and

$$L_{i\phi}(J)=\tfrac{1}{2}g_0^2\int J(x)\Delta(x-x')J(x')d^4x'. \quad (36)$$

In (35) the boson field $\phi(x)$ is treated as an elementary field. However, in (36) the field $\phi(x)$ is treated as a composite state. That is to say, the field $\phi(x)$ does not appear explicitly in the Lagrangian but manifests itself as a pole in the various tau functions of the fermion fields.

The Lagrangians discussed in this paper yield bound states that appear as poles in the various tau functions. This type of bound state is by far the simplest and most familiar. More complicated Lagrangians will yield bound states that will not appear as poles in the tau functions. Such bound states will manifest their presence as branch points, and will only confuse the context of this paper.

In Sec. II an expression for the bound-state operator was constructed from a knowledge of the two equivalent Lagrangians. Matrix elements of these bound-state operators did not in general agree with the ones calculated where $\phi(x)$ was treated as an elementary particle. This observation raises an interesting question as to whether or not the theory uniquely determines the properties of bound states. Only in the limit that $Z_3\rightarrow 0$ did *all* matrix elements become equivalent. Agreement might be expected in this limit because the effects of the free-field Lagrangian on the matrix elements vanish in this limit. It was just this term that accounted for the difference in the various matrix elements.

Closely related to the question of uniqueness is the construction of bound-state operators. We were able to construct such operators only because of a prior knowledge of the two equivalent Lagrangians. In general one does not have this information. It is usually the case that only the bound-state Lagrangian is given. Therefore, it is not a trivial task to construct such bound-state operators without assuming it is a bound state of a certain kind. These and other questions that were implied in the paper will be discussed elsewhere.

PHYSICAL REVIEW VOLUME 164, NUMBER 5 25 DECEMBER 1967

Connection between $SU(3)$ and $O(4)$*

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(Received 31 July 1967)

It is shown that the insertion of a barrier, in the isotropic harmonic oscillator and in the hydrogen atom, gives each system the higher symmetry usually associated with the other. The result is more general: The imposition of a reflection condition can change $SU(3)$ to $O(4)$, and vice versa. This may have implications for elementary-particle symmetries.

I. INTRODUCTION

In nonrelativistic quantum mechanics, the harmonic oscillator and the hydrogen atom are systems whose energy-level structures possess a higher symmetry than the symmetry of the space in which the motion takes place. There appear degeneracies which indicate invariance of the Hamiltonian under transformations in some higher-dimensional space. As is well known, the invariance group of the three-dimensional harmonic oscillator is $SU(3)$,[1] while that of the hydrogen atom is $O(4)$.[2] The purpose of this paper is to point out that physically selected sets of states of either system possess the higher symmetry usually associated with the other system. For example, under the condition of evenness or oddness on reflection in a plane, the remaining states of the three-dimensional oscillator possess the symmetry of $O(4)$, and those of the hydrogen atom, $SU(3)$. To some extent similar things happen in a space of n dimensions. The n-dimensional harmonic oscillator and the n-dimensional Schrödinger equation with an attractive $1/r$ potential have the invariance groups of $SU(n)$ and $O(n+1)$,[3] respectively. (The latter symmetry is also the natural symmetry of a particle constrained to move on a sphere in $n+1$ dimensions.) The imposition of selection rules under reflection on the "hydrogen atom" lead to degeneracies characteristic of certain representations of $SU(n)$. (The reverse situation is not clear to us.) This reciprocity property appears, in fact, to reside in the group representations themselves, and not only in the physical systems we have used to realize them. The relationship between $SU(3)$ and $O(4)$ may be re-

* Supported by the U. S. Office of Naval Research under Contract No. 00014-67-A-0305-0005 and by the National Science Foundation.

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[1] See, for example, J. P. Elliott, Proc. Roy. Soc. (London) **A245**, 128 (1958).

[2] A recent discussion is contained in M. Bander and C. Itzykson, Rev. Mod. Phys. **38**, 330 (1966).

[3] S. P. Alliluev, Zh. Eksperim. i Teor. Fiz. **33**, 200 (1957) [English transl.: Soviet Phys.—JETP **6**, 156 (1958)].

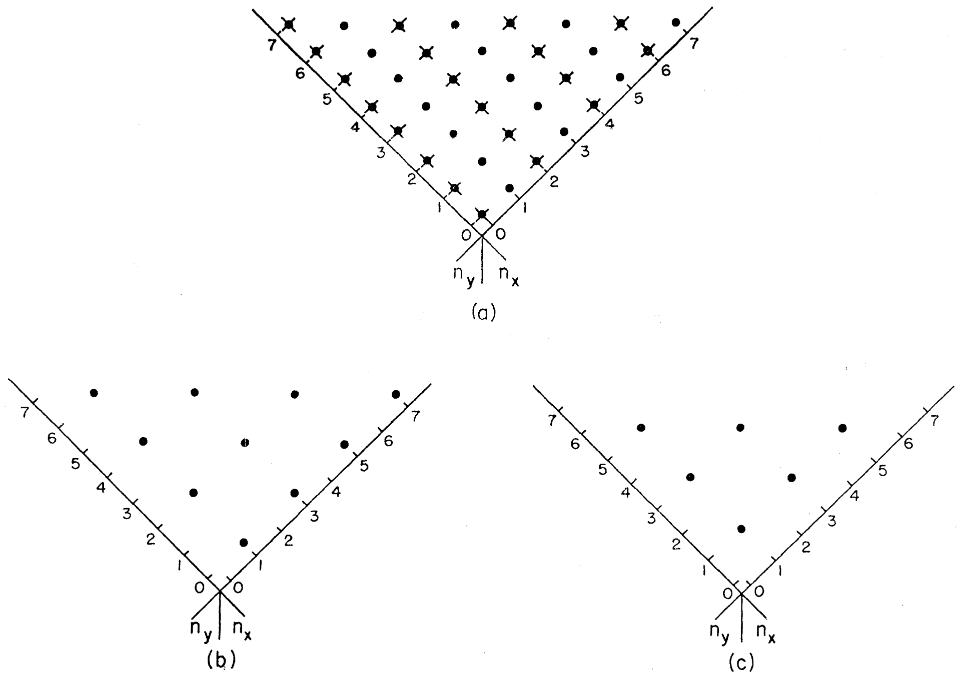

FIG. 1. States of the two-dimensional harmonic oscillator in Cartesian representation (n_xn_y). Of the complete set of states shown in (a), those excluded by a wall in the x direction are indicated by a cross. The remaining states are separated into two interleaving sets of levels, in (b) and (c), each exhibiting the multiplicity of $SU(2)$.

garded, perhaps, as a less rigid extension to a higher dimension of the homomorphism between $SU(2)$ and $O(3)$.

The relationship is obscure enough to have needed some accidental stimulus for its discovery. We were led to the hydrogen-atom problem by considering possible bound states of an electron around an impurity in a semiconductor.[4] When the impurity is close to the surface of the crystal, the electron is attracted both to the impurity and to its image in the wall, and is almost perfectly reflected from the wall itself, because of the high dielectric constant. Thus the allowed states are those of a hydrogen molecular ion with odd reflection symmetry about the line joining the impurity and its image. If the vacancy is right at the surface of the crystal, the system becomes that of a hydrogen *atom* with an impenetrable wall across the middle. That the problem was soluble in this manner, and that the resulting atom problem had an interesting higher symmetry, was realized by us some time ago. It is only now, however, that the reciprocal relationship with the oscillator has been appreciated and developed.

It seems to us that there are a number of lessons to be learned from this result, besides the direct, and somewhat useless, information about oscillators and hydrogen atoms with walls across them. After examining in detail the simplest cases, and verifying the group structure claimed from the observed multiplicities in Sec. II, we discuss some implications of the results in Sec. III.

[4] See, for example, M. D. Golinchuk and M. F. Deigen, Fiz. Tverd. Tela **5**, 405 (1963) [English transl.: Soviet Phys.—Solid State **5**, 295 (1963)]. We wish to acknowledge interesting discussions with Dr. F. Proix and Professor P. Handler and Professor J. Bardeen on this system and its properties.

II. SYMMETRIES AND GENERATORS

In the following subsections we examine a few of the simplest cases. The degenerate states for each level of either the harmonic oscillator or the hydrogen atom are separated into different sets according to the physical selection rule of putting an impenetrable barrier across the system. There remain, in all cases, more degeneracies than are required by the axial symmetry in space of the resulting system. By examining the multiplicities of the final system, we guess the higher symmetry implied by this degeneracy. The coupling of the degenerate states of one level leads us to expressions for generators of the symmetry, and we verify that the generators satisfy the appropriate commutation relations.

A. Two-Dimensional Harmonic Oscillator: $SU(2) \rightarrow SU(2)$

This, the simplest example, illustrates the method. In Fig. 1(a) the states of the two-dimensional harmonic oscillator are shown in Cartesian representation (n_xn_y). The degeneracy is an expression of the higher-symmetry $SU(2)$ possessed by this system, in addition to the $O(2)$ symmetry of any system isotropic in two dimensions. In terms of the harmonic-oscillator lowering and raising operators a_x, a_x^*, a_y, a_y^*, and the number operators $n_x=a_x^*a_x$, $n_y=a_y^*a_y$, the three generators T_+, T_-, and T_3 of the $SU(2)$ symmetry are

$$T_3=\tfrac{1}{2}(n_x-n_y),\quad T_+=a_x^*a_y,\quad T_-=a_xa_y^*. \tag{1}$$

These satisfy the required commutation rules

$$[T_\pm,T_3]=\mp T_\pm,\quad [T_+,T_-]=2T_3. \tag{2}$$

TABLE I. The representation S and generators $\mathbf{S}$ of the $SU(2)$ symmetry formed by selecting states of an original $SU(2)$ symmetry, which is also the higher symmetry of a two-dimensional harmonic oscillator. They are expressed in terms of the number operators n_x, n_y of the oscillator, and in terms of T and $\mathbf{T}$ of the original $SU(2)$. The selection, made by a reflection condition along the x axis, is as follows: (1) and (2) have odd symmetry, (3) and (4) have even symmetry; (2) and (3) have even occupation number n_x+n_y, (1) and (4) have odd n_x+n_y. The μ factors are given by Eq. (7) of the text.

Set	S	S_3	$S_+=S_-^*$
(1)	$\frac{1}{4}(n_x+n_y-1)=\frac{1}{2}(T-\frac{1}{2})$	$\frac{1}{4}(n_x-n_y-1)=\frac{1}{2}(T_3-\frac{1}{2})$	$\mu_1T_+\mu_2T_+$
(2)	$\frac{1}{4}(n_x+n_y-2)=\frac{1}{2}(T-1)$	$\frac{1}{4}(n_x-n_y)=\frac{1}{2}T_3$	$\mu_1T_+^2\mu_2$
(3)	$\frac{1}{4}(n_x+n_y)=\frac{1}{2}T$	$\frac{1}{4}(n_x-n_y)=\frac{1}{2}T_3$	$T_+\mu_1\mu_2T_+$
(4)	$\frac{1}{4}(n_x+n_y-1)=\frac{1}{2}(T-\frac{1}{2})$	$\frac{1}{4}(n_x-n_y+1)=\frac{1}{2}(T_3+\frac{1}{2})$	$T_+\mu_1T_+\mu_2$

In terms of M (the eigenvalue of T_3) and T [such that the eigenvalue of $\mathbf{T}^2$ is $T(T+1)$] the states can thus be expressed as $|TM\rangle$, where

$$T=\tfrac{1}{2}(n_x+n_y),\quad M=\tfrac{1}{2}(n_x-n_y). \tag{3}$$

(We use n_x, n_y for both the number operators and their eigenvalues.)

The introduction of an impenetrable barrier perpendicular to the x axis, which allows only states with odd reflection symmetry in that direction, i.e., n_x odd, removes the states indicated by crosses. The remaining states are immediately recognizable as two interleaved sets of states with the same degeneracy and multiplicity as the original set. They are shown separately in Figs. 1(b) and 1(c). We now conjecture that since each of these sets of states has the multiplicity of $SU(2)$, it is possible to find the generators of that symmetry, call them S_+, S_-, and S_3, and express them in terms of T_+, T_-, and T_3, the generators of the original higher-symmetry group. We now proceed to do this.

From the multiplicities we deduce the expressions for the operators S and S_3; they are given in the first two columns of Table I. For completeness, the two sets of states with n_x even are also included; the rows of Table I refer to the cases (1) n_x odd, n_x+n_y odd [Fig. 1(b)]; (2) n_x odd, n_x+n_y even [Fig. 1(c)]; (3) n_x even, n_x+n_y even; and (4) n_x even, n_x+n_y odd.

The extra factor $\frac{1}{2}$ in S_3, or M_s, reflects the fact that all transitions now involve $\Delta n_x=2$, and one is tempted to write for S_+ just T_+^2. It becomes obvious, however, on evaluating commutators of S_3, T_+^2, and T_-^2, that the algebra in this form does not close. A related difficulty is that, acting on a state, T_+^2 produces the proper transitions, but with the wrong constant factors. We have

$$T_+^2|n_xn_y\rangle=[(n_x+1)(n_x+2)n_y(n_y-1)]^{1/2}\times|n_x+2,\,n_y-2\rangle, \tag{4}$$

whereas the relationship proper to $SU(2)$ must be

$$S_+|SM_s\rangle=[(S-M_s)(S+M_s+1)]^{1/2}|SM_s+1\rangle. \tag{5}$$

From a comparison of Eqs. (4) and (5) we deduce the correct relationship

$$S_+|n_xn_y\rangle=T_+^2\left[\frac{(S-M)_s(S+M_s+1)}{(n_x+1)(n_x+2)n_y(n_y-1)}\right]^{1/2}\times|n_x+2,\,n_y-2\rangle. \tag{6}$$

This result is given in a more symmetrical form for each of the four cases in the third column of Table I. S_- is, of course, just the Hermitian conjugate of S_+. The operators μ_1 and μ_2 appearing in the table are

$$\begin{aligned}\mu_1&=(2n_x)^{-1/2}=[2(T+T_3)]^{-1/2},\\ \mu_2&=(2n_y)^{-1/2}=[2(T-T_3)]^{-1/2}.\end{aligned} \tag{7}$$

n_x, n_y are now the corresponding operators.

It can be verified that the commutation rules analogous to Eq. (2) for S_+, S_-, and S_3 are obeyed, and the algebra does close. The selected sets of states of Fig. 1(b) or 1(c), and the two sets even in n_x, are in fact bases of $SU(2)$ representations.

Thus all the states of the two-dimensional oscillator can be separated into these four interleaved sets of states, each of which separately are bases for representations of $SU(2)$,and this can be repeated. Other ways of regularly selecting states can also be treated in the same way. In fact, it is clear that our result really pertains to $SU(2)$ generally, and is not confined to the two-dimensional oscillator, which we have used as a realization of it. For future reference, the operators in Table I and Eq. (7) have also been expressed in terms of T, T_3 by means of Eq. (3). The operators $S_\pm$ depend on T, which labels the original representation. The operator $\mathbf{T}^2$ associated with T commutes with all the T generators, so T can be regarded as an operator or an eigenvalue. The expressions for the generators of the $SU(2)$ symmetries of the four sets of states are very similar. The differences are enough, however, to prevent commutators among the different sets of $SU(2)$ generators from being recognizably simple; the differences also prevent the newly found symmetry of the divided oscillator from being a symmetry of the original system.

In a similar manner, groups of states of $O(3)$ (integer l), selected by either a physical mechanism like an impenetrable barrier, or the requirement that $l-m_l$ is odd, become representations of $SU(2)$ (i.e., they include half-integral S). The connection is just that given in Table I with T integral.

B. Three-Dimensional Harmonic Oscillator: $SU(3)\to O(4)$

Before bisecting the three-dimensional oscillator, we mention a few facts about $SU(3)$ and $O(4)$ representations, and the relation of $SU(3)$ states to the oscillator.

The only representations of $SU(3)$ which appear are the triangular ones $(p,0)$ with dimensionality $\frac{1}{2}(p+1)\times(p+2)$, and all the results we quote are as yet applicable only to those representations. In terms of the oscillator, p is the number or energy operator

$$p=n_x+n_y+n_z. \qquad (8)$$

The basis states may be denoted by $|p;TM\rangle$. The isospin variables TM may be identified with the TM of Sec. IIA arising from the $SU(2)$ subgroup of the oscillator associated with the xy plane; the isospin generators are thus given by Eq. (1) and T by Eq. (3); for these triangular representations the hypercharge Y is just $2T-\frac{2}{3}p$.

The remaining generators of $SU(3)$ (the notation is that of Sharp and von Baeyer[5]) are $R_+=-a_z{}^*a_x$, $S_+=a_y{}^*a_z$ and their Hermitian conjugates R_-, S_-.

The generators of $O(4)$[6] may be taken as the components of two 3-vectors **J** and **K**; the former induces rotations in the 123 subspace of $O(4)$ and the components of the latter generate rotations in the 14, 24, 34 planes. The representations are conveniently labelled by (j^+j^-) corresponding to the commuting "angular momenta" $\mathbf{J}^+=\frac{1}{2}(\mathbf{J}+\mathbf{K})$ and $\mathbf{J}^-=\frac{1}{2}(\mathbf{J}-\mathbf{K})$; the dimensionality is $(2j^++1)(2j^-+1)$. The basis states of the representation are the eigenstates

$$\left|\begin{matrix} j^+ & j^- \\ j & m \end{matrix}\right\rangle$$

of the angular momentum **J**.

The enumeration of states and the selection of sets of states of the three-dimensional oscillator, shown in Table II, are analogous to those given in Sec. IIA.

The states selected by the condition that n_z be odd (an impenetrable barrier in the xy plane, for example) appear, from their multiplicities, to be bases for two interleaving sets of representations of $O(4)$. Square representations with $j^+=j^-=\frac{1}{4}(p-1)$ arise when p is odd; rectangular ones with $j^+=j^-+\frac{1}{2}=\frac{1}{4}p$ arise when p is even. Similarly, if even n_z states are selected, the multiplicities suggest that even p yields square representations with $j^+=j^-=\frac{1}{4}p$ and odd p yields rectangular ones with $j^+=j^-+\frac{1}{2}=\frac{1}{4}(p+1)$. We now obtain explicitly the generators of transitions within these sets of states and show that they satisfy the commutation rules of $O(4)$.

The $SU(2)$ subgroup of the oscillator associated with the xy plane remains after imposition of the reflection symmetry in the z direction, and it seems likely that those generators correspond to the angular momentum **J** needed in $O(4)$. We thus expect that

$$\mathbf{J}=\mathbf{T}. \qquad (9)$$

The operator K_3 has the selection rules $\Delta j=\pm 1, 0$ and $\Delta m=0$. Thus we are led to write

$$K_3=R_-S_+\Gamma_1+\Gamma_1S_-R_++J_3\Gamma_0. \qquad (10a)$$

[5] R. T. Sharp and H. von Baeyer, J. Math. Phys. 7, 1105 (1966).

[6] See, for example, P. Roman, *Theory of Elementary Particles* (North-Holland Publishing Company, Amsterdam, 1964).

TABLE II. States of the three-dimensional harmonic oscillator in Cartesian representation $(n_xn_yn_z)$. In (a), the etc. indicates permutations. The states in (b) are those remaining after inserting a wall in the x direction. The first column indicates the multiplicity of the level.

Mult.	Cartesian representation of states
	(a) Full three-dimensional oscillator
21	500 etc., 410 etc., 320 etc., 311 etc., 221 etc.
15	400 etc., 310 etc., 220 etc., 211 etc.
10	300 etc., 210 etc., 111
6	200 etc., 110 etc.
3	100 etc.
1	000
	(b) Remaining states
9	500, 320, 311, 302, 140, 131, 122, 113, 104
6	310, 301, 130, 121, 112, 103
4	300, 120, 111, 102
2	110, 101
1	100

The first term in the expression for K_3 increases j, the second decreases j, the third leaves j unchanged. The Γ factors, which depend on the diagonal operators, are expected from the experience of Sec. IIA; they are determined by requiring that the matrix elements of K_3 as calculated from the form (10a) agree with those given by Sharp.[7] Their values for the four sets of $O(4)$ representations are given in Table III.

From Eq. (10a) we deduce

$$K_+=R_-{}^2\Gamma_1-\Gamma_1S_-{}^2+J_+\Gamma_0, \qquad (10b)$$

$$K_-=-S_+{}^2\Gamma_1+\Gamma_1R_+{}^2+J_-\Gamma_0. \qquad (10c)$$

This follows from the fact that the respective terms of the three parts of Eq. (10) are the components of vectors with respect to **J**. That the triads **K** and **J** satisfy the correct commutation relations follows from the fact that they have been chosen to have the correct matrix elements; this can be verified by simple algebra. Thus the selected states are bases for representations of an $O(4)$ higher symmetry.

C. Hydrogen Atom: $O(4)\rightarrow SU(3)$

Before bisecting it, we recall some well-known facts about the bound nonrelativistic hydrogen atom without spin.[2] The constancy of the Lenz vector **K** is responsible for its $O(4)$ symmetry. In fact the commutation rules of **K**, suitably normalized, and the angular momentum **J** are just those of the $O(4)$ generators **K**, **J** described in Sec. IIB. Because **J** is perpendicular to **K**, the hydrogen atom is restricted to "square" representations, for which $j^+=j^-$; for square representations, we write $q=2j^+=2j^-$. Then $0\leq j\leq q$; the dimension is $(q+1)^2$.

From the group point of view, the bound hydrogen atom is equivalent to a particle constrained to move on a four-dimensional hypersphere. The particle on a hypersphere is also restricted to square representations.

[7] R. T. Sharp, J. Math. Phys. (to be published).

TABLE III. Representations (j^+,j^-) and factors associated with the generators of the $O(4)$ symmetry [see Eq. (10)] formed by selecting states of an original $SU(3)$ symmetry, also the higher symmetry of a three-dimensional harmonic oscillator. They are expressed in terms of p, the total occupation number, and T, the isospin of the original $SU(3)$. The selection, a reflection condition along the z axis, is as follows: (1) and (2) have odd symmetry, (3) and (4) have even symmetry; (2) and (3) have even p, (1) and (4) have odd p.

Set	j^+,j^-	Γ_1	Γ_0
(1)	$j^+=j^-=\frac{1}{4}(p-1)$	$-\frac{1}{2}\left[\frac{p+2T+3}{(2T+1)(2T+3)(p-2T)}\right]^{1/2}$	0
(2)	$j^+=j^-+\frac{1}{2}=\frac{1}{4}p$	$-\frac{1}{4}\left[\frac{p+2T+3}{(T+1)^2(p-2T)}\right]^{1/2}$	$\frac{p+1}{4T(T+1)}$
(3)	$j^+=j^-=\frac{1}{4}p$	$-\frac{1}{2}\left[\frac{p+2T+4}{(2T+1)+(2T+3)(p-2T-1)}\right]^{1/2}$	0
(4)	$j^+=j^-+\frac{1}{2}=\frac{1}{4}(p+1)$	$-\frac{1}{4}\left[\frac{p+2T+4}{(T+1)^2(p-2T-1)}\right]^{1/2}$	$\frac{p+2}{4T(T+1)}$

The introduction of an impenetrable barrier through the center of the hydrogen atom removes the states for which $j-m$ is even. Figure 2 illustrates the situation for $q=3$. The states for which $j-m$ is odd are marked O, those with $j-m$ even are marked ×. The pattern strongly suggests that the square representation q of $O(4)$ breaks up into two triangular representations $(q-1, 0)$ and $(q,0)$ of $SU(3)$. We shall verify this surmise by expressing the $SU(3)$ generators in terms of the $O(4)$ generators.

The $SU(3)$ isospin generators $T_\pm$, T_3 are readily expressed in terms of $J_\pm$, J_3, using the methods of Sec. IIA. Rows 2 ($j-m$ odd) and 3 ($j-m$ even) of Table II, with the substitution $T\to J$, $S\to T$, show the relationship. The hypercharge Y is diagonal and may be written down by inspection. The generators R_-, S_- are chosen as the linear combinations of K_+ and K_0J_+ which move a state to the right and up, or down, respectively; a factor Γ is supplied to give the correct matrix element of the generator. R_+, S_+ are the Hermitian conjugates of R_-, S_-. The $SU(3)$ commutation relations are necessarily satisfied since the generators have the prescribed matrix elements.

The results for the generators are

$$R_-=R_+^*=\{K_+(j+m+1)-K_0J_+\}\Gamma_R\,,\qquad\qquad (11)$$
$$S_-=S_+^*=\{K_+(j-m)+K_0J_+\}\Gamma_S\,.$$

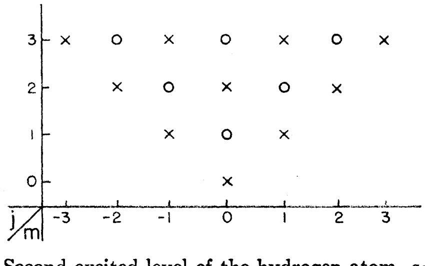

FIG. 2. Second excited level of the hydrogen atom, $q=3$ in our notation, in the (j,m) representation. The states marked with a cross, removed by a barrier in the xy plane, form a (3,0) basis of $SU(3)$. The remaining states, marked with a circle, form a (2,0) basis.

For the $SU(3)$ representation $(q-1, 0)$, we find

$$Y=2T-\tfrac{2}{3}(q-1)\,,$$
$$\Gamma_R=\left[\frac{2j+3}{2(j+m+2)(q+j+2)(2j+1)}\right]^{1/2},\qquad (12a)$$
$$\Gamma_S=\left[\frac{2j-1}{2(j-m)(q+j+1)(2j+1)}\right]^{1/2},$$

and for the representation $(q,0)$,

$$Y=2T-\tfrac{2}{3}q\,,$$
$$\Gamma_R=\left[\frac{2j+3}{2(j+m+1)(q+j+2)(2j+1)}\right]^{1/2},\qquad (12b)$$
$$\Gamma_S=\left[\frac{2j-1}{2(j-m-1)(q+j+1)(2j+1)}\right]^{1/2}.$$

Selection of hydrogen-atom states of odd or even parity (more precisely odd or even $q-j$) changes the symmetry from $O(4)$ to $SU(3)$ in just the same way as selecting states of odd or even parity in a plane. This is most easily seen by considering the equivalent particle on a hypersphere where the operation is equivalent to a cut perpendicular to the 4 axis.

With no additional computation, the hydrogen atom may be cut in two again by a second plane perpendicular to the first. Again, it is perhaps easier to think in terms of the equivalent particle on a hypersphere. The first cut, perpendicular to the 4 axis, converts the $O(4)$ symmetry to $SU(3)$ and makes the problem equivalent to the three-dimensional oscillator from the group point of view. The second cutting converts the symmetry back to $O(4)$ according to the results of Sec. IIB.

D. Other Examples

Unexplained degeneracies in a physical system have often turned out to be an indication of a hidden symmetry.[8] Many further examples of unexplained degeneracies come to mind.

The hydrogen atom in $n+1$ dimensions (or particle on a hypersphere in $n+1$ dimensions) is known to have the symmetry $O(n+1)$.[3] The levels are labelled by an integer q and have the degeneracies

$$N(q)=\frac{(2q+n-1)(q+n-2)!}{q!(n-1)!}.$$

The separation into states of even and odd parity with respect to one of the axes leads to two sets of levels with degeneracies

$$N_{\text{even}}(q)=\frac{(q+n-1)!}{q!(n-1)!},$$

$$N_{\text{odd}}(q)=\frac{(q+n-2)!}{(q-1)!(n-1)!}.$$

These multiplicities are those of the $(q,0,0\cdots)$ and $(q-1,0,0\cdots)$ representations of $SU(n)$, and this leads us to conjecture that the new symmetry is $SU(n)$. This is just the generalization of the result of Sec. IIC to an arbitrary number of dimensions.

Analogously, we might expect the bisected symmetric n-dimensional oscillator to exhibit $O(n+1)$ symmetry. This does not seem to be the case for $n>3$. Thus the levels of the four-dimensional oscillator, selected for evenness or oddness in one direction, exhibit degeneracies which we do not recognize as those of the representations of any group.

Another generalization of our results is the n-dimensional oscillator or hydrogen atom bisected more than once; or one may restrict the system to the region in the angle between two planes making an angle which is a rational fraction of π. These operations yield systems with unexplained degeneracies for which we do not in general know any symmetry.

The converse question also arises. In a given physical system does the selection of a subset of states with a definite group transformation property guarantee the existence of a corresponding physical symmetry condition on the system? For example, if all rth states are selected from a representation of $SU(2)$ [or $O(3)$], they transform among themselves according to a generally smaller representation of $SU(2)$; the new generators may be constructed from the old in a manner similar to that used in Sec. IIA. But we cannot think of any physical mechanism for selecting such states.

III. DISCUSSION

As a mathematical exercise, the possibility that representations of other groups are related in this or similar ways is interesting to pursue further. Of most topical interest, however, is the relationship between $SU(3)$ and $O(4)$.

In nonrelativstic situations involving a single particle, one would normally conclude from observing *multiplicities* characteristic of $SU(3)$ that the basic dynamical structure was that of the harmonic oscillator. We now see that an alternative conclusion could be that the structure was that of a hydrogen atom with a reflection symmetry. In these two cases, the relative positions of the different energy *levels*, which we know, would decide the matter, of course. But in situations like those encountered with the symmetries of elementary particles, the latter decision is much harder to make. It is tantalizing to think that the observed $SU(3)$ symmetry there might in fact arise from a basic $O(4)$ symmetry, with a strong added selection rule. A highly relativistic situation, as for example is encountered in the Bethe-Salpeter equation with strong binding, can lead naturally to a broken $O(4)$ symmetry.[9] The imposition of a reflection-symmetry rule, for example, connected with time parity, can then select out states which have $SU(3)$ multiplicity, and have energy splittings in a given multiplet. One might suggest that the isospin-hypercharge $SU(3)$ structure arises in such a way, from a dynamics involving a continuous four-dimensional space. Some of the difficulties with such an approach are associated with half-integral values of charges, and with representations with multiple weights.

[8] See, for example, H. V. McIntosh, Am. J. Phys. 27, 620 (1959).

[9] G. C. Wick, Phys. Rev. 96, 1124 (1954).