Eigenphases and the Generalized Breit-Wigner Approximation*

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The relation between the repulsion of S-matrix eigenphases and "crossing" branch points (complex energies at which two or more eigenphases are equal) is examined in some detail. It is exploited to obtain a unitary one-pole approximation to the S matrix for the description of an isolated resonance superimposed on a nonelastic background.

I. INTRODUCTION

At an energy between thresholds N and $N+1$ of a multichannel scattering system, the $N\times N$ open-channel S matrix in a single partial wave $S(E)$ [which we refer to as "the S matrix"] has N eigenvalues $\sigma_n(E)=e^{2i\delta_n(E)}$, where δ_n ["eigenphases"] are real for real E. The $\sigma_n(E)$ are of course functions of the elements $S_{ba}(E)$ of $S(E)$. If these S-matrix elements can be analytically continued into the complex energy plane, the eigenvalues $\sigma_n(E)$ can be as well. In particular, if the $S_{ba}(E)$ have a pole at an energy E_p, *one* of $\sigma_n(E)$ will have a pole there (only one, because normally $\det S$ has only single poles). If the pole occurs near the real axis it will manifest itself as a scattering resonance, and a conjecture seems to be abroad that only one eigenvalue will show a rapid energy dependence at real energies near the pole, enabling the resonance to be most economically parametrized in terms of a one-pole approximation to the resonating *eigenvalue*. Our purpose is to point out that this conjecture is quite false. Because of Wigner's eigenphase-repulsion phenomenon, *all N* of the eigenvalues will normally be active near the resonance, so that the one-resonating eigenphase approximation is inadequate. This is quite evident, e.g., in the two-channel example of Fig. 1(a), which occurs in d-α scattering.

Equivalently, we can say that a one-pole approximation for the *eigenvalues* $\sigma_n(E)$ fails because the $\sigma_n(E)$ have additional singularities, namely branch points. At each value of E, the σ_n are the N roots of the Nth degree equation

$$\det[S(E)-x\mathbf{1}]=0. \qquad (1)$$

Consequently, the solution of this equation, $x=\sigma(E)$, is an N-valued function, whose N values are the $\sigma_n(E)$. Therefore, except for trival cases, the function $\sigma(E)$ has branch points. Fig. 1(b) shows the distribution of singularities corresponding to the two-channel case of Fig. 1(a). The branch points of $\sigma(E)$ ($\square$) are closer to the real axis than the pole ($\times$), since the energy width of the "repulsion bend" in the eigenphases is narrower than the width of the resonance. Let us call $\sigma_1(E)$ the branch of $\sigma(E)$ which has the pole reached by a path from the real axis such as the one labeled "σ_1"; $\sigma_2(E)$ has the pole if a path "σ_2" going the other way around the branch point is taken. For real energies less than "A," $\sigma_1(E)$ is the "active" eigenvalue because it has a nearby pole, and so δ_1 has the resonance shape [see Fig. 1(a)], but δ_2 is "dormant." But when the energy A is passed, the roles are reversed; it is now σ_2 which has a nearby pole, and so for E greater than A, δ_2 has the resonance shape and δ_1 is dormant. One observes in Fig. 1(a) the "no-crossing" or "repulsion" of the phases; obviously this prevents the $\delta_n(E)$ from having normal resonance shapes.

In Sec. II we discuss some properties of the branch points of $\sigma(E)$. In Sec. III we show that branch points normally do occur in the vicinity of an isolated resonance, and in Sec. IV we explicitly do the algebra for the 2×2 case to locate the branch points. Finally, in Sec. V we exhibit a matrix $\hat{S}(E)$, simply related to $S(E)$, whose eigenvalues do *not* have branch points, and which can be used to obtain a simple one-pole approximation to $S(E)$ which is a unitary generalization of the Breit-Wigner form to the case of a nonelastic background.

II. SOME PROPERTIES OF THE BRANCH POINTS

(a) If one takes a path in the E plane which encircles a branch point, the branches of $\sigma(E)$, i.e. the $\sigma_n(E)$, are permuted.[1]

(b) Those $\sigma_n(E)$ which are permuted are equal at the branch point; consequently the branch points have been called "crossing points," E_C, by Goldberger and Jones.[2] In general only two of the σ_n will be equal at a point (the case that more than two are equal can always be considered as a limiting case of two different pairs being equal at nearby points); so, in general, the E_C are square-root branch points.

(c) The crossing points occur at complex conjugate energies. This follows from the statement of unitarity of S (reality of the eigenphases for real E) in the form

$$\sigma_n(E)\sigma_n{}^*(E^*)=1\,, \qquad (2)$$

with E and E^* at energies reached by direct continua-

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[1] Most of this material is contained in Ref. 2. We include it here mainly to make our presentation self-contained.

[2] M. L. Goldberger and C. E. Jones (to be published).

tion from the real axis between thresholds N and $N+1$, one of them being on the physical sheet.

(d) Two eigenphases can cross at real E only if a crossing point coincides with its conjugate there to form a "double" crossing point, which is not a branch point. The fact that this is "unlikely," and occurs only if S satisfies special conditions, is Wigner's "no-crossing" or "eigenphase repulsion" theorem: The eigenvalues $\sigma_n(E)=e^{2i\delta_n(E)}$ are normally not equal for real energies. Equivalently, the eigenphases $\delta_n(E)$ are not equal modulo π for real energies; they do not cross.

As a simple example, the 2×2 S matrix can be written in the form

$$\begin{pmatrix} \eta e^{2i\delta_1} & i(1-\eta^2)^{1/2}e^{i(\delta_1+\delta_2)} \\ i(1-\eta^2)^{1/2}e^{i(\delta_1+\delta_2)} & \eta e^{2i\delta_2} \end{pmatrix}, \tag{3}$$

with δ_1, δ_2, and η real, $0\le\eta^2\le1$, for E real. The conditions for the equality of its eigenvalues at a real energy are

$$\delta_1=\delta_2(\mathrm{mod}\,\pi) \quad \text{and} \quad \eta=1\,, \tag{4}$$

which are normally not both satisfied at the same real energy.

III. THE EXISTENCE OF CROSSING POINTS NEAR AN ISOLATED RESONANCE

In the vicinity of an isolated resonance we can approximate $S(E)$ by the one-pole form

$$S_{ba}(E)=B_{ba}-i[t_bt_a/(E_p-E)]\,, \tag{5}$$

where $E_p=E_0-\frac{1}{2}i\Gamma$. The first term is a constant "background"; the dyadic or "factorable residue" form of the second term ensures that $\det S$ [Eq. (3)] has only a single pole.

By the resonance being isolated, we mean that E_p is sufficiently far from other poles and threshold branch points that Eq. (5) is valid in a region $|E-E_p|<\Delta$ of the complex plane whose radius Δ is much larger than Γ. In the outer part of this region, where

$$\Delta\gg|E-E_p|\gg\Gamma\,,$$

we have

$$S_{ba}(E)\approx B_{ba}\,, \tag{6}$$

a constant, so the eigenphases $\delta_n(E)$ of S tend to the constant eigenphases β_i of B as E tends to "infinity" in any direction. In particular, if E_1 is a real, positive energy satisfying $\Delta\gg E_1\gg\Gamma$, the $\delta_n(E)$ remain constant as E passes from E_0-E_1 to E_0+E_1 along a large semicircle in either the upper or lower half of the E plane. On the other hand, the presence of the resonance forces $\sum_1^N \delta_n(E)$, which is half the phase of $\det S$, to rise by π as E increases from E_0-E_1 to E_0+E_1. As Weidenmüller has recently pointed out,[3] the eigenphases can accomplish this, respect the no-crossing theorem, and

[3] H. A. Weidenmüller, Phys. Letters **24B**, 441 (1967).

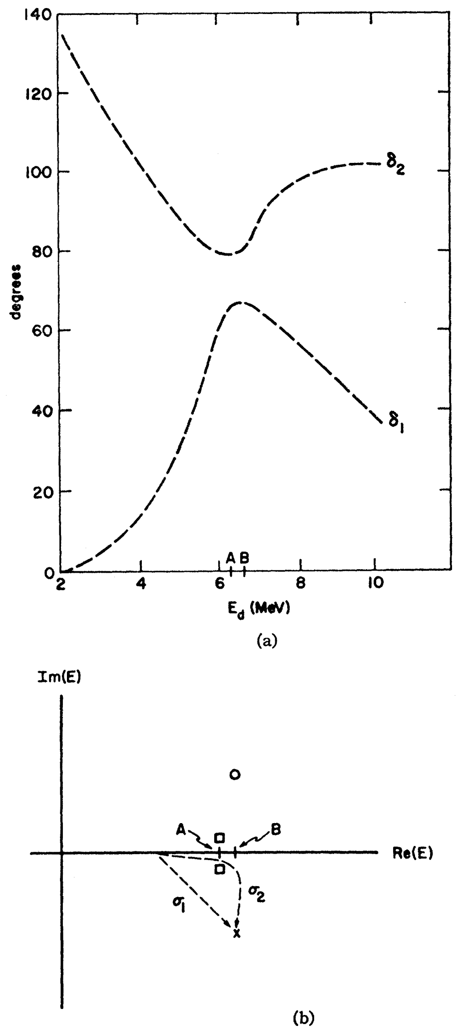

FIG. 1. (a) Eigenphase energy dependence for a two-channel $J=1^+$ resonance observed in d-α scattering [L. C. McIntyre and W. Haeberli, Nucl. Phys. **91**, 392 (1967)]. The eigenphase repulsion is centered at the energy marked A, and the center of the resonance is at B. Note that the background eigenphases are not constant, but decrease with energy much like hard-sphere phase shifts do. (b) Eigenphase singularities in the complex energy plane corresponding to the resonance of Fig. 1(a), showing the pole (×), conjugate zero (o), and crossing branch points (□) possessed by both eigenphases.

equal the β_i (modulo π) at both ends of the interval, only if their values at E_0+E_1 are some permutation of their values at E_0-E_1, as indicated in Fig. 2 for the case $N=3$. Thus $\delta_n(E_0+E_1)$ takes on different values depending on the path followed from E_0-E_1, and so must have branch points within both the upper and lower halves of the region $|E-E_0|<\Delta$. [The branch points in fact occur in complex conjugate pairs, Sec. II(c).] As we mentioned in Sec. II, we expect these branch points (i.e., the crossing points) to be simple and hence square-root, so that encircling one interchanges a pair of eigenvalues. Since it requires $N-1$ such interchanges to make a cyclic permutation of all N eigen-

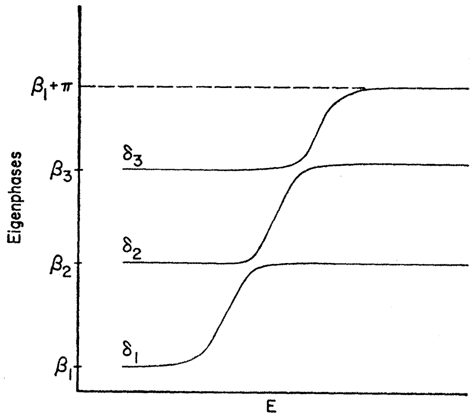

FIG. 2. Eigenphases for a typical 3-channel resonance, showing two eigenphase repulsions; constant backgrounds have been assumed for clarity. Note that as the resonance energy is passed, δ_1 rises from β_1 to β_2, δ_2 rises from β_2 to β_3, and δ_3 rises from β_3 to β_1 (mod π).

values, there will in general be $N-1$ square-root crossing points in each half plane.

IV. EXPLICIT EIGENVALUES FOR A TWO-CHANNEL RESONANCE

An isolated resonance coupled to two open channels provides an example simple enough to permit the algebra to be worked out explicitly. Without loss of generality we may write S in the Breit-Wigner (BW) form of Eq. (A4),

$$S_{ba}(E)=e^{i(\phi_b+\phi_a)}\left[\delta_{ba}-i\frac{\Gamma_b^{1/2}\Gamma_a^{1/2}}{E-E_p}\right], \qquad (7)$$

with ϕ_a and ϕ_b the real, constant background phases. The eigenvalues of a general 2×2 matrix are

$$\sigma_{\pm}(E)=\tfrac{1}{2}(S_{11}+S_{22})\pm\tfrac{1}{2}[(S_{11}-S_{22})^2+4S_{12}^2]^{1/2}, \qquad (8)$$

which for the Breit-Wigner matrix (7) become (defining $\lambda_j=e^{2i\phi_j}$)

$$\begin{aligned}\sigma_{\pm}(E)=\{&\lambda_1(E-E_p-i\Gamma_1)+\lambda_2(E-E_p-i\Gamma_2)\\ &\pm[(\lambda_1(E-E_p-i\Gamma_1)-\lambda_2(E-E_p-i\Gamma_2))^2\\ &-4\Gamma_1\Gamma_2\lambda_1\lambda_2]^{1/2}\}/2(E-E_p), \qquad (9)\end{aligned}$$

which exhibit the pole and square-root branch points explicitly. The latter can be made more evident by completing the square inside the square-root bracket, to give

$$\begin{aligned}\sigma_{\pm}(E)=\{&\lambda_1(E-E_p-i\Gamma_1)+\lambda_2(E-E_p-i\Gamma_2)\\ &\pm(\lambda_1-\lambda_2)[(E-E_c)(E-E_c^*)]^{1/2}\}/2(E-E_p), \qquad (10)\end{aligned}$$

with the crossing points occurring at the conjugate

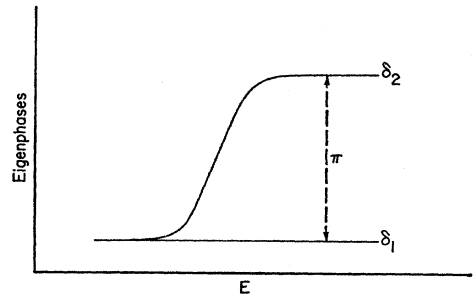

FIG. 3. A two-channel resonance with the special property that its eigenphases are equal off resonance. In this case the eigenphase "repulsion" occurs at an energy very distant from the resonance, and so allows the resonance to appear in only one of the eigenphases.

positions

$$\begin{aligned}\left\{\begin{matrix}E_c\\ E_c^*\end{matrix}\right\}&=E_0+\tfrac{1}{2}(\Gamma_1-\Gamma_2)\frac{i(\lambda_1+\lambda_2)}{\lambda_1-\lambda_2}\mp2\frac{(\Gamma_1\Gamma_2\lambda_1\lambda_2)^{1/2}}{\lambda_1-\lambda_2} \qquad (11)\\ &=E_0+\tfrac{1}{2}(\Gamma_1-\Gamma_2)\cot(\phi_1-\phi_2)\\ &\qquad\pm i(\Gamma_1\Gamma_2)^{1/2}\csc(\phi_1-\phi_2).\end{aligned}$$

Because of the crossing points, these eigenvalues clearly do not have a simple Breit-Wigner form in general, but there are two limiting cases in which they do. One is the case in which one channel is completely decoupled from the resonance, e.g. $\Gamma_2=0$. This makes the S of Eq. (7) diagonal, so the physical phases are the eigenphases, and if we associate σ_+ with channel 1, Eq. (9) becomes

$$\sigma_+(E)=e^{2i\phi_1}\left(\frac{E-E_p^*}{E-E_p}\right), \quad \sigma_-(E)=e^{2i\phi_2}. \qquad (12)$$

As is evident from Eq. (11), the two square-root branch points have in this case moved onto the real axis and "annihilated" one another there, allowing the two (decoupled) eigenphases to cross at that energy.

The opposite extreme is that in which the background phases ϕ_1 and ϕ_2 are equal. In this limit, according to Eq. (11), the crossing points move infinitely far from the pole, in a direction determined by Γ_2/Γ_1. Eq. (9) again reduces to a simple B W form, which we can write as

$$\sigma_-(E)=e^{2i\phi}\left(\frac{E-E_p^*}{E-E_p}\right), \quad \sigma_+(E)=e^{2i\phi}. \qquad (13)$$

The eigenphases now "cross" only at $E\to\pm\infty$; in the finite energy range near E_p the resonance is allowed to remain entirely in one eigenphase because it has just "room" enough to rise by π without intersecting the other, as indicated by Fig. 3.

In the general case, the branch points are near the pole and off the real axis, causing the eigenphases to repel one another for real E. The energy interval over which the repulsion occurs is approximately equal to $\mathrm{Im}E_c$, i.e., $(\Gamma_1\Gamma_2)^{1/2}/\sin(\phi_1-\phi_2)$, and so depends on both the background phases and the relative coupling of the two channels to the resonance.

V. GENERALIZED BREIT-WIGNER APPROXIMATION

The traditional Breit-Wigner approximation, Eq. (7), is applicable only to an isolated resonance superimposed on an *elastic* background. Its generalization to the case of a nonelastic background was given some time ago by Davies and Baranger,[4] whose argument is reproduced in simplified form in Appendix A. We outline here an alternative derivation which shows that the result, Eq. (18), can be obtained as one-resonating eigenphase approximation, not to $S(E)$, but to a closely related matrix $\hat{S}(E)$; further details can be found elsewhere.[5]

Near an isolated resonance we assume S to have the one-pole form

$$S(E)=B-it\tilde{t}/(E-E_p)\,, \tag{14}$$

with B a constant background matrix, t a complex column vector, $\tilde{t}$ its transpose, and $E_p=E_0-\frac{1}{2}i\Gamma$; the channelwise factorability of the residue at the pole follows from the assumption of a simple pole, i.e., a pole in only one eigenvalue per sheet.

The only question at issue is the relation which must hold between B and t in order that S be unitary. It can be obtained by noting that $S(E)$ can be unitary identically in E only if B is as well. B is also symmetric because S is, so its constant matrix of eigenvectors V is real and orthogonal. Denote by $e^{2i\beta}=\tilde{V}BV$ the diagonal matrix of eigenvalues of B, and consider the matrix

$$\begin{aligned}\hat{S}(E)&\equiv e^{-i\beta}\tilde{V}S(E)Ve^{-i\beta} \qquad\qquad (15\mathrm{a})\\ &=1-i\Gamma/(E-E_p)u\tilde{u}\,,\end{aligned}$$

with

$$u=\Gamma^{-1/2}e^{-i\beta}\tilde{V}t. \tag{15b}$$

Although the transformation is not unitary, $\hat{S}$ is readily seen to be symmetric and unitary if and only if S is; since it differs from $S(E)$ by a constant transformation, its elements clearly have no branch points.

The merit of $\hat{S}(E)$ is that its background eigenphases are equal, so that their crossing points are infinitely far from the pole, allowing the resonance to remain entirely in one of its eigenphases. That this indeed occurs is apparent by inspection of Eq. (15), which shows u to be an eigenvector of $\hat{S}$, with eigenvalue $\hat{\sigma}=1-i\Gamma/(E-E_p)=(E-E_p{}^*)/(E-E_p)$, while the rest of the eigenvectors, being orthogonal to u, have eigenvalues "dormant" at $\hat{\sigma}=1$.

[4] K. T. R. Davies and M. Baranger, Ann. Phys. (N. Y.) **19**, 383 (1962).

[5] K. W. McVoy, in *Fundamentals in Nuclear Theory* (International Atomic Energy Agency, Vienna, 1967).

The condition that $\hat{S}$ be unitary is that u be real and normalized, $\tilde{u}u=u^\dagger u=1$, or, from Eq. [15(b)],

$$t^\dagger t=\sum_{a=1}^{N}|t_a|^2=\Gamma; \tag{16}$$

since $|t_a|$ is the partial width for decay of the resonance into channel a, this is merely the statement that the sum of the partial widths equals the total width. From Eq. [15(b)] the reality of u can be written in terms of t as

$$e^{i\beta}\tilde{V}t^*=e^{-i\beta}\tilde{V}t,$$

which, since $B=Ve^{2i\beta}\tilde{V}$, is equivalent to

$$Bt^*=t. \tag{17}$$

This is our desired result, for Eqs. (16) and (17), which guarantee the unitarity of $\hat{S}$ and so of S, are expressed entirely in terms of the components of S. Consequently, in summary, if $B_{ba}=|B|e^{2i\phi_{ba}}$ and $\Gamma_a=|t_a|$, the desired generalization of the Breit-Wigner approximation can be written

$$S_{ba}(E)=e^{2i\varphi_{ba}}\left[|B_{ba}|-ie^{i\alpha_{ba}}\frac{\Gamma_b{}^{1/2}\Gamma_a{}^{1/2}}{E-E_p}\right], \tag{18}$$

where α_{ba} is the phase of t_bt_a relative to that of B_{ba}, i.e., the "phase of the resonance relative to the background." The matrix is unitary provided the conditions (16) and (17) are satisfied, and reduces to the BW approximation, Eq. (7), if $|B_{ba}|=\delta_{ba}$ (no direct reactions) and $\alpha_{ba}=0$, for all b and a.

The resonance zeros of the $S_{aa}(E)$ of the BW approximation all have the same real part as the pole, E_0. This is not true of the corresponding zeros in Eq. (18), which are found to be "tipped" relative to a vertical line through the pole by the angles α_{aa}. Further discussion of these tip angles, as well as a consideration of overlapping resonances, can be found in Ref. 5.

VI. CONCLUSION

Our principal result is that the eigenphases, because of their branch points, generally have a more complicated energy dependence than the physical S-matrix elements, and for this reason it would normally seem unwise to use them for representing experimental data. A two-channel isolated resonance is something of an exception, since the expression for the energy dependence of the eigenphases in terms of the resonance parameters, Eq. (9), is not significantly more cumbersome than the Breit-Wigner expression for the $S_{ba}(E)$. [For example, the experimental data of Fig. 1(a), were analyzed directly in terms of the eigenphases, rather than by the Breit-Wigner expression.] However, if the resonance is coupled to more than two open channels, it would seem that the physical S-matrix elements, parametrized by Eq. (18), provide a more convenient description than the corresponding eigenphases.

APPENDIX

In this appendix we derive the form of $S(E)$ in the vicinity of an isolated resonance. This result has been given previously by Davies and Baranger,[4] who term it the "generalized Breit-Wigner formula."

By only requiring that the form be valid in the vicinity of a resonance, we take its only energy dependence to be in a resonance denominator, so we can write

$$S_{ba}^{\mathrm{BW}}(E)=B_{ba}+\frac{T_{ba}}{E_p-E}, \tag{A1}$$

where B_{ba} [="background"], T_{ba}, and $E_p=E_0-i\Gamma/2$ are constants.

The determinant of S in general has only simple poles; equivalently, the pole occurs in only a single eigenvalue of S. This requires that T_{ba} be a dyad; write it $T_{ba}=t_b t_a$.

Finally, S must be unitary, $S^\dagger S=1$, i.e.

$$\sum_b B_{cb}{}^\dagger B_{ba}+\frac{-i(E-E_0)(B_{cb}{}^\dagger t_b t_a-t_c{}^*t_b{}^*B_{ba})+[t_c{}^*t_b{}^*t_b t_a-\frac{1}{2}\Gamma(B_{cb}{}^\dagger t_b t_a+t_c{}^*t_b{}^*B_{ba})]}{|E_p-E|^2}=\delta_{ca}, \text{ for all } E.$$

This provides the conditions

$$B_{cb}{}^\dagger B_{ba}=\delta_{ca},\quad B_{cb}{}^\dagger t_b=t_c{}^*,\quad \textstyle\sum_b |t_b|^2=\Gamma. \tag{A2}$$

It might be noted that

$$\det S^{\mathrm{BW}}(E)=1+i\Gamma/(E_p-E)=(E_p{}^*-E)/(E_p-E). \tag{A3}$$

In a representation in which B is diagonal, we can write

$B_{ba}=e^{2i\beta_a}\delta_{ba}$ (β_a real) and $t_a=e^{i\beta_a}\Gamma_a{}^{1/2}$ (Γ_a real, $\sum \Gamma_a=\Gamma$), so

$$S_{ba}^{\mathrm{BW}}(E)=e^{i(\beta_a+\beta_b)}\left[\delta_{ba}+\frac{i\Gamma_a{}^{1/2}\Gamma_b{}^{1/2}}{E_p-E}\right]. \tag{A4}$$

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Superconvergent Dispersion Relations and $I=2$ Electromagnetic Mass Differences of Hadrons

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Explicit calculations have been presented for the $I=2$ electromagnetic mass difference between particles of a given isomultiplet using the superconvergent-dispersion-relation approach suggested by Harari. We obtain, in particular, the electromagnetic mass differences (i) $\pi^+-\pi^0$, (ii) $\rho^+-\rho^0$, (iii) $\Sigma^++\Sigma^--2\,\Sigma^0$, (iv) $N^{*++}+N^{*0}-2N^{*+}$, and (v) $Y_1{}^{*+}+Y_1{}^{*-}-2Y_1{}^{*0}$. The agreement of our results with experiments is excellent.

1. INTRODUCTION

THE calculation of electromagnetic mass differences between members of various isomultiplets has attracted a lot of attention in recent times. It is well known that attempts to calculate electromagnetic mass differences, taking into account only certain low-lying pole terms in the self-energy diagram, lead to confusing results. The notorious wrong sign is obtained for the mass differences $n-p$ and K^+-K^0, while the correct sign and magnitude is predicted for the mass difference $\pi^+-\pi^0$. Recently, Harrai[1] put forward a simple criterion based on the use of superconvergent dispersion relations to understand this anomaly. As is well known, in perturbation theory the electromagnetic self-energy of a strongly interacting particle is given by[2]

$$\Delta M=\frac{1}{8\pi^2}\int\frac{T_{\mu\nu}(q^2,\nu)}{q^2-i\epsilon}g_{\mu\nu}d^4q, \tag{1}$$

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[2] W. N. Cottingham, Ann. Phys. (N. Y.) 25, 424 (1963).