ will be of type 1 or 2 in Table II, depending on whether $D^s(g)$ is potentially real or pseudoreal. If Eqs. (4.29) and (4.37) are substituted into Table II, the entries 4 and 8 in Table IV result, as $D^s(g)$ is potentially real or pseudoreal.

We now consider the only remaining possibility, namely that relation (4.33a) holds. By reference to Table III, we see that $D^s(g)$ is then of type 1 or 2 with respect to extension by $\mathcal{T}_0$, the matrix which effects the equivalence (4.33a) being T_s:

$$D^{s*}(F^{-1}\cdot g)=T_s^{-1}D^s(g)T_s\,. \tag{4.38}$$

If we take the complex conjugate of this equation and multiply on the left by T_s^* and on the right by T_s^T, we obtain

$$D^{s*}(g)=T_s{}^*D^s(F^{-1}\cdot g)T_s{}^T, \tag{4.39}$$

which establishes the equivalence of relations (4.33a) and (4.33b). By comparing these last two equations with (4.31), we find that $\Theta_{12}=\lambda T_s$ and $\Theta_{21}=\mu T_s{}^T$, where λ and μ are arbitrary numbers, and so

$$\Theta_t=\begin{pmatrix} 0 & \lambda T_s \\ \mu T_s{}^T & 0 \end{pmatrix},$$

since we have already observed that when Eq. (4.33) holds $\Theta_{11}=\Theta_{22}=0$. From the unitarity of Θ_t, we conclude that $|\lambda|^2=|\mu|^2=1$. Because the phase of T_s is arbitrary, we may choose $\lambda=1$ and

$$\Theta_t=\begin{pmatrix} 0 & T_s \\ \mu T_s{}^T & 0 \end{pmatrix}. \tag{4.40}$$

If this equation and Eq. (4.29b) are substituted into Eq. (4.6b), multiplied in the left by Θ_t, one obtains

$$\begin{pmatrix} T_s & \\ & \mu T_s{}^T D^{s*}(f_0) \end{pmatrix}=\begin{pmatrix} \mu D^s(f)T_s{}^T & \\ & T_s \end{pmatrix}.$$

Using the result of Table III, namely

$$T_sT_s{}^*=\pm D^s(f_0), \tag{4.41}$$

one finds that $\mu=\pm 1$ and hence, from Eq. (4.40),

$$\Theta_t=\begin{pmatrix} 0 & T_s \\ \pm T_s{}^T & 0 \end{pmatrix}, \tag{4.42}$$

the upper or lower sign holding, according as $D^s(g)$ is of type 1 or 2 with respect to extension by $\mathcal{T}_0$. At this point we could already make the final entries in Table IV. However, one can obtain a more symmetric form for the corepresentation by making a change of basis. If Eqs. (4.39) and (4.41) are substituted into Eqs. (4.29), one finds for $D_t(h)$:

$$D^t(g)=\begin{pmatrix} D^s(g) & \\ & T_s{}^TD^{s*}(g)T_s \end{pmatrix}, \tag{4.43a}$$

$$D^t(\mathcal{P}_0)=\begin{pmatrix} & \pm T_sT_s{}^* \\ 1 & \end{pmatrix}. \tag{4.43b}$$

By making the change of basis

$$D^t(h)\to D^{t\prime}(h)=UD^t(h)U^\dagger,$$

with

$$U=\begin{pmatrix} 1 & \\ & \pm T_s{}^* \end{pmatrix}, \tag{4.44}$$

one finds in the new basis, dropping primes,

$$D^t(g)=\begin{pmatrix} D^s(g) & \\ & D^{s*}(g) \end{pmatrix}, \tag{4.45a}$$

$$D^t(\mathcal{P}_0)=\begin{pmatrix} & T_s \\ \pm T_s{}^* & \end{pmatrix}. \tag{4.45b}$$

The matrix Θ_t which effects the transformation (4.6) in the new basis is simply

$$\Theta_t=\begin{pmatrix} & \pm 1 \\ 1 & \end{pmatrix}. \tag{4.46}$$

In Eqs. (4.45) and (4.46) the upper or lower sign holds according as $D^s(g)$ is of type 1 or 2 with respect to extension by $\mathcal{T}_0$, and the matrix Θ_t is symmetric or antisymmetric accordingly. By substituting into Table II the representation (4.45) for $D^t(h)$ and Eq. (4.46) for Θ_t, one obtains entry 11 in Table IV for the upper sign, while for the lower sign, the following corepresentation results:

$$D^u(g)=\begin{bmatrix} D^s(g) & & & \\ & D^{s*}(g) & & \\ & & D^s(g) & \\ & & & D^{s*}(g) \end{bmatrix}, \tag{4.47a}$$

$$D^u(\mathcal{P}_0)=\begin{bmatrix} & T_s & & \\ -T_s{}^* & & & \\ & & & T_s \\ & & -T_s{}^* & \end{bmatrix}, \tag{4.47b}$$

$$D^u(\Theta_0)=\begin{bmatrix} & & & 1 \\ & & -1 & \\ & -1 & & \\ 1 & & & \end{bmatrix}. \tag{4.47c}$$

This corepresentation may be given a slightly more symmetric form. By making the change of basis according to Eq. (3.9), with

$$U=\begin{bmatrix} 1 & & & \\ & i & & \\ & & i & \\ & & & 1 \end{bmatrix},$$

one obtains entry 12 in Table IV in which $(-iT_s)$ has been replaced by T_s, as may be done since the phase of T_s may be chosen arbitrarily. The analysis of case 3, $D^s(F^{-1}\cdot g)$ not$\sim D^s(g)$, is now complete, and with it Table IV, giving all types of irreducible corepresentations $D^u(k)$ of $K=\{G, G\mathcal{P}_0, G\Theta_0, G\mathcal{T}_0=G\mathcal{P}_0\Theta_0\}$ in terms of the irreducible representations $D^s(g)$ of G. This table expresses the principal results of the present investigation.

In deriving and classifying the corepresentations of the group K, we have selected out a particular parity operator $\mathcal{P}_0$. However, a principal tenet of the present approach is that one could have equally well chosen another parity operator differing by a factor of $g_1\in G$,

$$\mathcal{P}_0{}'\equiv g_1\mathcal{P}_0=\mathcal{P}_0F^{-1}\cdot g_1. \tag{4.48}$$

TABLE V. Irreducible corepresentations of the group $K=\{G,G\mathcal{P}_0,G\Theta_0,G\mathcal{P}_0\Theta_0\}$ when $\mathcal{P}$ is of geometric type, $\mathcal{P}g\mathcal{P}^{-1}=g$, $\mathcal{P}^2=(-1)^{2j}$.

Types 1 and 2 $D^{s*}(g)=\Theta_s^{-1}D^s(g)\Theta_s$ $\Theta_s\Theta_s^*=1$	Types 3 and 4 $D^{s*}(g)=\Theta_s^{-1}D^s(g)\Theta_s$ $\Theta_s\Theta_s^*=-1$	Types 5 and 6 $D^{s*}(g)$ not$\sim D^s(g)$
$D^u(g)=D^s(g)$	$D^u(g)=D^s(g)\otimes\begin{pmatrix}1 & \\ & 1\end{pmatrix}$	$D^u(g)=\begin{pmatrix}D^s(g) & \\ & D^{s*}(g)\end{pmatrix}$
$D^u(\mathcal{P}_0)=\pm 1$	$D^u(\mathcal{P}_0)=\pm 1\otimes\begin{pmatrix}1 & \\ & 1\end{pmatrix}$	$D^u(\mathcal{P}_0)=\pm\begin{pmatrix}1 & \\ & 1\end{pmatrix}$
$D^u(\Theta_0)=\Theta_s$	$D^u(\Theta_0)=\Theta_s\otimes\begin{pmatrix} & -1\\ 1 & \end{pmatrix}$	$D^u(\Theta_0)=\begin{pmatrix} & 1\\ 1 & \end{pmatrix}$

(This is the generalization to arbitrary groups and general parity operator of the familiar freedom of choice of absolute intrinsic parity in different charge sectors.) One may easily verify that the classification of the corepresentation into the types 1 to 13 of Table IV remains invariant under this change of parity operator. The same table may be used to give the corepresentations of K in terms of the new parity of operator $\mathcal{P}_0'$ of Eq. (4.48) by making the substitutions

$$\mathcal{P}_0\to\mathcal{P}_0', \tag{4.49a}$$

$$F\cdot g\to F'\cdot g=g_1F\cdot gg_1^{-1}, \tag{4.49b}$$

$$f_0\to f_0'=g_1f_0F^{-1}\cdot g_1, \tag{4.49c}$$

$$P_s\to P_s'=D^s(g_1)P_s=P_sD^s(F^{-1}\cdot g_1), \tag{4.49d}$$

$$T_s\to T_s'=D^s(g_1)T_s=T_sD^{s*}(F^{-1}\cdot g_1), \tag{4.49e}$$

$$\Theta_s\to\Theta_s'=\Theta_s. \tag{4.49f}$$

V. COMPARISON WITH PREVIOUS ANALYSES

Upon completing the classification of the corepresentations, it is instructive to fit previous analyses, which are familiar to the reader, into the present scheme.

Wigner[12] has given an analysis without explicitly considering internal symmetry groups, and also without including all the implications of local field theory. If we assume that the internal group consists only of the identity, then $D^s(g)=1$, and so Θ_s exists and $\Theta_s=1$. Furthermore, all particles will correspond to Majorana fields for which

$$\mathcal{P}^2=(-1)^{2j}, \tag{5.1}$$

as first observed by Racah,[20] and as we shall verify in the following section. We propose to call this the "geometric" parity type. Upon introducing $\mathcal{P}_0=\sigma\mathcal{P}$, where, as we recall, $\sigma=1$ for a tensor particle and $\sigma=i$ for a spinor particle, we find $\mathcal{P}_0^2=1$, and hence P_s exists and is given by

$$P_s=1. \tag{5.2}$$

Consequently all particles fall into types 1 and 2 of Table V, with

$$D^u(g)=1,\quad D^u(\Theta_0)=1,\quad D^u(\mathcal{P}_0)=\pm 1. \tag{5.3}$$

For this simplest of all situations, because $\mathcal{P}=\sigma\mathcal{P}_0$ all tensor particles have intrinsic parity ± 1, and all spinor particles $\pm i$.

Another possibility is that there exists a parity operator $\mathcal{P}$, which commutes with all elements g of the internal symmetry group, G,

$$g\mathcal{P}=\mathcal{P}g, \tag{5.4}$$

and which satisfies the "geometric" condition (5.1). Then Eq. (5.2) also holds, and the corepresentation $D^u(k)$ is of type 1 or 2 of Table IV if $D^s(g)$ is potentially real, or of type 5 or 6 of Table IV if $D^s(g)$ is pseudoreal, or of type 9 or 10 of Table IV if $D^s(g)$ is complex. These corepresentations are given explicitly in Table 5. We see by inspection of this table and from $\mathcal{P}=\sigma\mathcal{P}_0$ that all tensor particles have intrinsic parity ± 1, and all spinor particles $\pm i$, and also that *all particles have the same intrinsic parity as their corresponding antiparticles*. However, if all spinor particles carry a conserved quantum number, as is believed to be the case for all those observed up to now, then by introducing a new parity operator

$$\mathcal{P}'=\exp(i\pi F/2)\mathcal{P}, \tag{5.5}$$

where F may be baryon number, or lepton number, or muon number, as appropriate, then the intrinsic parity is ± 1 for spinor particles also, but spinor antiparticles then have opposite intrinsic parity with respect to the corresponding particles.

It is traditional to introduce a "charge-conjugation" operator C in discussions of parity, time-reversal, and internal symmetries. Although from the present point of view there is no *a priori* reason why such an operator should exist (or why not several), one may suppose, as a specific additional assumption that the internal symmetry group G is obtained as the minimal extension of another group N by the linear operator C,

$$G=\{N,NC\}, \tag{5.6}$$

with

$$C^2=1. \tag{5.7}$$

[20] G. Racah, Nuovo Cimento 14, 322 (1937).

TABLE VI. Irreducible representations $D^s(g)$ of the group $G=\{N,NC\}$, where $CnC^{-1}=n^*$ and $C^2=1$. The elements n of N are linear operators, as is C.

Types 1 and 2 $D^r(n^*)=D^{r*}(n)=C_r^{-1}D^r(n)C_r$ $C_r^2=1$	Type 3 $D^r(n^*)=D^{r*}(n)$ not$\sim D^r(n)$
$D^s(n)=D^r(n)$	$D^s(n)=\begin{pmatrix} D^r(n) & \\ & D^{r*}(n)\end{pmatrix}$
$D^s(C)=\pm C_r$	$D^s(C)=\begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$

To complete the specification of the minimal extension, the automorphism induced by C in N must be given.

One would like to suppose that charge conjugation reverses the sign of conserved additive quantum numbers F_i,

$$CF_i=-F_iC\,, \tag{5.8}$$

where the Hermitian operators F_i are the infinitesimal generators of the compact Lie group contained in N. However, it is not possible to maintain Eq. (5.8) for all the F_i when the Lie group is noncommutative, for it contradicts

$$[F_i,F_j]=iC_{ijk}F_k \tag{5.9}$$

when the structure constants C_{ijk} do not vanish. [Because of the i and the reality of the C_{ijk}, this objection does not hold for the antilinear operator Θ, which, as we have seen in Sec. I, does satisfy Eq. (5.8) for all F_i.] Consequently we choose a complete commuting set of F_i and suppose that Eq. (5.8) holds for these. Such a choice is mathematically arbitrary but familiar in physics where, for example, T_3 in SU_2, and T_3 and Y in SU_3 are customarily singled out.

A natural way to achieve Eq. (5.8) for a complete commuting set of F_i's is to make the further suppositions that N is a Kronecker product of one parameter gauge groups and of the classical unitary Lie groups[21] and that for the group elements n and n^* (corresponding to the matrix n and its complex conjugate in the defining representation)

$$CnC^{-1}=n^*\in N\,, \tag{5.10}$$

so that the signs of the F_i (which are the phases of the diagonal n's) are reversed. This is not a matrix equation but rather, in accordance with our convention, an equation for the operators in Hilbert space that correspond to the group elements n and n^* and the automorphism C. Equations (5.7) and (5.10) complete the specification of the minimal extension (5.6) of N by C to give G.

We may now apply the results of Sec. II and given in Table I to obtain the irreducible representations $D^s(g)$ of G in terms of the irreducible representations $D^r(n)$ of N, assumed known. For the classical unitary Lie groups we are considering, one may choose a basis for each class of equivalent representations which has the property[22]

$$D^r(n^*)=[D^r(n)]^*\,. \tag{5.11}$$

The irreducible representations of $D^s(g)$ for $G=\{N,NC\}$ are given in Table VI and are obtained from Table I by the substitutions $g\to n$, $h\to g$, $F^{-1}\cdot g\to n^*$, $\mathcal{P}_0\to C$, $f_0\to 1$, $s\to r$, and $t\to s$. We notice that types 1 and 2, which are self-conjugate under C, are distinguished only by the sign of the representative of C and may be regarded as having opposite intrinsic charge-conjugation parity.[23]

The classification of the representations $D^s(g)$ depends on the existence or nonexistence of a unitary matrix C_r satisfying

$$D^{r*}(n)=C_r^{-1}D^r(n)C_r\,, \tag{5.12}$$

and whose phase may be chosen, by virtue of the analog of Eq. (2.8), such that it satisfies

$$C_r^2=1\,. \tag{5.13}$$

However, Eq. (5.12) has the same form as Eq. (3.37) for Θ_s, and we may immediately conclude that Eq. (3.38) also holds for C_r,

$$C_rC_r^*=\pm 1\,, \tag{5.14}$$

and C_r is symmetric or antisymmetric, according as the upper or lower sign holds. From the last two equations we have

$$C_r^*=C_r^T=\pm C_r\,. \tag{5.15}$$

Hence, with the phase of C_r chosen to satisfy Eq. (5.13), the unitary matrix C_r is real and symmetric, or pure imaginary and antisymmetric, depending on whether $D^r(g)$ is potentially real or pseudoreal. In the former case C_r may be diagonalized by a real orthogonal transformation. (We restrict to such transformations so that Eq. (5.11) remains true in the new basis). Since $C_r^2=1$, its diagonal form is

$$C_{rij}=\epsilon_i\delta_{ij}\,, \tag{5.16a}$$

where ϵ_i is a sign. In the latter case, the dimension of C_r and $D^r(g)$ is necessarily even [take the determi-

[21] These are defined in H. Weyl, *The Classical Groups* (Princeton University Press, Princeton, New Jersey, 1946). This assumption is quite general and includes the gauge group U_1, the special unitary groups, SU_n, the symplectic groups, and the orthogonal groups.

[22] This property is established in Ref. 21, although not explicitly stated as a theorem. It may be understood from the fact that all irreducible representations of the one parameter gauge groups and the classical unitary groups are obtainable by reduction of tensor products of n and n^*. This property does not hold in general for the representations of other groups.

[23] This is familiar in SU_2 as isotopic parity, L. Michel, Nuovo Cimento **10**, 319 (1953), or $\mathcal{G}$-parity, T. D. Lee and C. N. Yang, *ibid.* **3**, 749 (1956); Y. Dothan [*ibid.* **30**, 399 (1963)] has used the method of group extension to obtain the charge-conjugation parity for self-conjugate SU_3 multiplets, which he calls unitary parity. K. Tanabe and K. Shima [J. Math Phys. **8**, 657 (1967)] have found out for which groups there exists an operator corresponding to this parity, analogous to the isoparity or G-parity operator $\mathcal{G}=C\exp(-i\pi T_2)$ in the case of the isospin group.

TABLE VII. Irreducible corepresentations of the group $K=\{G,G\mathcal{P}_0,G\Theta_0,G\mathcal{P}_0\Theta_0\}$ for $G=\{N,NC\}$ and parity of geometric type, in terms of the irreducible representations $D^r(n)$ of N which satisfy $D^r(n^*)=D^{r*}(n)$.

Types 1, 2, 3, 4 $D^{r*}(n)=C_r^{-1}D^r(n)C_r$ $C_r=C_r^T=C_r^*$	Types 5 and 6 $D^{r*}(n)=C_r^{-1}D^r(n)C_r$ $C_r^T=C_r^*=-C_r$	Types 7 and 8 $D^{r*}(n)$ not$\sim D^r(n)$
$D^u(n)=D^r(n)$	$D^u(n)=D^r(n)\otimes\begin{pmatrix}1&\\&1\end{pmatrix}$	$D^u(n)=\begin{pmatrix}D^r(n)&\\&D^{r*}(n)\end{pmatrix}$
$D^u(C)=\pm C_r$	$D^u(C)=C_r\otimes\begin{pmatrix}1&\\&-1\end{pmatrix}$	$D^u(C)=\begin{pmatrix}&1\\1&\end{pmatrix}$
$D^u(\mathcal{P}_0)=\pm 1$	$D^u(\mathcal{P}_0)=\pm\begin{pmatrix}1&\\&1\end{pmatrix}$	$D^u(\mathcal{P}_0)=\pm\begin{pmatrix}1&\\&1\end{pmatrix}$
$D^u(\Theta_0)=C_r$	$D^u(\Theta_0)=C_r\otimes\begin{pmatrix}&-1\\1&\end{pmatrix}$	$D^u(\Theta_0)=\begin{pmatrix}&1\\1&\end{pmatrix}$

nant of Eq. (5.15) with the lower sign to show this] and C_r may be brought, again by a real orthogonal transformation, to the form[24]

$$C_r=i\begin{bmatrix}0&-1&&&\\1&0&&&\\&&0&-1&\\&&1&0&\\&&&&\ddots\end{bmatrix}. \qquad (5.16b)$$

One can also give Θ_s a canonical form[25] by a transformation (3.9). If Θ_s is symmetric it can be transformed into the identity matrix; if it is antisymmetric, it can be brought into form (5.16b). For groups other than $SU(2)$, these results are more detailed than those given in Ref. 19.

Let us consider the extension of an internal group G of the type described above, $G=\{N,NC\}$, by a parity operator $\mathcal{P}$ of geometric type, $\mathcal{P}g=g\mathcal{P}$, $\mathcal{P}^2=(-1)^{2j}$, and by the CPT operator Θ. As before we have $P_s=1$. If $D^r(n)$ is potentially real, then $D^s(g)$ is potentially real with $\Theta_s=C_r$, since $D^{s*}(n)=D^{r*}(n)=C_r^{-1}D^r(n)C_r=C_r^{-1}D^s(n)C_r$ and

$$D^{s*}(C)=C_r^*=C_r=C_r^{-1}C_rC_r=C_r^{-1}D^s(C)C_r.$$

Consequently the irreducible corepresentations $D^u(k)$ are of types 1 and 2 of Table V, and are recorded in Table VII. If $D^r(n)$ is pseudoreal, then $D^s(g)$ is complex. Assume there exists a Θ_s satisfying $D^{s*}(g)=\Theta_s^{-1}D^s(g)\Theta_s$. Then, for $g=n$, we find from Table VI that $D^s(n)=D^r(n)$ and hence that $\Theta_s=C_r$. Upon setting $g=C$, we find

$$D^{s*}(C)=C_r^*=-C_r=C_r^{-1}D^s(C)C_r=C_r,$$

which is a contradiction and hence no Θ_s exists and the representation $D^s(g)$ is complex. In this case the irreducible corepresentations $D^u(k)$ are of type 5 and 6 in Table V, and we have explicitly

$$D^u(n)=\begin{pmatrix}D^r(n)&\\&D^{r*}(n)\end{pmatrix}, \qquad (5.17a)$$

$$D^u(C)=\begin{pmatrix}C_r&\\&C_r^*\end{pmatrix}, \qquad (5.17b)$$

$$D^u(\mathcal{P}_0)=\pm\begin{pmatrix}1&\\&1\end{pmatrix}, \qquad (5.17c)$$

$$D^u(\Theta_0)=\begin{pmatrix}&1\\1&\end{pmatrix}. \qquad (5.17d)$$

Upon effecting the equivalence transformation $D^u(k)\to D^{u\prime}(k)$ according to Eq. (3.9) with

$$U=\begin{pmatrix}1&\\&C_r\end{pmatrix},$$

one finds the corepresentations listed as types 5 and 6 in Table VII. Finally, if $D^r(n)$ is complex, then $D^s(g)$ is potentially real, with $\Theta_s=D^s(C)$. The resulting corepresentations for $D^u(k)$ are listed as types 7 and 8 in Table VII. The situation which was assumed to hold before the discovery known as nonconservation of parity is described in the first and last columns of Table VII, both of which correspond to cases 1 and 2 of the general Table IV. Parity is accounted for simply by associating an intrinsic parity to each particle; extension by the CPT operator Θ does not increase the multiplicity of the multiplet and Θ_0 may be represented by the same matrix as the charge conjugation operator C. One may introduce a new time reversal operator $\mathcal{T}'=C\mathcal{P}\Theta$, instead of $\mathcal{T}=\mathcal{P}\Theta$, which appears geometric because it commutes with the commuting set of additive quantum numbers. However, it will not, in general, commute with the remaining generators of the internal symmetry group. As an example, take the familiar $SU(2)$ isospin group, with generators T_1, T_2, T_3 and elements $u=\exp(i\boldsymbol{\omega}\cdot\mathbf{T})$. From $\Theta u=u\Theta$, we find

$$\Theta T_i\Theta^{-1}=-T_i,$$

[24] F. R. Gantmacher, *The Theory of Matrices* (Chelsea Publishing Company, New York, 1964), Vol. I, p. 293.
[25] B. Zumino, J. Math. Phys. 3, 1055 (1962).

from

$$\mathcal{P}u=u\mathcal{P},$$
$$\mathcal{P}T_i\mathcal{P}^{-1}=T_i;$$

from $Cu=u^*C$, we have in the usual representation

$$CT_1C^{-1}=-T_1,$$
$$CT_2C^{-1}=T_2,$$
$$CT_3C^{-1}=-T_3,$$

and hence, for $\mathcal{T}'=C\mathcal{P}\Theta$,

$$\mathcal{T}'T_1\mathcal{T}'^{-1}=T_1,$$
$$\mathcal{T}'T_2\mathcal{T}'^{-1}=-T_2,$$
$$\mathcal{T}'T_3\mathcal{T}'^{-1}=T_3.$$

VI. EXAMPLES FROM FIELD THEORY

In this section we give field theoretic examples illustrating the various types obtained in Secs. III and IV from group theoretic considerations and collected in Tables II and IV. Many of these examples describe situations which are not known to occur in nature, corresponding to the fact that many of the types in Tables II and IV are not known to occur in nature. As a matter of fact, if we consider the approximation in which only the strong and the electromagnetic interactions are effective, so that the theory is invariant separately under the traditional P and C, we find ourselves always in types 1 and 2 of Table IV. If we add the familiar CP invariant weak interaction, that violates separately the traditional P and C, we obtain the situation described by types 1 and 2 and by type 11 of Table IV. If in addition to these interactions there is also a CP violating interaction,[3] so that no good parity operator exists, we are in the situations described by Table II (taking the second interpretation of the table, namely $H\rightarrow G$, the group of internal symmetries), more precisely by types 1 and 3 of that table. The other types of Tables II and IV do not occur in any of the approximate descriptions presently used in particle physics. Our purpose in giving field theoretic examples is to show that it is possible, within the framework of Lagrangian field theory, to realize all types given in the tables. The explanation, if any, for the fact that some types are not found in nature should, therefore, not be sought in requirements, like e.g., the locality requirement, which are satisfied in Lagrangian field theory.

Our method will be to construct an interaction Lagrangian whose symmetry properties are such as to force the fields to transform according to one of the multiplet types tabulated previously. Such a Lagrangian is easily found for each multiplet type. From the field transformation law one may immediately deduce the corresponding particle transformation law if one makes the assumption, which is implicit in the framework of standard Lagrangian field theory, that, apart from possible bound states, there is a particle corresponding to each field which appears in the Lagrangian.

In order to show that the various group theoretic types can occur for both integral and half-integral spin, we construct our examples with Dirac (or Majorana) spinors and with vector fields. It would not be difficult to translate them into examples involving fields having other spin values. For spinors the operation

$$\psi\rightarrow-\psi, \tag{6.1}$$

which changes the sign of all spinor fields, is always an element of the internal symmetry group, since one can realize it by performing a rotation by the angle 2π about an arbitrary axis. The operation (6.1) changes simultaneously the sign of all spinor fields; relative sign differences between spinor fields may have an intrinsic meaning.

The irreducible corepresentations of an internal symmetry group g, to which one has adjoined the CPT operator Θ and possibly a parity operator $\mathcal{P}$, operate on multiple component fields as follows. The basis for the representation will be a set of local fields, for instance of spinor fields ψ_α, and the effect of a transformation of the group g is described by

$$g\psi_\alpha(\mathbf{r},t)g^{-1}=\psi_\beta(\mathbf{r},t)D^u{}_{\beta\alpha}(g), \tag{6.2}$$

while the effect of CPT and of parity, when there is a parity invariance, are described by[26]

$$\Theta\psi_\alpha(\mathbf{r},t)\Theta^{-1}=\psi_\beta(-\mathbf{r},-t)\gamma_5D^u{}_{\beta\alpha}(\Theta_0) \tag{6.3}$$

and

$$\mathcal{P}\psi_\alpha(\mathbf{r},t)\mathcal{P}^{-1}=\psi_\beta(-\mathbf{r},t)\gamma_0D^u{}_{\beta\alpha}(\mathcal{P}_0). \tag{6.4}$$

In Eqs. (6.2)–(6.4), in which the index α distinguishes the different fields of the multiplet, the Dirac indices have not been explicitly indicated. As implied by the notation, for a suitable choice of the basis fields ψ_α, the matrices, $D^u(g)$, $D^u(\Theta_0)$ and $D^u(\mathcal{P}_0)$ can be made to have exactly the forms described in Tables II and IV for the various types. It would indeed be possible to repeat step by step the developments of Secs. III and IV in terms of representations of the form (6.2)–(6.4) on multiple component fields with the result that the same classification of types would emerge. For vector fields the analogs of Eqs. (6.2)–(6.4) are

$$gV_{\mu\alpha}(\mathbf{r},t)g^{-1}=V_{\mu\beta}(\mathbf{r},t)D^u{}_{\beta\alpha}(g), \tag{6.5}$$

$$\Theta V_{\mu\alpha}(\mathbf{r},t)\Theta^{-1}=-V_{\mu\beta}(-\mathbf{r},-t)D^u{}_{\beta\alpha}(\Theta_0), \tag{6.6}$$

and

$$\mathcal{P}V_{\mu\alpha}(\mathbf{r},t)\mathcal{P}^{-1}=-\epsilon_\mu V_{\mu\beta}(-\mathbf{r},t)D^u{}_{\beta\alpha}(\mathcal{P}_0). \tag{6.7}$$

[26] For convenience we use the Majorana representation in which all four matrices γ_μ are real. In this representation $\gamma_0^2=-1$, $\gamma_1^2=\gamma_2^2=\gamma_3^2=1$; the matrix γ_0 is antisymmetric (and anti-Hermitian), the matrices γ_1, γ_2, and γ_3 are symmetric (and Hermitian). We define $\gamma_5=\gamma_0\gamma_1\gamma_2\gamma_3$, so that γ_5 is real, antisymmetric (and anti-Hermitian) and $\gamma_5^2=-1$. The projection operator which enters in the usual form of the weak interactions is then $\frac{1}{2}(1+i\gamma_5)$.

Here $\epsilon_0=-1$, $\epsilon_1=\epsilon_2=\epsilon_3=1$, and no summation over the index μ is intended.

A different choice of basis fields would give the matrices $D^u{}_{\beta\alpha}$ a different form and may in concrete cases be considered preferable according to one's taste and background. We wish to point out here that it is always possible by means of a unitary change of basis to transform the matrix $D^u{}_{\beta\alpha}(\Theta_0)$ into the unit matrix and the matrices $D^u{}_{\beta\alpha}(g)$ and $D^u{}_{\beta\alpha}(\mathcal{P}_0)$ into real matrices. This follows immediately from Eqs. (1.10), (1.11), and (1.15) which imply, respectively,

$$D^u(\Theta_0)D^{u*}(\Theta_0)=1\,, \tag{6.8}$$

$$D^u(g)D^u(\Theta_0)=D^u(\Theta_0)D^{u*}(g)\,, \tag{6.9}$$

and

$$D^u(\mathcal{P}_0)D^u(\Theta_0)=D^u(\Theta_0)D^{u*}(\mathcal{P}_0)\,. \tag{6.10}$$

The new basis fields can be taken as Hermitian fields. The dimension of the corepresentation does not change, of course, by this change of basis. Its irreducibility is connected now to the irreducibility of the representation of the group g extended by $\mathcal{P}_0$ given by the matrices $D^u(g)$ and $D^u(\mathcal{P}_0)$ with the restriction that they be real matrices.

For the case of a single free Hermitian spinor field ϕ the parity operation can only be defined as

$$P:\ \ \phi(\mathbf{r},t)\rightarrow\pm\phi(-\mathbf{r},\,t)\gamma_0\,, \tag{6.11}$$

where the matrix γ_0 is real (in the Majorana representation which we are using). No complex phase can be introduced in Eq. (6.11) since it would spoil the reality properties of the Majorana field ϕ. For a complex spinor field χ the analogous definition

$$P:\ \ \chi(\mathbf{r},t)\rightarrow\pm\chi(-\mathbf{r},\,t)\gamma_0 \tag{6.12}$$

has the advantage of commuting with the operation of charge conjugation defined (in the Majorana representation) by

$$C:\qquad \chi\rightarrow\chi^\dagger. \tag{6.13}$$

This property

$$PC=CP \tag{6.14}$$

singles out the above definition of parity as being in some sense purely geometric. Its convenience was particularly emphasized by Racah.[20] The square of this parity operator is -1 for spinor fields, and in general

$$P^2=(-1)^{2j}; \tag{6.15}$$

consequently the eigenvalues of P are $\pm i$ for one particle states having half-integer spin. When the fields are in interaction the parity operation defined in Eqs. (6.11) and (6.12) may not be a symmetry of the theory and the correct parity operation may require an extra phase factor or may involve a linear combination of different fields and their Hermitian adjoints. These cases are all covered by Eq. (6.4) where the "nongeometric" part of the parity transformation is contained in the matrix $D^u(\mathcal{P}_0)$. The same considerations can be made in connection with Eq. (6.7).

We now proceed to the construction of examples. Examples for Table II (when no parity invariance is present) are easy to construct. In order to obtain a Lagrangian which does not admit a parity transformation we may take the interaction

$$\chi^\dagger\gamma_0\gamma_\mu(1+i\gamma_5)\phi W_\mu+\text{H. c.} \tag{6.16}$$

involving two spinors ϕ and χ and a vector W_μ. Clearly (6.16) violates the traditional C and P invariances. We can violate also CP invariance if we add another interaction which, for instance, conserves P but violates C, such as

$$\chi^\dagger\gamma_0\gamma_\mu\chi v_\mu\,, \tag{6.17}$$

where v_μ is a Hermitian vector field which, under C, does *not* change sign[27]

$$C:\ \ v_\mu\rightarrow v_\mu\,, \tag{6.18}$$

but which behaves as usual under P. We restrict the internal group by requiring ϕ to be a Majorana field

$$\phi=\phi^\dagger. \tag{6.19}$$

The internal symmetry group of the theory is now

$$g:\ \ \chi\rightarrow e^{i\alpha}\chi,\quad W_\mu\rightarrow e^{i\alpha}W_\mu \tag{6.20}$$

together with the transformation which changes the sign of all spinor fields. (Since this transformation is always an element of the internal symmetry group, in the following we shall not mention it explicitly every time). The transformation Θ is:

$$\begin{aligned}\Theta:\ \ &\chi(\mathbf{r},t)\rightarrow\chi^\dagger(-\mathbf{r},\,-t)\gamma_5,\quad \chi^\dagger(\mathbf{r},t)\rightarrow\chi(-\mathbf{r},\,-t)\gamma_5\\ &W_\mu(\mathbf{r},t)\rightarrow -W_\mu{}^\dagger(-\mathbf{r},\,-t),\ W_\mu{}^\dagger(\mathbf{r},t)\rightarrow -W_\mu(-\mathbf{r},\,-t).\end{aligned} \tag{6.21}$$

In the basis $\psi=(\chi,\chi^\dagger)$ the irreducible corepresentation is given by Eqs. (6.2) and (6.3) with

$$D^u(g)=\begin{pmatrix}e^{i\alpha} & 0\\ 0 & e^{-i\alpha}\end{pmatrix} \tag{6.22}$$

and

$$D^u(\Theta_0)=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}. \tag{6.23}$$

Clearly we find ourselves in type 3 of Table II for spin $\frac{1}{2}$. If we take the basis $V_\mu=(W_\mu,W_\mu{}^\dagger)$, we must use Eqs. (6.5) and (6.6) and we see that the expressions (6.22) and (6.23) still apply: we have an example of type 3 for spin 1.

We may now impose the reality conditions

$$\chi=\chi^\dagger,\quad W_\mu=W_\mu{}^\dagger \tag{6.24}$$

[27] What we mean is that the other interactions in which the field v_μ enters are invariant under the operator C, provided it operates on v_μ as in (6.18). A simple example of such a vector field would be $v_\mu=\partial(\phi^2)/\partial x_\mu$, where ϕ is a Hermitian scalar field.

which restrict the internal group to the simultaneous sign change

$$\chi\to-\chi,\quad W_\mu\to -W_\mu. \tag{6.25}$$

Here we take the basis $\psi=\chi$ or the basis $V_\mu=W_\mu$, and we have an example of type 1 of Table II for spin $\frac{1}{2}$ and 1 respectively. In this simple example $\Theta_t=1$.

An example of type 2 of Table II can be constructed in an analogous way. Let W_μ and χ be isospinors and take a linear combination of the interaction

$$\chi^\dagger\gamma_0\gamma_\mu(1+i\gamma_5)\phi W_\mu+\text{H. c.} \tag{6.26}$$

and the interaction

$$\chi\gamma_0\gamma_\mu\chi v_\mu, \tag{6.27}$$

where v_μ has the property given in Eq. (6.18).

In Eqs. (6.26) and (6.27) a sum over the isospin indices is understood. With the restriction (6.19) the Lagrangian would be invariant under a $U(2)$ group operating on the fields χ and W_μ. We can restrict the internal group to be an $SU(2)$ group by adding the further interaction

$$\chi\epsilon\gamma_0\gamma_\mu\chi v_\mu+\text{H. c.}, \tag{6.28}$$

where ϵ is the 2 by 2 matrix

$$\epsilon=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}. \tag{6.29}$$

The internal group g is now given by

$$g:\ \chi\to\chi u,\quad W_\mu\to W_\mu u, \tag{6.30}$$

where u is an $SU(2)$ matrix. The transformation Θ is

$$\Theta:\ \chi\to\chi^\dagger\gamma_5,\quad W_\mu\to -W_\mu^\dagger \tag{6.31}$$

with the appropriate changes of sign in the coordinates, which we shall not indicate explicitly from now on. In the basis $\psi=(\chi,\chi^\dagger\epsilon)$, we have again Eqs. (6.2) and (6.3) with

$$D^u(g)=\begin{pmatrix}u & 0\\ 0 & u\end{pmatrix},\quad D^u(\Theta_0)=\begin{pmatrix}0 & -\epsilon\\ \epsilon & 0\end{pmatrix}. \tag{6.32}$$

We are in type 2 of Table II, for spin $\frac{1}{2}$ and $D^s(g)=u$, $\Theta_s=\epsilon$. Similarly, for spin 1, we take the basis

$$V_\mu=(W_\mu,W_\mu^\dagger\epsilon).$$

We have described the above examples in some detail in order to show how, by suitably putting together various interactions, one can restrict the symmetries of the Lagrangian to agree with a particular type in the table. Observe that the field theoretic bases for the various types agree with the following general forms. For type 1, $\psi=\chi$, where χ represents a set of fields on which g acts irreducibly:

$$g:\ \chi\to\chi D^s(g). \tag{6.33}$$

In this case one can choose a basis such that $\Theta_s=1$, $D^s(g)$ is real, and χ are Hermitian fields. For type 2, $\psi=(\chi,\chi^\dagger\Theta_s^{-1})$; for type 3, $\psi=(\chi,\chi^\dagger)$. These forms are completely general.

We shall now describe briefly field-theoretic examples for the various types of Table IV. All types will be covered, but not in the order in which they are given in the table. All fields are massive and, unless explicitly stated, different fields are assumed to have different masses. (The masses appear in the free Lagrangian which we do not write out.) The very familiar interaction

$$\chi^\dagger\gamma_0\gamma_\mu\chi A_\mu \tag{6.34}$$

between a spinor and a Hermitian vector A_μ field (which we take to be massive) provides an example of type 1 for both spin $\frac{1}{2}$ and spin 1. We have here a phase group and a charge-conjugation operation. In the basis $\psi=(\chi,\chi^\dagger)$, Eqs. (6.2)–(6.4) apply, with

$$D^u(g)=\begin{pmatrix}e^{i\alpha} & 0\\ 0 & e^{-i\alpha}\end{pmatrix},\quad \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$$
$$D^u(\Theta_0)=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},\quad D^u(\mathcal{P}_0)=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}. \tag{6.35}$$

In the basis $V_\mu=A_\mu$, Eqs. (6.5)–(6.7) apply with

$$D^u(g)=1,\quad -1,\quad D^u(\Theta_0)=1,\quad D^u(\mathcal{P}_0)=1. \tag{6.36}$$

An example of type 2 for spin 1 is given by the interaction

$$i\chi^\dagger\gamma_0\gamma_\mu\gamma_5\chi B_\mu, \tag{6.37}$$

which forces the Hermitian 4-vector B_μ to be a pseudovector. Here, in the basis $V_\mu=B_\mu$, the internal symmetry group has a trivial representation.

$$D^u(g)=1,\quad 1,\quad D^u(\Theta_0)=1,\quad D^u(\mathcal{P}_0)=-1. \tag{6.38}$$

In the case of spinors, type 1 and type 2 can only be distinguished by considering the relative sign between two spinors. For instance, in the interaction

$$g_1\chi^\dagger\gamma_0\gamma_\mu\chi_1A_\mu+ig_2\chi^\dagger\gamma_0\gamma_\mu\gamma_5\chi_2A_\mu+\text{H. c.}, \tag{6.39}$$

the two spinors χ_1 and χ_2 will transform with opposite sign under parity.

Types 5 and 6. Let N, N', K_μ, and K_μ' be isospinors, $\Lambda=\Lambda^\dagger$ an isoscalar. Take a linear combination of the interactions

$$\begin{aligned}&K_\mu^\dagger\Lambda\gamma_0\gamma_\mu N+\text{H. c.},\\ &iK_\mu'^\dagger\Lambda\gamma_0\gamma_\mu\gamma_5N+\text{H. c.},\\ &iK_\mu^\dagger\Lambda\gamma_0\gamma_\mu\gamma_5N'+\text{H. c.},\\ &v_\mu N\epsilon\gamma_0\gamma_\mu N+\text{H. c.},\\ &N^\dagger\gamma_0\gamma_\mu Nv_\mu,\end{aligned} \tag{6.40}$$

where v_μ transforms as in (6.18). The parities of K_μ and K_μ' are opposite and so are those of N and N'. The internal group is the $SU(2)$ group on N, N', K_μ, K_μ'. Take the basis $V_\mu=(K_\mu,K_\mu^\dagger\epsilon)$ for spin 1 and

$\psi=(N,N^{\dagger}\epsilon)$ for spin $\frac{1}{2}$ and similarly with the primed fields.

Types 9 and 10. One may take any Lagrangian with a one-parameter gauge group which is invariant under the traditional parity transformation but not under the traditional C. The two signs correspond to the usual distinction, e.g., between vectors and pseudovectors or, for spinors, between spinors having different parity signs.

Type 11. We may take here the usual weak interactions with an intermediate boson

$$\bar{\chi}_1\gamma_\mu(1+i\gamma_5)\chi_2W_\mu+\text{H. c.} \tag{6.41}$$

Here $\mathcal{P}$ is the traditional "CP" and there are two phase groups. In the basis $V_\mu=(W_\mu,W_\mu^{\dagger})$ we have an example for spin 1, with

$$D^u(g)=\begin{pmatrix} e^{-i\alpha} & 0\\ 0 & e^{i\alpha}\end{pmatrix},\quad D^u(\mathcal{P}_0)=\begin{pmatrix} 0 & -1\\ -1 & 0\end{pmatrix},\quad D^u(\Theta_0)=\begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix}, \tag{6.42}$$

while in the basis $\psi=(\chi_2,\chi_2^{\dagger})$ we have an example for spin $\frac{1}{2}$, with

$$D^u(g)=\begin{pmatrix} e^{i(\alpha+\beta)} & 0\\ 0 & e^{-i(\alpha+\beta)}\end{pmatrix},\quad D^u(\mathcal{P}_0)=\begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix},\quad D^u(\Theta_0)=\begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix}, \tag{6.43}$$

and in the basis $\psi'=(\chi_1,\chi_1^{\dagger})$ also an example for spin $\frac{1}{2}$ with

$$D^u(g)=\begin{pmatrix} e^{i\beta} & 0\\ 0 & e^{-i\beta}\end{pmatrix},\quad D^u(\mathcal{P}_0)=\begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix},\quad D^u(\Theta_0)=\begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix}. \tag{6.44}$$

Case 13. Let the vector fields U_μ and W_μ have the same mass and take a linear combination of the interactions

$$\chi_1^{\dagger}\gamma_0\gamma_\mu(1+i\gamma_5)\chi_2U_\mu+\text{H. c.} +\chi_1^{\dagger}\gamma_0\gamma_\mu(1-i\gamma_5)\chi_2^{\dagger}W_\mu+\text{H. c.} \tag{6.45}$$

and

$$\chi_1^{\dagger}\gamma_0\gamma_\mu\chi_1v_\mu, \tag{6.46}$$

where v_μ is again as in (6.18). The internal symmetry group is given by

$$\chi_1\to e^{i\beta}\chi_1,\quad \chi_2\to e^{i\alpha}\chi_2,\quad U_\mu\to e^{-i\alpha+i\beta}U_\mu,\quad W_\mu\to e^{i\alpha+i\beta}W_\mu. \tag{6.47}$$

In the basis $V_\mu=(W_\mu,U_\mu,W_\mu^{\dagger},U_\mu^{\dagger})$ we have

$$D^u(g)=\begin{pmatrix} e^{i\alpha+i\beta} & & & \\ & e^{-i\alpha+i\beta} & & \\ & & e^{-i\alpha-i\beta} & \\ & & & e^{i\alpha-i\beta}\end{pmatrix},\quad D^u(\mathcal{P}_0)=\begin{pmatrix} & 1 & & \\ 1 & & & \\ & & & 1\\ & & 1 & \end{pmatrix},\quad D^u(\Theta_0)=\begin{pmatrix} & & 1 & \\ & & & 1\\ 1 & & & \\ & 1 & & \end{pmatrix}, \tag{6.48}$$

which shows that we are in type 13 for spin 1. To obtain an example of type 13 for spin $\frac{1}{2}$, just switch the roles of the vectors U_μ, W_μ, and of the spinor χ_2. The interaction which replaces (6.45) is now (χ_1, U, and W are spinors and K_μ a vector)

$$\chi_1^{\dagger}\gamma_0\gamma_\mu(1+i\gamma_5)UK_\mu+\text{H. c.}, +\chi_1\gamma_0\gamma_\mu(1-i\gamma_5)WK_\mu^{\dagger}+\text{H. c.}, \tag{6.49}$$

while we still keep (6.46). The internal group is

$$\chi_1\to e^{i\beta}\chi_1,\quad K_\mu\to e^{i\alpha}K_\mu,\quad U\to e^{-i\alpha+i\beta}U,\quad W\to e^{i\alpha+i\beta}W. \tag{6.50}$$

In the basis $\psi=(V,U,V^{\dagger},U^{\dagger})$, the matrices D^u have the form given in (6.48). Observe that in type 13, both Θ and $\mathcal{P}$ cause doubling of the original multiplet.

Type 4. Take the interaction (6.45) [plus (6.46)] but impose the restriction that U_μ and W_μ be Hermitian,

$$U_\mu=U_\mu^{\dagger},\quad W_\mu=W_\mu^{\dagger}. \tag{6.51}$$

This restricts the internal group from (6.47) to a group of sign changes. In the basis $V_\mu=(W_\mu,U_\mu)$ we have

$$D^u(g)=\begin{pmatrix} 1 & 0\\ 0 & 1\end{pmatrix},\ \begin{pmatrix} 1 & 0\\ 0 & -1\end{pmatrix},\ \begin{pmatrix} -1 & 0\\ 0 & -1\end{pmatrix},\ \begin{pmatrix} -1 & 0\\ 0 & 1\end{pmatrix}, \tag{6.52}$$

$$D^u(\mathcal{P}_0)=\begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix},\quad D^u(\Theta_0)=\begin{pmatrix} 1 & 0\\ 0 & 1\end{pmatrix}.$$

Similarly, for spin $\frac{1}{2}$, take the interaction (6.49) [plus (6.46)] and impose the further restriction that U and W be Majorana spinors

$$U=U^{\dagger},\quad W=W^{\dagger}. \tag{6.53}$$

In the basis $\psi=(W,U)$ the matrices D^u are given by (6.52).

TABLE VIII. Describes the field-theoretic basis appropriate to the various types of Table 4. The fields χ, χ_1, and χ_2 are the multiple-component fields on which the internal-symmetry group G acts irreducibly, e.g. $g\chi g^{-1}=\chi D^s(g)$.

Type	Basis
1, 2	$\psi=\chi$
3, 5, 6, 7	$\psi=(\chi,\chi^\dagger\Theta_s^{-1})$
9, 10, 11	$\psi=(\chi,\chi^\dagger)$
4	$\psi=(\chi_1,\chi_2)$
8	$\psi=(\chi_1,\chi_2,\chi_1^\dagger\Theta_s^{-1},\chi_2^\dagger\Theta_s^{-1})$
13	$\psi=(\chi_1,\chi_2,\chi_1^\dagger,\chi_2^\dagger)$
12	$\psi=(\chi_1,\chi_2T_s,\chi_2^\dagger T_s^*,\chi_1^\dagger)$

Type 3. Consider again the interaction (6.45), but impose the condition $W_\mu=iU_\mu$. It is easily seen that, in terms of the field $\Omega=\chi_2+i\chi_2^\dagger$, the interaction (6.45) takes the form

$$\chi_1^\dagger\gamma_0\gamma_\mu(\Omega+i\gamma_5\Omega^\dagger)W_\mu+\text{H. c.} \tag{6.54}$$

With this interaction the parity transformation is

$$\mathcal{P}:\ \Omega\to\pm i\Omega\gamma_0,\ \chi_1\to\eta\chi_1\gamma_0,\ W_\mu\to\pm i\eta W_\mu\epsilon_\mu\,, \tag{6.55}$$

where η is a phase. It should be noted that an interaction like the above forces a particular phase in the parity transformation of a spinor field. (Similarly, it is possible to give interactions or Hermiticity conditions which require a spinor field Σ to transform as[28]

$$\mathcal{P}:\ \Sigma\to\pm\Sigma\gamma_0. \tag{6.56}$$

The internal group consists of the transformations

$$\chi_1\to e^{i\beta}\chi_1,\quad W_\mu\to e^{i\beta}W_\mu,\quad \Omega\to\Omega\,, \tag{6.57}$$

plus the usual sign change for spinors. In the basis $\psi=(\Omega,\Omega^\dagger)$ and choosing the lower sign in (6.55), we have

$$D^u(g)=\begin{pmatrix}1&0\\0&1\end{pmatrix},\ \begin{pmatrix}-1&0\\0&-1\end{pmatrix},$$
$$D^u(\mathcal{P}_0)=\begin{pmatrix}-i&0\\0&+i\end{pmatrix},\quad D^u(\Theta_0)=\begin{pmatrix}0&1\\1&0\end{pmatrix}. \tag{6.58}$$

We are in type 3 for spin $\frac{1}{2}$. If we take the same interaction (6.54) and impose the further restriction $\chi_1=\chi_1^\dagger$, the group (6.57) is restricted to a sign group and the phase η in (6.55) must be real. In the basis $V_\mu=(W_\mu,W_\mu^\dagger)$, the matrices D^u have the same form (6.58). We are in type 3 for spin 1.

Type 12. An example can be developed in complete analogy with that for 13, but using fields U_μ, W_μ, and χ_1 which are isospinors, while χ_2 is an isoscalar (for the spin 1 example) and fields U, W, and χ_1 which are isospinors while K_μ is an isoscalar (for the spin $\frac{1}{2}$ example). Let us concentrate on the spin 1 example. In addition to the interactions analogous to (6.45) and (6.46) (with sums over the isospin indices), we add the interaction

$$\chi_1\epsilon\gamma_0\gamma_\mu\chi_1A_\mu+\text{H. c.} \tag{6.59}$$

The internal group is thereby restricted to

$$\chi_1\to\chi_1u,\quad \chi_2\to\chi_2e^{i\alpha},$$
$$U_\mu\to U_\mu ue^{-i\alpha},\quad W_\mu\to W_\mu ue^{i\alpha}, \tag{6.60}$$

where u is an $SU(2)$ matrix. In the basis

$$V_\mu=(W_\mu,U_\mu\epsilon,U_\mu^\dagger\epsilon,W_\mu^\dagger)$$

we have

$$D^u(g)=\begin{pmatrix}e^{i\alpha}u&&&\\&e^{-i\alpha}u^*&&\\&&e^{i\alpha}u&\\&&&e^{-i\alpha}u^*\end{pmatrix},$$
$$D^u(\mathcal{P}_0)=\begin{pmatrix}&-\epsilon&&\\\epsilon&&&\\&&&\epsilon\\&&-\epsilon&\end{pmatrix}, \tag{6.61}$$
$$D^u(\Theta_0)=\begin{pmatrix}&&&1\\&&1&\\&1&&\\1&&&\end{pmatrix}.$$

The case of spin $\frac{1}{2}$ can be treated in a similar way. In type 12, as in type 13, both Θ and $\mathcal{P}$ are responsible for doubling of the dimension of the original multiplet.

Type 8. One can proceed as for type 12, but, instead of adding the interaction (6.59), let us add, e.g.

$$U_\mu\in(\partial_\lambda U_\mu-\partial_\mu U_\lambda)A_\lambda$$
$$+\text{H. c.}+W_\mu\in(\partial_\lambda W_\mu-\partial_\mu W_\lambda)A_\lambda+\text{H. c.}\,, \tag{6.62}$$

which eliminates the phases from the internal symmetry group. In the basis $V_\mu=(W_\mu,U_\mu,W_\mu^\dagger\in,U_\mu^\dagger\in)$, we have

$$D^u(g)=\begin{pmatrix}u&&&\\&u&&\\&&u&\\&&&u\end{pmatrix},\ \begin{pmatrix}-u&&&\\&u&&\\&&-u&\\&&&u\end{pmatrix},$$
$$\begin{pmatrix}u&&&\\&-u&&\\&&u&\\&&&-u\end{pmatrix},\ \begin{pmatrix}-u&&&\\&-u&&\\&&-u&\\&&&-u\end{pmatrix},$$
$$D^u(\mathcal{P}_0)=\begin{pmatrix}&1&&\\1&&&\\&&&1\\&&1&\end{pmatrix}, \tag{6.63}$$
$$D^u(\Theta_0)=\begin{pmatrix}&&-\epsilon&\\&&&-\epsilon\\\epsilon&&&\\&\epsilon&&\end{pmatrix}.$$

We are in type 8 for spin 1. The example for spin $\frac{1}{2}$ can be developed in a similar way.

[28] That different spinor fields can transform under parity in different ways, as described in Eqs. (6.54) and (6.55) was especially emphasized by C. N. Yang and J. Tiomno, Phys. Rev. 79, 495 (1950).

Type 7. For spin $\frac{1}{2}$ one may take an isospinor N which by parity transforms as

$$N \to \pm iN\gamma_0, \quad (6.64)$$

and one should add an interaction which restricts the internal symmetry group to be just $SU(2)$. Then, in the basis $\psi = (N, N^\dagger \epsilon)$,

$$D^u(g) = \begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix}, \quad D^u(\mathcal{P}_0) = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}, \quad D^u(\Theta_0) = \begin{pmatrix} 0 & -\epsilon \\ \epsilon & 0 \end{pmatrix}. \quad (6.65)$$

A Lagrangian which achieves the above can be easily constructed following a method similar to that used for type 3. Similarly, for spin 1, one may use a vector K_μ which is an isospinor and which by parity transforms as

$$K_\mu \to iK_\mu \epsilon_\mu. \quad (6.66)$$

In the basis $V_\mu = (K_\mu, K_\mu{}^\dagger \epsilon)$ the matrices D^u are again given by (6.65).

The basis fields for the examples discussed above for Table IV fit into general forms analogous to those described after Eq. (6.33) for the types of Table II. These general forms are collected in Table VIII, which is a sort of summary of the results of this section.

PHYSICAL REVIEW VOLUME 164, NUMBER 5 25 DECEMBER 1967

Partial-Wave Analysis in Terms of the Homogeneous Lorentz Group

R. Delbourgo,* Abdus Salam,† and J. Strathdee
International Atomic Energy Agency, International Centre for Theoretical Physics, Trieste, Italy
(Received 10 July 1967)

The aim of this paper is to generalize Toller's work on elastic forward scattering and to expand the general two-body amplitude for all values of momentum transfer in terms of unitary representations of $SO(3,1)$. The authors' concern is covariant inclusion of spin.

1. INTRODUCTION

In a series of recent papers, Toller[1] has made a fundamental advance in noticing and exploiting the extra $O(3,1)$ invariance possessed by the elastic forward-scattering amplitude. The new invariance leads him to an expansion of the amplitude in terms of unitary representations of the group $SO(3,1)$. This is in contrast to the normal partial-wave analysis which is an expansion in terms of unitary representations of $SO(3)$—a much smaller structure. The new expansion—embodying the higher symmetry—leads to newer insights; for example, if the new partial-wave amplitude, labeled with the four-dimensional generalized angular momentum σ, is assumed to be meromorphic for complex σ, one finds that to each pole in the σ-plane there corresponds a family of integrally spaced daughter poles in the complex J plane for the partial-wave amplitude a^J. This parent-daughter phenomenon anticipated in the works of Gribov and Volkov,[2] Domokos and Suranyi,[3] and rediscovered recently by Freedman and Wang,[4] finds its most complete expression in Toller's development insofar as, in contrast to the other authors, Toller takes full account of the very essential complications introduced by spin.

The aim of the present paper is to generalize Toller's work on elastic forward scattering and to expand the general two-body amplitude for all values of momentum transfer in terms of unitary representations of $SO(3,1)$. That such a program is feasible and that it may be expected to lead to new results has already been demonstrated by Oakes[5] and Domokos[6] for scattering of equal- or unequal-mass particles when no spins are involved. In this simple case, the amplitude is a function of scalar products of incoming and outgoing momenta. Such a function (or rather its analytic continuation to a Euclidean metric) can always be expanded in terms of a *complete set* of four-dimensional Gegenbauer polynomials.

Our concern in this paper is covariant inclusion of spin. One simple suggestion for doing this would be to separate out all spin-dependent factors and to write the general amplitude in terms of scalar amplitudes of

* Imperial College, London, United Kingdom.
† On leave of absence from Imperial College, London, United Kingdom.

[1] M. Toller, Nuovo Cimento 37, 631 (1965); Internal Reports University of Rome Nos. 76 (1965) and 84 (1965) (unpublished); M. Toller and A. Sciarrino, Internal Report, University of Rome No. 108 (1966) (unpublished).

[2] R. J. N. Volkov and V. N. Gribov, Zh. Eksperim. i Teor. Fiz. 44, 1068 (1963) [English transl.: Soviet Phys.—JETP 17, 720 (1963)].

[3] G. Domokos and P. Suranyi, Nucl. Phys. 54, 529 (1964); Ya. A. Smorodinski, M. Ulhir, and P. Winternitz, Dubna Report No. E-1591 (unpublished).

[4] D. Z. Freedman and J. M. Wang, Phys. Rev. 153, 1596 (1967); M. L. Goldberger and E. Jones, Phys. Rev. Letters 17, 105 (1966).

[5] R. Oakes, Phys. Letters 24B, 154 (1967).

[6] G. Domokos, Phys. Rev. 159, 1387 (1967).