Equivalent Field Theories and Bound States*

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It was previously shown how one can eliminate an arbitrary number of fields from a general Lagrangian and obtain a new one. Regardless of which Lagrangian was used to evaluate the matrix elements for the elementary fields, equivalent results were obtained. It is the principal object of this paper to investigate the matrix elements of the so-called redundant fields, i.e., the fields that do not explicitly appear in the Lagrangian. We see that complete agreement among all matrix elements calculated with either of the Lagrangians is achieved only in the limit that the Lagrangian becomes local. It is also shown how one can construct an expression for the redundant fields in terms of the elementary fields. In the limit that $Z_3 \to 0$, this expression for the redundant fields agrees with those discussed by Nishijima, Zimmermann, and Haag.

I. INTRODUCTION

THE numerous resonances in elementary particle physics forces one to study all theoretical facets of composite particles. In particular, for a Lagrangian field theory to treat all these resonances as elementary is unrealistic. That is, to introduce an elementary field in the Lagrangian for each experimental resonance would make the theory absurd; it would degenerate to the status of a mere phenomenological model—a theory which would not predict charge, mass, or other intrinsic properties of elementary particles. On the other hand, if one is to treat some or all of the particles as composite, then the interaction must be more complex than the historical Yukawa coupling. To perform calculations with such complex couplings becomes very difficult if not infeasible. Therefore, one must in some way approximate these interactions so that the properties of the bound states are not lost, yet the problem is tractable. This task, needless to say, is a very formidable one which cannot be approached in a trivial way, but a solution to the problem must be obtained if field theory is to survive as a useful tool in high-energy physics. This is one in a series of articles that, in a modest way, is attempting to shed some light on the above problem by studying the properties of bound states.

In a previous article[1] a method for studying composite particles was developed. It was shown how to eliminate an arbitrary number of fields from a general Lagrangian and obtain a new Lagrangian. We referred to the eliminated fields as redundant fields. These redundant fields correspond to composite states. That is to say, we call a particle or field composite if there does not explicitly appear a field or linear combination of fields for it in the Lagrangian. This definition of compositeness is certainly the most obvious in the context of a Lagrangian field theory. We see that compositeness is definitely a function of the Lagrangian. In fact, a particle may be considered composite for one Lagrangian, whereas an equivalent theory can be constructed in which the particles are elementary. Examples of such Lagrangians have been discussed.[1,2] The Lagrangians considered in the previous references were, in general, nonlocal. Upon requiring them to be local, we showed that the wave-function renormalization constant vanished. It is this limit, i.e., $Z_3 \to 0$, that is usually meant when a particle is considered to be composite.

This procedure of constructing equivalent theories is very useful in discussing the properties of composite states. In Ref. 1 it was shown how one could construct equivalent field theories in which the particle was considered as composite in one case and as elementary in the other case. What was meant by equivalence was that the matrix elements for the elementary fields in the two theories were the same. Nothing was ever said about the equivalence of the redundant or composite states in the two theories. The purpose of this present paper is to investigate the matrix elements containing the redundant fields.

In order to evaluate the matrix elements containing the redundant fields, we must find an expression for them in terms of the elementary fields. The construction of such an operator for the redundant fields is analogous to the work of Nishijima, Zimmermann, Haag,[3] and others where they studied bound-state operators in the Heisenberg representation. The construction of operators for the redundant fields may be obtained by comparing the equations of motion calculated from the two different Lagrangians.

With this expression for the redundant fields, its various matrix elements will be evaluated. Comparisons will then be made with the analogous matrix elements calculated with the original Lagrangian, that is, from the Lagrangian where the fields have not been eliminated. It will be seen that in general these matrix elements calculated from the two different Lagrangians will not agree. This is in contrast to the matrix elements for the elementary fields. However, in the special limit that the Lagrangian is made local, the matrix elements for the redundant fields will agree in the two cases.

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[1] Robert L. Zimmerman, Phys. Rev. **141**, 1554 (1966).

[2] Robert L. Zimmerman, Phys. Rev. **146**, 955 (1966).

[3] K. Nishijima, Progr. Theoret. Phys. (Kyoto) **10**, 549 (1953); **12**, 279 (1954); **13**, 305 (1955); W. Zimmermann, Nuovo Cimento, **10**, 597 (1958); R. Haag, Phys. Rev. **112**, 669 (1958).

Only in the limit that the new Lagrangian is made local will the two Lagrangians be truly equivalent. That is to say, only in this limit is agreement achieved for *all* matrix elements calculated from the two different Lagrangians.

The outline of the paper is as follows. In Sec. II we present a short review of the construction of equivalent Lagrangians and their properties. We then obtain an expression for the redundant fields in terms of the elementary fields. In Sec. III we calculate the matrix elements for the scalar field from a Lagrangian in which the scalar field is elementary. In Sec. IV the matrix elements for the scalar field are constructed from a Lagrangian in which they are redundant or composite. Comparisons between the matrix elements of Secs. III and IV are made and the conclusion follows in Sec. V.

II. EQUIVALENT LAGRANGIANS

In this section we will briefly review the method of eliminating various fields from a Lagrangian and the construction of an expression for the redundant fields in terms of elementary fields.

Let us consider the Lagrangian density

$$L=L_0(\phi_1\cdots\phi_n,\psi_1\cdots\psi_m,\bar\psi_1\cdots\bar\psi_m)+L_i(\phi_1\cdots\phi_n,\psi_1\cdots\psi_m,\bar\psi_1\cdots\bar\psi_m)\,,\quad(1)$$

where L_0 is the noninteracting Lagrangian density and L_i is the interacting Lagrangian density.

The physical vacuum expectation value of a time-ordered product can be expressed as

$$\langle 0|T(A_1(x_1)A_2(x_2)\cdots A_l(x_l))|0\rangle =\frac{\int A_1(x_1)\cdots A_l(x_l)\exp[i\int d^4x\,L(\phi_1\cdots\phi_n,\psi_1\cdots\psi_m,\bar\psi_1\cdots\bar\psi_m)]\delta\phi_1\cdots\delta\phi_n\delta\psi_1\cdots\delta\psi_m\delta\bar\psi_1\cdots\delta\bar\psi_m}{\int\exp[i\int d^4x\,L(\phi_1\cdots\phi_n,\psi_1\cdots\psi_m,\bar\psi_1\cdots\bar\psi_m)]\delta\phi_1\cdots\delta\phi_n\delta\psi_1\cdots\delta\psi_m\delta\bar\psi_1\cdots\delta\bar\psi_m}\,,\quad(2)$$

where $A_1(x_1),A_2(x_2)\cdots A_l(x_l)$ are any set of field operators acting at the space-time points $x_1,x_2\cdots x_l$.

Let us define a new Lagrangian density

$$L_{\cdots\sigma_i\cdots,\psi_e\cdots,\psi_e\cdots}(\phi_1\cdots\phi_{i-1}\phi_{i+1}\cdots\phi_n,\psi_1\cdots\psi_{e-1},\psi_{e+1}\cdots\bar\psi_1\cdots\bar\psi_{e-1},\bar\psi_{e+1}\cdots)$$

by the relation

$$\exp[id^4x\,L_{\cdots\phi_1\cdots,\psi_e\cdots,\bar\psi_e\cdots}(\;)]=\frac{\int\exp[i\int d^4x\,L(\phi_1\cdots\phi_n,\psi_1\cdots\psi_m,\bar\psi_1\cdots\bar\psi_m)]\cdots\delta\phi_i\cdots\delta\psi_e\cdots\delta\bar\psi_e}{\int\exp[i\int d^4x\cdots L_0(\phi_i)\cdots L_0(\psi_e\bar\psi_e)]\cdots\delta\phi_i\cdots\delta\psi_e\cdots\delta\bar\psi_e}\,.\quad(3)$$

The Lagrangian density

$$L_{\cdots\phi_i\cdots,\psi_e\cdots,\bar\psi_e\cdots}(\phi_1\cdots\phi_{i-1},\phi_{i+1}\cdots\psi_{e-1}\psi_{e+1}\cdots\bar\psi_{e-1}\bar\psi_{e+1}\cdots)$$

is a function of only the fields appearing in the parentheses. We have integrated over the fields that are indicated by the subscripts and they will be referred to as the redundant fields.

Let us consider the case where all the $A_i(x_i)$ do not contain the redundant fields. It was then shown in Ref. 1 that the evaluation of the matrix element $\langle 0|T(A_1\cdots A_N)|0\rangle$ was the same whether or not we used the original Lagrangian density in Eq. (1) or the new one in Eq. (3). We now consider the question whether or not the evaluation of $\langle 0|T(A_1\cdots A_N)|0\rangle$ is still independent of the two Lagrangians when the $A_i(x_i)$ are allowed to contain the redundant fields.

For the sake of clarity the remainder of the paper is limited to a discussion of a special class of Lagrangians and redundant fields. The general case contains new problems and will only obscure the present discussion. The problems that arise in the more general case will be discussed elsewhere.

Let us limit our considerations to the case where the Lagrangian density is of the form

$$L(\phi,\psi,\bar\psi)=+\bar\psi(x)(\not p+m_0)\psi(x)+\tfrac12\phi(x)(\Box-\mu_0^2)\phi(x)+g_0J(x)\phi(x)\,,\quad(4)$$

and $J(x)$ is only a function of the field ψ and $\bar\psi$. We could now consider ψ and $\bar\psi$ to be redundant fields and eliminate them from Eq. (4) by means of (3). In the following, however, we will limit ourselves to the case where ϕ is considered the redundant field. This corresponds to the more familiar case where the scalar field is a bound state of fermions and antifermions.

Using Eq. (3) and eliminating $\phi(x)$ from the Lagrangian in Eq. (4), we obtain

$$L_\phi(J(x))=\bar\psi(x)(\not p+m)\psi(x)+\tfrac12 g_0^2\int J(x)\Delta(x-x')J(x')d^4x'\,,\quad(5)$$

where

$$\Delta(x-x')=\frac{1}{(2\pi)^4}\int\frac{e^{ip\cdot(x-x')}}{p^2+\mu_0^2-i\epsilon}d^4p\,.$$

As was shown in the previous article,[1] if the $A_i(x_i)$ do not contain $\phi(x)$, then the evaluation of the matrix element $\langle 0|T(A_1\cdots A_l)|0\rangle$ is the same whether or not we used the Lagrangian density in Eqs. (5) or (4). In the next two sections we will investigate the matrix elements $\langle 0|T(A_1\cdots A_l)|0\rangle$ when $A_i(x_i)$ is a function of the redundant field ϕ. We will evaluate it with the Lagrangians in Eqs. (4) and (5) and compare their results. We will see that these matrix elements are equivalent only in the limit that the Lagrangian density in Eq. (5) becomes local; that is, in the limit that the bare charge and mass go to infinity in such a way that their ratio is finite:

$$\lambda_0 = \lim_{\mu_0\to\infty,\mu_0\to\infty} g_0^2/\mu_0^2 < \infty\,. \tag{6}$$

In this limit, the Lagrangian density in Eq. (5) becomes

$$L_\phi(J(x)) = \bar{\psi}(x)(p+m)\psi(x) + \lambda_0 J(x)J(x)\,. \tag{7}$$

As is the case with all local field theories, the Lagrangian density in Eq. (7) involves products of field operators at the same space-time point. Consequently, its meaning is ambiguous and divergent equations may result. In order to give it meaning we must displace the various coordinates of the operators in the term $J(x)J(x)$. For example, if

$$J(x) = \bar{\psi}(x)\Gamma\psi(x)\,, \tag{8}$$

then this will be understood to signify implicitly

$$J(x) = \bar{\psi}(x+\zeta)\Gamma\psi(x-\zeta)\,, \tag{9}$$

where ζ is an infinitesimal displacement. The limiting case of $\zeta \to 0$ may then be considered for appropriate ratios where the divergences cancel.

In order to evaluate the matrix elements containing the redundant fields by means of the Lagrangian density in Eq. (5) or (7), we must be able to express the redundant fields $\phi(x)$ in terms of the elementary fields $\bar{\psi}(x)$ and $\psi(x)$. So let us show how one can get such an expression for the redundant field.

The relation for the redundant field $\phi(x)$ in terms of the elementary fields $\bar{\psi}(x)$ and $\psi(x)$ may be obtained by equating the interacting terms in the equations of motion derived from the interacting Lagrangian densities in Eqs. (4) and (5). In this manner, we obtain the following expression for the redundant field $\phi(x)$ in terms of elementary fields implicitly contained in the operator $J(y)$:

$$\phi(x) = \tfrac{1}{2} g_0 \int \Delta(x-y) J(y) d^4y\,. \tag{10}$$

If more care had been exercised in the notation, our Lagrangians should have normal-ordering signs about them. If this is done, then Eq. (10) should read

$$\phi(x) = \tfrac{1}{2} g_0 \int \Delta(x-y) :J(y): d^4y\,. \tag{11}$$

The normal-ordering sign is necessary here in order that

$$\langle 0|\phi(x)|0\rangle = 0\,. \tag{12}$$

This expression corresponds to the unrenormalized field. We can introduce the renormalized field in the usual manner:

$$\phi_R(x) = \phi(x)/Z_3\,, \tag{13}$$

where

$$Z_3 = \langle 0|\phi(0)|b,P\rangle (2\pi)^{3/2}(2P_0)^{1/2}\,, \tag{14}$$

and using Eq. (11) we may express Z_3 by

$$\sqrt{Z_3} = \tfrac{1}{2} g_0 (2\pi)^{3/2} 2P_0 \int \Delta(-y) \langle 0| :J(y): |b,P\rangle\,. \tag{15}$$

$|b,P\rangle$ stands for the one-particle eigenstate of the field $\phi(x)$ with energy momentum P_μ; b indicates any other quantum numbers that are necessary to characterize the state. Therefore, the operator for the renormalized field $\phi_R(x)$ becomes

$$\phi_R(x) = \frac{\int \Delta(x-y) :J(y): d^4y}{(2\pi)^{3/2}(2P_0)^{1/2} \int \Delta(-y)\langle 0| :J(y): |b,P\rangle}\,. \tag{16}$$

In the limit that makes the Lagrangian density in Eq. (5) local, we obtain

$$\phi_R(x) = \frac{:J(y):}{(2\pi)^{3/2}(2P_0)^{1/2}\langle 0| :J(0,\zeta): |b,P\rangle}\,. \tag{17}$$

Recalling that the product of operators at the same space-time point is not defined, Eq. (17) really means

$$\phi_R(x) = \lim_{\zeta\to 0} \frac{:J(y,\zeta):}{(2\pi)^{3/2}(2P_0)^{1/2}\langle 0| :J(0,\zeta): |bP\rangle}\,, \tag{18}$$

where the operators in $J(y,\zeta)$ are not at the same space-time points.

This result is a generalization of the bound-state operator studied by Nishijima, Zimmermann, and Haag.[3]

III. GREEN'S FUNCTIONS FOR ELEMENTARY FIELDS

In this section we are going to consider those matrix elements containing boson fields and evaluate them by means of the Lagrangian density in Eq. (4). Our principal concern is to compare these matrix elements with the equivalent ones calculated in the next section from the Lagrangian density in Eq. (5). In order to draw a comparison it will be necessary to recast the matrix elements in this section into a form not containing boson fields. This can conveniently be done by means of the

Matthews-Salam equations[4]

$$i(\Box+m^2)\langle 0|T(\phi(x)A_1\cdots A_l)|0\rangle =\langle 0|T\{[\delta/\delta\phi(x)]A_1\cdots A_l\}|0\rangle +ig_0\langle 0|T(J(x)A_1\cdots A_l)|0\rangle. \quad (19)$$

Let us now express the tau functions containing boson fields into a form of only fermion fields.

The simplest matrix element containing the boson fields is

$$\langle 0|T(\phi(x)A_1(x_1)\cdots A_l(x_l))|0\rangle, \quad (20)$$

where the $A_1(x_1)$ are not functions of $\phi(x)$.

Using the Matthews-Salam relations, (20) becomes

$$1(\Box+m^2)_x\langle 0|T(\phi(x)A_1(x_1)\cdots A_l(x_l))|0\rangle =g_0\langle 0|T(J(x)A_1(x_1)\cdots A_l(x_l))|0\rangle. \quad (21)$$

Using the Green's function

$$i(\Box+m^2)\Delta(x-y)=\delta^4(x-y), \quad (22)$$

we can invert Eq. (21) to give

$$\langle 0|T(\phi(x)A_1(x_1)\cdots A_l(x_l))|0\rangle=ig_0\int d\omega_1\Delta(x-\omega_1) \times\langle 0|T(J(\omega_1)A_1(x_1)\cdots A_l(x_l))|0\rangle. \quad (23)$$

A convenient expression of Eq. (23) in terms of renormalized fields $\phi_R(x)$ can be obtained by noticing that

$$\phi_R(x)=\phi(x)/\sqrt{Z_3}, \quad (24)$$

where

$$\sqrt{Z_3}=(2\pi)^{3/2}(2P_0)^{1/2}\langle 0|\phi(0)|b,P_\mu\rangle. \quad (25)$$

Analogously to Eq. (15), $\sqrt{Z_3}$ can be expressed as

$$\sqrt{Z_3}=ig_0(2\pi)^{3/2}(2P_0)^{1/2} \times\int d\omega_1\Delta(-\omega_1)\langle 0|T(J(\omega_1))|b,P_\mu\rangle. \quad (26)$$

Substituting Eqs. (24) and (26) into Eq. (23), we get the matrix element for the renormalized field:

$$\langle 0|T(\phi_R(x)A_1(x_1)\cdots A_l(x_l))|0\rangle=\frac{\int d\omega_1\Delta(x-\omega_1)\langle 0|T(J(\omega_1)A_1(x_1)\cdots A_l(x_l))|0\rangle}{(2\pi)^{3/2}(2P_0)^{1/2}\int d\omega_1\Delta(-\omega_1)\langle 0|T(J(\omega_1))|b,P_\mu\rangle}. \quad (27)$$

We have succeeded in expressing the matrix element containing one renormalized boson field in terms of only fermion fields. This is the form needed for comparison with the results of the next section.

Let us now consider the next hardest matrix element, that is, the matrix element containing two boson fields:

$$\langle 0|T(\phi(x)\phi(y)A_1(x_1)\cdots A_l(x_l))|0\rangle. \quad (28)$$

As above, these boson fields may be eliminated, with the aid of the Matthews-Salam relation, to obtain

$$(\Box+m^2)_x(\Box+m^2)_y\langle 0|T(\phi(x)\phi(y)A_1(x_1)\cdots A_l(x_l))|0\rangle=-i(\Box+m^2)_y\delta^4(x-y)\langle 0|T(A_1(x_1)\cdots A_l(x_l))|0\rangle +g_0^2\langle 0|T(J(x)J(y)A_1(x_1)\cdots A_l(x_l))|0\rangle. \quad (29)$$

Converting the differential equation into an integral equation, we get

$$\langle 0|T(\phi(x)\phi(y)A_1(x_1)\cdots A_l(x_l))|0\rangle=\Delta(x-y)\langle 0|T(A_1(x_1)\cdots A_l(x_l))|0\rangle \times g_0^2\int\int d\omega_1 d\omega_2\Delta(x-\omega_1)\Delta(y-\omega_2)\langle 0|T(J(\omega_1)J(\omega_2)A_1(x_1)\cdots A_l(x_l))|0\rangle. \quad (30)$$

In terms of the renormalized field $\phi_R(x)$, Eq. (30) becomes

$$\langle 0|T(\phi_R(x)\phi_R(y)A_1(x_1)\cdots A_l(x_l))|0\rangle=Z_3\Delta(x-y)\langle 0|T(A_1(x_1)\cdots A_l(x_l))|0\rangle -\frac{g_0^2}{Z_3}\int d\omega_1 d\omega_2\Delta(x-\omega_1)\Delta(y-\omega_2)\langle 0|T(J(\omega_1)J(\omega_2)A_1(x_1)\cdots A_l(x_l))|0\rangle, \quad (31)$$

where Z_3 is given by Eq. (26).

In the same manner we may obtain an expression for an arbitrary matrix element in terms of only ψ and $\bar{\psi}$. After this is done, then the comparison of the various matrix elements calculated by means of the Lagrangian density in Eqs. (5) and (4) is trivial. The reason it is trivial is that, as was shown in Ref. (1), the matrix elements containing only fermion fields are the same whether Eqs. (5) or (4) are used to evaluate them.

IV. GREEN'S FUNCTION FOR REDUNDANT FIELDS

In this section let us consider the matrix elements containing the redundant field and evaluated by the Lagrangian density in (5). Since the Lagrangian density contains only fermion fields, the redundant field in the matrix elements must be replaced by some appropriate function of fermion fields. Such a relation between the

[4] P. T. Matthews and A. Salam, Nuovo Cimento, 2, 120 (1955).

redundant and fermion fields is given in (16). Consider the tau function that contains only one redundant field:

$$\langle 0|T(\phi_R(x)A_1(x_1)\cdots A_l(x_l))|0\rangle. \tag{32}$$

Substituting Eq. (16) for $\phi_R(x)$, we obtain

$$\langle 0|T(\phi_R(x)A_1(x_1)\cdots A_l(x_l))|0\rangle=\frac{\int\Delta(x-y)\langle 0|T(:J(y):A_1(x_1)\cdots A_l(x_l))|0\rangle d^4y}{(2\pi)^{3/2}(2P_0)^{1/2}\int\Delta(-y)\langle 0|:J(y):|b,P\rangle d^4y}, \tag{33}$$

where it is implicitly implied that the Lagrangian density in (5) is used to evaluate Eq. (33).

The matrix element containing two redundant fields at x and y is given by

$$\langle 0|T(\phi_R(x)\phi_R(y)A_1(x_1)\cdots A_l(x_l))|0\rangle=\frac{\int\Delta(x-\omega_1)\Delta(y-\omega_2)\langle 0|T(:J(\omega_1):J(\omega_2):A_1(x_1)\cdots A_l(x_l))|0\rangle d^4\omega_1d^4\omega_2}{(2\pi)^3 2P_0\int\Delta(-\omega_1)\Delta(-\omega_2)\langle 0|:J(\omega_1):|b,P\rangle\langle 0|:J(\omega_2):|b,P\rangle d^4\omega_1d^4\omega_2}. \tag{34}$$

In a similar manner the expression of all higher-order matrix elements containing redundant fields may be constructed.

Let us now compare the results of the matrix elements discussed in this and the previous section. Compare Eq. (27) with Eq. (33):

$$\langle 0|T(\phi_R(x)A_1(x_1)\cdots A_l(x_l))|0\rangle=\frac{\int\Delta(x-\omega_1)\langle 0|T(:J(\omega_1):A_1(x_1)\cdots A_l(x_l))|0\rangle d^4\omega_1}{(2\pi)^{3/2}2P_0\int\Delta(-\omega_1)\langle 0|:J(\omega_1):|b,P\rangle d^4\omega_1} \tag{33}$$

$$=\frac{\int\Delta(x-\omega_1)\langle 0|T(J(\omega_1)A_1(x_1)\cdots A_l(x_l))|0\rangle d^4\omega_1}{(2\pi)^{3/2}2P_0\int\Delta(-\omega_1)\langle 0|T(J(\omega_1))|b,P\rangle d^4\omega_1}, \tag{27}$$

and Eq. (31) with Eq. (34):

$$\langle 0|T(\phi_R(x)\phi_R(y)A_1(x_1)\cdots A_l(x_l))|0\rangle=\frac{\int\Delta(x-\omega_1)\Delta(y-\omega_2)\langle 0|T(:J(\omega_1):J(\omega_2):A_1(x_1)\cdots A_l(x_l))|0\rangle d^4\omega_1d^4\omega_2}{(2\pi)^3 2P_0\int\Delta(-\omega_1)\Delta(-\omega_2)\langle 0|:J(\omega_1):|bP\rangle\langle 0|:J(\omega_2):|bP\rangle d^4\omega_1d^4\omega_2} \tag{34}$$

$$=\frac{-\Delta(x-y)\langle 0|T(A_1(x_1)\cdots A_l(x_l))|0\rangle}{g_0^2(2\pi)^3 2P_0\int\Delta(-\omega_1)\Delta(-\omega_2)\langle 0|T(J(\omega_1))|b,P\rangle\langle 0|T(J(\omega_2))|b,P\rangle d^4\omega_1d^4\omega_2}$$

$$+\frac{\int\Delta(x-\omega_1)\Delta(x-\omega_2)\langle 0|T(J(\omega_1)J(\omega_2)A_1(x_1)\cdots A_l(x_l))|0\rangle d^4\omega_1d^4\omega_2}{(2\pi)^3 2P_0\int\Delta(-\omega_1)\Delta(-\omega_2)\langle 0|T(J(\omega_1))|bP\rangle\langle 0|T(J(\omega_2))|bP\rangle d^4\omega_1d^4\omega_2}. \tag{31}$$

Recall that Eqs. (27) and (31) are evaluated by means of the Lagrangian density in Eq. (4), whereas Eqs. (33) and (34) are evaluated by the Lagrangian density in (5). However, since all the matrix elements contain only elementary fermion fields, their evaluation is the same whether or not the Lagrangian density in Eq. (4) or (5) is used.

From these observations we see that Eqs. (33) and (27) are equivalent. The only difference is the normal-ordering signs, and these just make the product of the operators at the same space-time point well defined. However, Eqs. (34) and (31) do not agree and, in general, neither will the higher-order matrix elements. With a little consideration this result is to be expected. In deriving the Lagrangian density in Eq. (5), we have decoupled it from the boson field. The information concerning the boson field has been neglected. Therefore we would only expect to obtain agreement when the boson field does not affect the results. This situation is realized in the limit where Z_3 vanishes.

Let us now consider the limit where Z_3 vanishes. We force the Lagrangian density in Eq. (5) to be local, i.e.,

$$g_0^2,\mu_0^2\rightarrow\infty,$$

such that

$$g_0^2/\mu_0^2\rightarrow\lambda_0<\infty$$

implies

$$g_0^2\Delta(x-y)\underset{x\neq y}{\longrightarrow}0.$$

Therefore, in the limit that $Z_3\rightarrow 0$, the first term in Eq. (31) drops out and equivalence with Eq. (34) is established. In a similar manner, all higher-order matrix elements of the redundant field agree in this limit.

This finishes the proof that all matrix elements calculated from either the Lagrangian in (4) or (5) are equivalent in the limit that $Z_3\rightarrow 0$. For $Z_3\neq 0$, only the matrix elements for the elementary fields are the same.

V. CONCLUSION

In Secs. III and IV we considered the specific interacting Lagrangian densities

$$L_i(\phi,J)=g_0J(x)\phi(x) \tag{35}$$

and

$$L_{i\phi}(J)=\tfrac{1}{2}g_0{}^2\int J(x)\Delta(x-x')J(x')d^4x'. \qquad (36)$$

In (35) the boson field $\phi(x)$ is treated as an elementary field. However, in (36) the field $\phi(x)$ is treated as a composite state. That is to say, the field $\phi(x)$ does not appear explicitly in the Lagrangian but manifests itself as a pole in the various tau functions of the fermion fields.

The Lagrangians discussed in this paper yield bound states that appear as poles in the various tau functions. This type of bound state is by far the simplest and most familiar. More complicated Lagrangians will yield bound states that will not appear as poles in the tau functions. Such bound states will manifest their presence as branch points, and will only confuse the context of this paper.

In Sec. II an expression for the bound-state operator was constructed from a knowledge of the two equivalent Lagrangians. Matrix elements of these bound-state operators did not in general agree with the ones calculated where $\phi(x)$ was treated as an elementary particle. This observation raises an interesting question as to whether or not the theory uniquely determines the properties of bound states. Only in the limit that $Z_3\rightarrow 0$ did *all* matrix elements become equivalent. Agreement might be expected in this limit because the effects of the free-field Lagrangian on the matrix elements vanish in this limit. It was just this term that accounted for the difference in the various matrix elements.

Closely related to the question of uniqueness is the construction of bound-state operators. We were able to construct such operators only because of a prior knowledge of the two equivalent Lagrangians. In general one does not have this information. It is usually the case that only the bound-state Lagrangian is given. Therefore, it is not a trivial task to construct such bound-state operators without assuming it is a bound state of a certain kind. These and other questions that were implied in the paper will be discussed elsewhere.

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Connection between $SU(3)$ and $O(4)$*

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It is shown that the insertion of a barrier, in the isotropic harmonic oscillator and in the hydrogen atom, gives each system the higher symmetry usually associated with the other. The result is more general: The imposition of a reflection condition can change $SU(3)$ to $O(4)$, and vice versa. This may have implications for elementary-particle symmetries.

I. INTRODUCTION

In nonrelativistic quantum mechanics, the harmonic oscillator and the hydrogen atom are systems whose energy-level structures possess a higher symmetry than the symmetry of the space in which the motion takes place. There appear degeneracies which indicate invariance of the Hamiltonian under transformations in some higher-dimensional space. As is well known, the invariance group of the three-dimensional harmonic oscillator is $SU(3)$,[1] while that of the hydrogen atom is $O(4)$.[2] The purpose of this paper is to point out that physically selected sets of states of either system possess the higher symmetry usually associated with the other system. For example, under the condition of evenness or oddness on reflection in a plane, the remaining states of the three-dimensional oscillator possess the symmetry of $O(4)$, and those of the hydrogen atom, $SU(3)$. To some extent similar things happen in a space of n dimensions. The n-dimensional harmonic oscillator and the n-dimensional Schrödinger equation with an attractive $1/r$ potential have the invariance groups of $SU(n)$ and $O(n+1)$,[3] respectively. (The latter symmetry is also the natural symmetry of a particle constrained to move on a sphere in $n+1$ dimensions.) The imposition of selection rules under reflection on the "hydrogen atom" lead to degeneracies characteristic of certain representations of $SU(n)$. (The reverse situation is not clear to us.) This reciprocity property appears, in fact, to reside in the group representations themselves, and not only in the physical systems we have used to realize them. The relationship between $SU(3)$ and $O(4)$ may be re-

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[1] See, for example, J. P. Elliott, Proc. Roy. Soc. (London) **A245**, 128 (1958).
[2] A recent discussion is contained in M. Bander and C. Itzykson, Rev. Mod. Phys. **38**, 330 (1966).
[3] S. P. Alliluev, Zh. Eksperim. i Teor. Fiz. **33**, 200 (1957) [English transl.: Soviet Phys.—JETP **6**, 156 (1958)].