Approximation in the Unitarity Condition and the Neutral Scalar Theory*

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A method for the approximate determination of the elastic scattering amplitude is applied to the static neutral scalar meson model of meson-nucleon scattering. It is shown that under appropriate conditions the approximate solution reduces to the exact solution in the static limit of the usual neutral scalar theory. The effect of recoil and more general static interactions on bound-state poles is discussed in relation to difficulties previously encountered in applying this approximation method to the equal-mass case. The latter difficulty can be avoided at the cost of other problems with the bound-state poles.

1. INTRODUCTION

THE problem of computing the scattering amplitude for a process starting from the assumptions of an analytic, unitary, crossing symmetric S matrix[1] has not been satisfactorily solved even for elastic two-body processes. It seems worthwhile to investigate all likely approximations in order to assess their relative strengths and weaknesses. It is the purpose of this paper to continue the investigation of an approximation suggested previously.[2]

One method of checking the validity of an approximation is to apply it to an exactly soluble problem and to compare the exact and approximate solutions. This is the approach adopted in the present paper. In particular, the static neutral scalar model of meson-nucleon scattering can be solved exactly,[3] and recoil corrections have been investigated.[4] In what follows we will apply the approximation of Ref. 2 to the scattering of a neutral scalar meson from a spinless neutral nucleon. In Sec. 2, the approximate solution will be derived and the static limit of the solution and recoil corrections will be determined. The evaluation of the static limit of the solution involves the estimation of a number of integrals; these estimates are made in the Appendix. Section 3 presents the conclusion of the investigation.

2. STATIC LIMIT OF THE APPROXIMATE SOLUTIONS

The basic development of the equations has been made in a previous paper,[2] so only a brief statement of the results obtained there will be given here. We take the meson mass to be μ and the nucleon mass to be M and start with the fixed-momentum-transfer dispersion relation[5]

$$T(s,t)=\frac{g^2}{M^2-s}+\frac{g^2}{M^2-u}+\frac{1}{\pi}\int_{(M+\mu)^2}^{\infty}ds'\,\mathrm{Im}T(s't)\left[\frac{1}{s'-s}+\frac{1}{s'-u}\right]. \quad (1)$$

The approximation made is that

$$\mathrm{Im}T(s,t)=+\frac{\{[s-(M+\mu)^2][s-(M-\mu)^2]\}^{1/2}}{16\pi s}\times K(s,t)\,|T(s,t)|^2, \quad (2)$$

with $K(s,t)$ some known function, so that in general $\mathrm{Im}T$ will not exactly satisfy the unitarity condition. With this assumption the dispersion relation can be solved for $T(s,t)$. The solution is

$$T(s,t)=N(s,t)/D(s,t), \quad (3)$$

where

$$D(s,t)=\Lambda(t)-\frac{(s-u)}{16\pi^2}\int_{(M+\mu)^2}^{\infty}ds'\times\frac{\{[s'-(M+\mu)^2][s'-(M-\mu)^2]\}^{1/2}}{s'(2s'-2M^2-2\mu^2+t)}\times K(s',t)\left[\frac{g^2}{M^2-s'}+\frac{g^2}{s'+t-M^2-2\mu^2}+\lambda(t)\right]\times\left(\frac{1}{s'-s}-\frac{1}{s'-u}\right), \quad (4)$$

and

$$N(s,t)=\frac{g^2}{M^2-s}+\frac{g^2}{M^2-u}+\lambda(t).$$

The residues of the poles at M^2 must be g^2, which gives one condition

$$D(M^2,t)=1 \quad (5)$$

toward determining the two functions $\lambda(t)$ and $\Lambda(t)$ which appear in the solution; the complete determination of these functions must depend on information

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[1] For a description of the foundations and philosophy of this approach see, for example, the article by G. F. Chew, in *Strong Interaction Physics*, edited by M. Jacobs and G. F. Chew (W. A. Benjamin, Inc., New York, 1964).

[2] D. W. Schlitt, Phys. Rev. **137**, B666 (1965).

[3] L. Castillejo, R. H. Dalitz, and F. J. Dyson, Phys. Rev. **101**, 453 (1956); hereafter cited as CDD.

[4] V. Barger and E. Kazes, Phys. Rev. **132**, 896 (1963).

[5] We use the usual Lorentz-invariant variables s, t, and u, where $u=2M^2+2\mu^2-s-t$ and $\hbar=c=1$.

which is not contained in the fixed dispersion relation and unitarity in the s and u channels.

The next step is to investigate the properties of the solution where $M \gg \mu$. The appropriate energy variable for this investigation is the laboratory energy ω defined by

$$s=M^2+\mu^2+2M\omega. \tag{6}$$

If we are to expect the solution to reproduce the static limit of the neutral scalar model, we should make the assumptions that are necessary to reduce Eq. (1) to the Low equations for that model. From previous investigations[4] appropriate restrictions appear to be

$$\mu(\mu/M)^{1/2} \leqslant \omega \leqslant (\mu M)^{1/2}$$

and

$$|t| \leqslant \mu^2.$$

In terms of the new variable ω, the numerator function has the form

$$N=\frac{g^2}{2M\omega}\left[\frac{1}{(1+(t/2M\omega)-(\mu^2/2M\omega))}-\frac{1}{(1+\mu^2/2M\omega)}\right]+\lambda(t). \tag{7}$$

This involves the dimensionless quantities $\mu^2/2M\omega$ and $t/M\omega$. With the above restrictions on ω and t we have

$$(\mu/M)^{3/2} \leqslant (\mu^2/M\omega) \leqslant (\mu/M)^{1/2} \ll 1$$

and

$$|t|/M\omega \leqslant (\mu/M)^{1/2} \ll 1.$$

As a consequence of this, the multinomial theorem can be used to expand the two terms in power series. Making these expansions, the pole terms become

$$\frac{g^2}{2M\omega}\left[\sum_{q_1,q_2=0}^{\infty}\frac{(-1)^{q_2}\Gamma(q_1+q_2+1)}{2^{q_1+q_2}\Gamma(q_1+1)\Gamma(q_2+1)}\left(\frac{\mu^2}{M\omega}\right)^{q_1}\left(\frac{t}{M\omega}\right)^{q_2}-\sum_{q_1=0}^{\infty}\frac{(-1)^{q_1}}{2^{q_1}}\left(\frac{\mu^2}{M\omega}\right)^{q_1}\right].$$

The result involves the combination (g^2/M^2) which is assumed to remain constant as $\mu/M \to 0$ in extracting the static limit. The result of the expansion is

$$N(s,t)=\left(\frac{g^2}{2M^2}\right)\left(\frac{M}{\omega}\right)\times\left[\sum_{q_1,q_2=0}^{\infty}\frac{(-1)^{q_2}\Gamma(q_1+q_2+1)}{2^{q_1+q_2}\Gamma(q_1+1)\Gamma(q_2+1)}\left(\frac{\mu^2}{M\omega}\right)^{q_1}\left(\frac{t}{M\omega}\right)^{q_2}-\sum_{q_1=0}^{\infty}\frac{(-1)^{q_1}}{2^{q_1}}\left(\frac{\mu^2}{M\omega}\right)^{q_1}\right]+\lambda(t). \tag{8}$$

Because of our choice of limits on ω and t,

$$(\mu/M)^{3/2} \leqslant (\mu^2/M\omega) \leqslant (\mu/M)^{1/2};\ |t|/M\omega \leqslant (\mu/M)^{1/2}.$$

The denominator function presents a more complicated problem. In the expansion of the integrand in D it is convenient to expand three pieces separately and then recombine them. First there is the factor

$$\{[s'-(M+\mu)^2][s'-(M-\mu)^2]\}^{1/2}/s' = \frac{(\omega'^2-\mu^2)^{1/2}}{\omega'+\frac{1}{2}M+\mu^2/2M}. \tag{9}$$

No power-series expansion is possible here because the limits on ω' are determined by the range of integration. Next, the bracket with g^2 and λ is the same as the numerator function except ω' replaces ω; so we use Eq. (8). Finally there is the combination

$$\frac{(s-u)}{(2s'-2M^2-2\mu^2+t)}\left[\frac{1}{(s'-s)}-\frac{1}{(s'-u)}\right]=\frac{2\omega^2}{2M\omega'(\omega'^2-\omega^2)}+\frac{1}{2M\omega'}\sum_{q_1=2}^{\infty}\frac{(-1)^{q_1}(1-2^{1-q_1})}{2^{q_1}}\left(\frac{t}{M\omega'}\right)^{q_1}-\frac{1}{2M(\omega'+\omega)}\sum_{q_1,q_2=1}^{\infty}\frac{(-1)^{q_1}\Gamma(q_1+2)}{2^{q_1}\Gamma(q_2+1)\Gamma(q_1-q_2+2)}\times\left(\frac{t}{M(\omega+\omega')}\right)^{q_1}\left(\frac{\omega}{\omega'}\right)^{q_2}. \tag{10}$$

This result is obtained by expansion in terms of $(t/M\omega')$ and $[t/M(\omega+\omega')]$ and then recombining the terms. The recombining could be performed after substitution in the integrand, in which case factors like (ω/ω'), which are not small, would not appear.

After the above expressions are introduced into the integrand and all the products multiplied out, integrals of two types occur in D. It is convenient to introduce a notation for these integrals. We define

$$i_n=\mu^{n-2}\omega M\int_{\mu}^{\infty}\frac{d\omega'(\omega'^2-\mu^2)^{1/2}K(M^2+\mu^2+2M\omega',\ t)}{(\omega'+\frac{1}{2}M+\mu^2/2M)\omega'^n(\omega'^2-\omega^2)},\quad n\geqslant 0 \tag{11}$$

or, changing variables,

$$i_n=\int_0^1 dx\,\rho(x)\frac{x^nK(M^2+\mu^2+2M\mu/x,\ t)}{[(\mu/\omega)+x][(\mu/\omega)-x]},$$

where

$$\rho(x)=(1-x^2)^{1/2}/[(\mu/M)+(x/2)+(\mu^2x/2M^2)].$$

The second type of integral can be written in terms of

$$j_{nm}=\mu^{n-2}\omega^m M\int_\mu^\infty \frac{d\omega'(\omega'^2-\mu^2)^{1/2}K(M^2+\mu^2+2M\omega',t)}{(\omega'+\frac{1}{2}M+\mu^2/2M)\omega'^n(\omega'+\omega)^m}$$
$$=\int_0^1 dx\,\rho(x)x^{n+m-2}K(M^2+\mu^2+2M\mu/x,t)/(\mu/\omega+x)^m, \quad (12)$$

with $n, m\geqslant 0$, and $n+m>1$.

Notice that the integrals depend on ω and also on t through the t dependence of K. These integrals, and in particular their behavior as $\mu/M\to 0$, are discussed in the Appendix.

After performing the operations indicated above and introducing the notation for the integral, we obtain

$$D(s,t)=\Lambda(t)-\frac{\lambda(t)}{16\pi^2}\Bigg[2\left(\frac{\mu}{M}\right)i_1+\sum_{q_1=2}^{\infty}\frac{(-1)^{q_1}(1-2^{1-q_1})}{2^{q_1}}\left(\frac{\mu}{M}\right)\left(\frac{t}{M\mu}\right)^{q_1}j_{q_1+1,0}$$
$$-\sum_{q_1q_2=1}^{\infty}\frac{(-1)^{q_1}\Gamma(q_1+2)}{2^{q_1}\Gamma(q_2+1)\Gamma(q_1-q_2+2)}\left(\frac{\mu}{M}\right)\left(\frac{t}{M\omega}\right)^{q_1-q_2+1}\left(\frac{t}{M\mu}\right)^{q_2-1}j_{q_2,q_1+1}\Bigg]$$
$$-\frac{1}{16\pi^2}\left(\frac{g^2}{2M^2}\right)\Bigg[\sum_{q_1q_2=0}^{\infty}\frac{(-1)^{q_2}\Gamma(q_1+q_2+1)}{2^{q_1+q_2-1}\Gamma(q_1+1)\Gamma(q_2+1)}\left(\frac{\mu}{M}\right)^{q_1}\left(\frac{t}{M\mu}\right)^{q_2}i_{q_1+q_2+2}$$
$$-\sum_{q_1=0}^{\infty}\frac{(-1)^{q_1}}{2^{q_1-1}}\left(\frac{\mu}{M}\right)^{q_1}i_{q_1+2}+\sum_{q_1q_2=0;\,q_3=2}^{\infty}\frac{(-1)^{q_2+q_3}(1-2^{1-q_3})\Gamma(q_1+q_2+1)}{2^{q_1+q_2+q_3}\Gamma(q_1+1)\Gamma(q_2+1)}\left(\frac{\mu}{M}\right)^{q_1}\left(\frac{t}{M\mu}\right)^{q_2+q_3}j_{q_1+q_2+q_3+2,0}$$
$$-\sum_{q_1=0;\,q_2=2}^{\infty}\frac{(-1)^{q_1+q_2}(1-2^{1-q_2})}{2^{q_1+q_2}}\left(\frac{\mu}{M}\right)^{q_1}\left(\frac{t}{M\mu}\right)^{q_2}j_{q_1+q_2+2,0}$$
$$-\sum_{q_1,q_2=0;\,q_3,q_4=1}^{\infty}\frac{(-1)^{q_2+q_3}\Gamma(q_1+q_2+1)\Gamma(q_3+2)}{2^{q_1+q_2+q_3}\Gamma(q_1+1)\Gamma(q_2+1)\Gamma(q_4+1)\Gamma(q_3-q_4+2)}\left(\frac{\mu}{M}\right)^{q_1}\left(\frac{t}{M\mu}\right)^{q_2+q_4-1}\left(\frac{t}{M\omega}\right)^{q_3-q_4+1}j_{q_1+q_2+q_4+1,q_3+1}$$
$$+\sum_{q_1=0;\,q_2,q_3=1}^{\infty}\frac{(-1)^{q_1+q_2}\Gamma(q_2+2)}{2^{q_1+q_2}\Gamma(q_3+1)\Gamma(q_2-q_3+2)}\left(\frac{\mu}{M}\right)^{q_1}\left(\frac{t}{M\omega}\right)^{q_2-q_3+1}\left(\frac{t}{M\mu}\right)^{q_3-1}j_{q_1+q_3+1,q_2+1}\Bigg]. \quad (13)$$

The final step for this section is to extract the leading terms as $\mu/M\to 0$. This is done using the results of the Appendix, Eqs. (A4), (A5), and (A6). From Eq. (13) we keep only the static limit and the leading recoil corrections. For the numerator function we have

$$N(s,t)\approx\lambda(t)+\frac{g^2}{2M^2}\left(\frac{M}{\omega}\right)\left\{\left[\frac{\mu^2}{M\omega}-\frac{1}{2}\frac{t}{M\omega}\right]+\left[\frac{1}{4}\left(\frac{t}{M\omega}\right)^2-\frac{1}{2}\left(\frac{\mu^2}{M\omega}\right)\left(\frac{t}{M\omega}\right)\right]\right\}. \quad (14)$$

The denominator is

$$D(s,t)\approx\Lambda(t)-\frac{\lambda(t)}{16\pi^2}\left\{2\left(\frac{\mu}{M}\right)i_1+\left(\frac{\mu}{M}\right)\left(\frac{t}{M\omega}\right)j_{1,2}\right\}$$
$$-\frac{1}{16\pi^2}\left(\frac{g^2}{2M^2}\right)\left\{\left[2\left(\frac{\mu}{M}\right)-\left(\frac{t}{M\mu}\right)\right]i_3+\left[\left(\frac{\mu}{M}\right)\left(\frac{t}{M\omega}\right)-\frac{1}{2}\left(\frac{t}{M\mu}\right)\left(\frac{t}{M\omega}\right)\right]j_{3,2}\right\}. \quad (15)$$

Terms of the same order of magnitude are grouped in square brackets and in each curly bracket the largest terms appear first.

3. CONCLUSIONS

First consider the static limit. Even in the limit, the numerator will contain t dependence, so the solution can not be the pure S-wave solution of the usual model. This seems to be a common problem in taking the static limit of dispersion relations. The same effect is present in Ref. 4 where they find that their solutions also contain both S- and P-wave contributions in the static limit.

Since the dispersion relation, Eq. (1), does not contain explicit information about the interaction, while the derivation of the Low equation solved by CDD does depend on the form of the interaction, we argue that the static limit of Eq. (1) may include the effects of more general interactions than the one in the usual static model. In particular, the presence of P-wave contributions in the static limit is a reflection of this. In order to compare the solution (14) and (15) with the solution of CDD we must not only eliminate the recoil terms but, like Barger and Kazes, make the ad hoc assumption that the P wave vanishes. The result of these assumptions is the solution

$$T_{\text{static}}=16\pi^2[16\pi^2\lambda_0-2(\mu/M)i_1]^{-1}, \tag{16}$$

where we have introduced the (now angle-independent) constant

$$\lambda_0=\Lambda/\lambda.$$

Barger and Kazes[6] in similar circumstances find

$$T_{\text{static}}=-\left[B_0+\frac{2\omega^2}{16\pi^2}\int_\mu^\infty \frac{d\omega'(\omega'-\mu^2)^{1/2}R_0(\omega')}{(\omega'+\frac{1}{2}M)\omega'(\omega'^2-\omega^2)}\right]^{-1}, \tag{17}$$

where B_0 is a constant and R_0 can be interpreted as the ratio of the total to elastic S-wave cross section. The two solutions are identical if

$$B_0=-\lambda_0 \quad \text{and} \quad K(s,t)=R_0(\omega). \tag{18}$$

From the definition of K it is seen that the identification is quite reasonable. The solution (16) is also of the form which was obtained in CDD. The conclusion from this is that the approximation method of Ref. 2 does reduce to the exact solution of the static neutral scalar model when it should, but that even in the static limit the solution corresponds to a more general situation than the usual model.

Next it is of some interest to assess the influence of the recoil corrections and the effect of the g^2/M^2 interaction, which was eliminated from the static limit by our assumption of no P-wave scattering. The inclusion of these effects to leading order leads to the approximate solutions

$$N\approx\lambda(t)+(g^2/2M^2)(M/\omega)[(\mu^2/M\omega)^2-(t/2M\omega)],$$

$$D\approx\Lambda(t)+(\lambda(t)/16\pi^2)\times[2(\mu/M)i_1+(\mu/M)(t/M\omega)j_{1,2}], \tag{19}$$

since the other terms in (14) and (15) are of higher order. If this is assumed to be the solution for all t and the condition $D(M^2,t)=1$ is imposed, then

$$\Lambda(t)=1+\lambda(t)O((\mu/M)^3), \tag{20}$$

since i_1 and $j_{1,2}$ are both $O((\mu/M)^3)$ at $s=M^2$.

The point which is of most interest here is the behavior of the zeros of D as t is varied. The zeros of D are of interest in bootstrap arguments, and in Ref. 2 it was observed that the approximation led to zeros which changed location as t was changed.

In the solution (19) it is possible that the zeros will move as t is changed, but the additional freedom provided by the choice of $\lambda(t)$ allows us to avoid this problem. Unfortunately, this is not the end of the discussion, for there is another problem. The numerator has a term linear in t, so even if the t dependence of $\lambda(t)$ and D is of the same order as neglected terms and thus ignorable, there will be both S- and P-wave scattering, and both will share the same D and thus have a pole at the same location. While this is not as undesirable a result as the moving zeros, it is still peculiar.

We are compelled to conclude that the zeros of D present a significant challenge to the validity of the approximation under investigation. It is hoped that the imposition of unitarity will introduce a t dependence in D which will remove the difficulties encountered in Ref. 2. Such a calculation is underway. The problems encountered in the static limit described above have the additional complication that any effect of order $(\mu/M)^2$ must be neglected and further investigation of the problem is probably unwarranted.

A final comment concerns the relation between the solution (19) and the results of Ref. 4. Unlike the static limits in Eqs. (16) and (17), there is no simple identification possible between the two results. If the S-wave part dominates strongly, then the static-limit relations of Eq. (18) should hold and the two will be similar; however, the P-wave parts will be quite different because the integrals in the denominator which give the right-hand cut contribution to the P wave will be very different.

To summarize the conclusions, we have found that, in the static limit, the results of the approximation of unitarity, Barger and Kazes, and CDD agree in an understandable way. The inclusion of recoil corrections introduces difficulties if D is allowed to have zeros. The connection between the solutions derived here and Barger and Kazes's solutions is not easy to establish.

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APPENDIX

In Eqs. (11) and (12) the integrals i_n and $j_{n,m}$ were introduced. It is necessary to know how they behave as $\mu/M\to 0$ throughout the allowed range of ω and t. In doing this we will assume that $K(s',t)$ is a bounded positive-definite function for values of s' and t required in Eq. (13). First, just the integrals will be estimated, and then an improved estimate will be made for some cases, making use of the combination of factors which occur in Eq. (13). Before proceeding with this, we note

[6] Ref. 4, Eq. (30). A difference of a factor of 16π in the definition of T has been taken into account.

two relationships between the integrals:

$$i_{n+1}=(\mu/\omega)i_n-j_{n+1,1} \tag{A1}$$

and

$$j_{n,m-1}=(\mu/\omega)j_{n,m}+j_{n+1,m}. \tag{A2}$$

Also, with the above assumption about K,

$$j_{nm}>0.$$

For the integral i_n we have

$$i_n<C\int_0^1 dx\,\rho(x)x^n/(\mu^2/\omega^2-x^2)\,,$$

where, here and in the following, C is an unspecified finite positive constant. Recall also that

$$\rho(x)=(1-x^2)^{1/2}/(\mu/M+\tfrac{1}{2}x+\mu^2x/2M^2)\,.$$

When $n>2$ nothing peculiar happens as $\mu/M\rightarrow 0$, but for $n=1$ and 2 the integral diverges for ω at the upper end of its range. For $n=2$ we have

$$i_2<C\int_0^1 dx(1-x^2)^{1/2}/(\mu/\omega-x)\,.$$

The integral can be performed and the large ω behavior extracted to give

$$i_2<C\ln(\omega/\mu)<C\ln(M/\mu)\,.$$

A similar attack when $n=1$ leads to

$$i_1<C\left(\frac{\omega}{\mu}\right)\int_0^1 dx(1-x^2)^{1/2}\Big/\left(\frac{\mu}{\omega}-x\right)$$

$$<C(M/\mu)^{1/2}\ln(M/\mu)\,.$$

In the case of the j_{nm} we have

$$j_{nm}<C\int_0^1 dx\,\rho(x)x^{n+m-2}(\mu/\omega+x)^{-m}.$$

For $n\geqslant 3$ there are no problems; the cases $n=1$ and 2 we treat separately. The problems and approach are similar to the above. For $n=2$,

$$j_{2m}<C\int_0^1 dx(1-x^2)^{1/2}/(\mu/M+x/2)<C\ln(M/\mu)\,,$$

and for $n=1$,

$$j_{1m}<C\int_0^1 dx(1-x^2)^{1/2}/(\mu/M+x/2)(\mu/\omega+x)$$

$$<C(\omega/\mu)\int_0^1 dx(1-x^2)/(\mu/M+x/2)$$

$$<C(M/\mu)^{1/2}\ln(M/\mu)\,.$$

The above estimates can be improved upon by noticing that the terms in the sums where $m\neq 0$ always have the form

$$(t/M\omega)^{m-K}j_{N,m}\,,$$

with $1\leqslant K\leqslant m$, $m\geqslant 2$ and $N\geqslant 1$. This can be combined with the inequality

$$j_{N+K-m,m}=\int_0^1 dx\,\rho(x)K(M^2+\mu^2+2M\mu/x,\,t)$$

$$\times x^{N-2}\left(\frac{x^K}{(\mu/\omega+x)^m}\right)<\left(\frac{\omega}{\mu}\right)^m J_{N,0}\,,\quad K\geqslant 1 \tag{A3}$$

to give the estimate

$$\left(\frac{t}{M\omega}\right)^{m-K} j_{N,m}<\left(\frac{t}{M\mu}\right)^{m-K}\left(\frac{\omega}{M}\right)^K\left(\frac{M}{\mu}\right)^K j_{N+m,0}$$

$$<C\left(\frac{\mu}{M}\right)^{m-(3/2)K}.$$

This gives an improved estimate when $m\geqslant 2K$.

Summarizing the results, we have the inequalities

$$\begin{aligned} &i_n<C\,,\quad n>2;\\ &i_2<C\ln(M/\mu)\,,\\ &i_1<C(M/\mu)^{1/2}\ln(M/\mu)\,,\end{aligned} \tag{A4}$$

and

$$\begin{aligned} &j_{n,m}<C\,,\quad n>2;\\ &j_{2,m}<C\ln(M/\mu)\,,\\ &j_{1,m}<C(M/\mu)^{1/2}\ln(M/\mu)\,.\end{aligned} \tag{A5}$$

Finally, when $m\geqslant 2K$,

$$(t/M\omega)^{m-K}j_{N,m}<C(\mu/M)^{m-(3/2)K}. \tag{A6}$$