Rapidly Decreasing Form Factors and Infinitely Composite Particles*

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The problem of understanding the asymptotic behavior of hadron electromagnetic form factors is considered. It is shown that if the hadrons form a bootstrap, each hadron must be a bound state of an infinite number of particles, or infinitely composite. A simple model containing an infinitely composite particle is constructed, and the form factor of the particle is calculated. The form factor decreases exponentially for large spacelike momentum transfers. On the basis of this, it is suggested that a criterion for a bootstrap should be exponentially decreasing form factors for all hadrons.

I. INTRODUCTION

A. Form Factors

THIS paper is the beginning of an attempt to gain theoretical understanding of strong-interaction scattering amplitudes and form factors at large momentum transfers. The experiments which have been carried out in this region have shown some striking features which invite theoretical explanation. In the best known cases of pp,[1–3] πp,[4] and $p\bar{p}$[5] collisions, high-energy large-angle elastic-scattering measurements show cross sections which are falling extremely rapidly with energy at fixed angles. Aside from this common feature of rapid decrease with energy, the three cases differ in detail, it being possible to fit the pp case with a simple formula proposed by Orear,[6] whereas the behavior of the πp and $p\bar{p}$ cross sections is more complicated. With regard to form factors, the latest evidence on the electromagnetic structure of the proton shows that the form factors of the proton decrease very rapidly at large momentum transfer, consistent with either inverse fourth power or exponential decrease.[7,8]

Theoretical work up to now has been largely confined to approaches based on statistical considerations[9] or qualitative suggestions, and while a number of interesting ideas have been advanced,[10] it is safe to say that no really convincing explanation of the behavior of hadron scattering amplitudes or form factors at large momentum transfers exists. Aside from its intrinsic interest, the elucidation of such questions may be of great importance for further progress in the subject of low-energy dynamics. Recent experience has shown that many low-energy strong-interaction quantities are much more sensitive to the details of far-away singularities than was originally hoped.[11] Consequently, a better understanding of high-energy boundary conditions on scattering amplitudes may be an important ingredient for further progress in this direction.

In this paper we concentrate on form factors which represent the simplest of the broad class of problems discussed above. Fundamental to our considerations is the familiar notion that a composite particle should have a form factor which decreases at large spacelike momentum transfer. Such behavior is readily interpreted semiclassically and may well be a characteristic of any composite particle in quantum mechanics, regardless of the specific dynamical context. Simple cases where it can be studied in detail are particles which are bound states of two elementary particles, either in a nonrelativistic Schrödinger equation framework or relativistically, using the Bethe-Salpeter equation. Our purpose here is to apply these ideas to the problem of understanding hadron form factors, under the assumption that the hadrons form a mutually self-consistent set of composite particles. From this viewpoint, the observed rapid decrease of the electromagnetic form factors of the nucleon should be a consequence of the fact that the nucleon is a composite particle.

B. Infinitely Composite Particles

It is clear at the outset that if the collection of hadrons forms a bootstrap or a mutually self-consistent set of composite particles, they have a unique property not shared by simpler systems in which composite particles occur. This is that the set of particles being generated is the same as the set of particles being bound together. This unique feature has a great impact on the question of choosing a set of channels to be used in generating a given particle. The problem of generating

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[9] See, for example, G. Fast and R. Hagedorn, Nuovo Cimento **27**, 208 (1963); G. Fast, R. Hagedorn, and L. W. Jones, Nuovo Cimento **27**, 856 (1963).

[10] See especially T. T. Wu and C. N. Yang, Phys. Rev. **137**, B708 (1965).

[11] J. R. Fulco, G. L. Shaw, and D. Y. Wong, Phys. Rev. **137**, B1242 (1965).

the nucleon as a composite particle provides an excellent illustration. The traditional approach to this problem has been to use only the πN channel, attempting to ignore multiparticle channels. The numerical failure of this approximation is well known by now.[12] What is less widely appreciated is that the approximation is *a priori* inconsistent. This can be seen as follows: Consider πN scattering, ignoring the coupling to any other channel, and assume there is a bound state N' generated in the πN channel. [See Fig. 1(a).] The fact that N and N' cannot be self-consistent particles follows by considering $\pi\pi N$ scattering which will certainly occur in this approximation. [See Fig. 1(b).] There will be poles in each of the two πN subenergies at the mass of the N'. The residue at the pole in either one of the two initial subenergies is proportional to the amplitude for $\pi N' \to \pi\pi N$.[13] In the general case of nonzero residues, N and N' cannot represent the same particle, i.e., they are not self-consistent, since the transition $\pi N' \to \pi\pi N$ occurs, whereas the πN system has not been allowed to couple to $\pi\pi N$. This is quite independent of the mass ratio of N and N' or the value of the $\pi NN'$ coupling constant. Stated less formally, the two-body approximation for the nucleon bootstrap is inconsistent in that it attempts to generate the nucleon as a two-body bound state[14] while at the same time it treats the external nucleon as an elementary particle. Ignoring the coupling of πN to $\pi\pi N$ is tantamount to treating the external nucleon as elementary.

What is needed is an improved approximation which allows the external nucleon to be treated as composite at the same time as the particle represented by the direct channel pole is generated. A first step can be taken by allowing the πN system to couple to $\pi\pi N$, but ignoring any channels of greater particle multiplicity. This allows the external nucleon to be treated as a two-body bound state. While this approximation represents an improvement at the $\pi\pi N$ level, it will be inconsistent at the $3\pi N$ level, just as the two-body approximation was inconsistent at the $\pi\pi N$ level. Using the same terminology as above, in this approximation the external nucleon is a two-body bound state whereas the particle being generated in the direct channel is, in fact, a three-body bound state.

It is clear that in order to achieve complete consistency between the external nucleon and the particle associated with the pole in the direct channel, the number of π's treated must become infinite. In other words, a necessary condition for consistent treatment of the nucleon as a composite particle is that it be a bound state of an infinite number of particles or an "infinitely composite" particle.

[12] P. W. Coulter and G. L. Shaw, Phys. Rev. **141**, 1419 (1966).

[13] These residues could vanish identically only under very pathological circumstances.

[14] In what follows an n-body bound state means a particle which is dynamically generated in a set of channels with particle multiplicities up to and including n, but no greater.

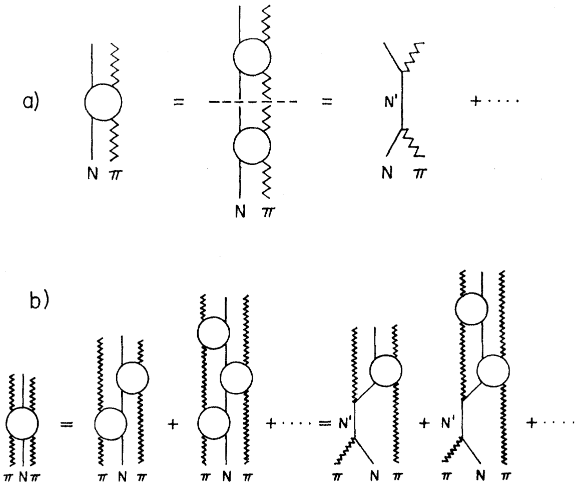

FIG. 1. (a) The generation of N' as a πN bound state in the elastic unitarity or two-body approximation. (b) Some terms in the $\pi\pi N$ amplitude in this approximation, showing explicitly terms responsible for the nonvanishing $\pi N' \to \pi\pi N$ amplitude.

So far the discussion has centered on the nucleon. By analogous arguments one can establish that any particular hadron in a theory in which all are composite must be infinitely composite. The πN problem merely provides an example where the problem of consistency of the external particles and those being generated arises immediately rather than at some higher stage in the bootstrap.

The present paper makes an attempt to explore the consequences of these ideas by considering an analytically tractable model which contains an infinitely composite particle. As mentioned earlier, the emphasis is on gaining insight into the asymptotic behavior of form factors. In Sec. II, the model is constructed by using the πN bootstrap problem discussed above as a guide. In Sec. III, the electromagnetic form factor of the composite "nucleon" is calculated. The form factor shows an exponential decrease for large spacelike momentum transfers. This model is highly idealized. Nonrelativistic kinematics is used. In addition, a number of other simplifying assumptions is introduced in order to achieve an analytically tractable situation. In Sec. IV arguments are presented which suggest that the rapid decrease found in the model is not dependent on the simplifying assumptions made, but only on the presence of an infinitely composite particle. If so, then a bootstrap among the hadrons should manifest itself by showing exponentially decreasing form factors for all hadrons.

II. MODEL FOR AN INFINITELY COMPOSITE NUCLEON

In Sec. I it was pointed out that for the consistent treatment of the nucleon as a composite particle, even if only nucleons and pions are present, it is necessary that the nucleon be allowed to couple to channels containing arbitrarily large numbers of particles. In this section we construct a model which attempts to take this requirement seriously, while relaxing other requirements which would have to be imposed in an exact bootstrap theory. The model is constructed by imposing the requirement of self-consistency for a composite particle in a stepwise way.

The starting point is a system consisting of an elementary spinless charged "nucleon" N_1 and an elementary spinless neutral "pion" π, each having the same mass. They scatter under the action of forces which represent strong interactions to generate an s-wave bound state N_2. This starting point is a highly simplified version of the familiar one-channel bootstrap calculations which attempt to generate the nucleon using the pion-nucleon channel. The idea is to proceed from this point in stages as the number of π's treated increases, attempting at each stage to make the external nucleon and the one generated as a bound state self-consistent. At the nth stage, for example, the bound state N_{n-1} of N_1 and $n-2$ π's which was generated at the previous stage is used as an external particle which scatters with an additional π to generate a new bound state N_n, bound of N_1 and $n-1$ π's. In the limit as $n\to\infty$, it becomes possible to make the particles N_{n-1} and N_n fully self-consistent. It is clear, of course, that such a theory does not create a bootstrap world in which all particles are composite. The particles N_1 and π are treated as elementary throughout. However, loosely speaking, the present theory does contain one composite particle N_∞ which is consistent with itself and therefore imitates to some extent a true bootstrap situation where all particles are composite and mutually self-consistent.

In order to be able to carry out the program outlined in the preceding paragraph analytically, it is necessary to make a number of special assumptions. The first of these is that nonrelativistic kinematics is used and the forces between N_1 and π are described by a potential $\lambda_2 V(r)$. To avoid having to solve the n-body Schrödinger equation in full generality, it is further assumed that at the nth stage the forces between N_{n-1} and π can be described by a potential of the same form $\lambda_n V(r)$, acting between the π and the center of mass of N_{n-1}. This means of course that the π's are not really treated as identical particles in this model. The coupling constant λ_n is adjusted so that N_n has the same wave function regarded as a bound state of N_{n-1} and π as does N_{n-1}, regarded as a bound state of N_{n-2} and π. In the limit as $n\to\infty$, λ_{n-1}/λ_n and B_{n-1}/B_n both approach unity, where B_n represents the energy of N_n in the πN_{n-1} system. This means that the force between N_n and π, and the position and residue[15] of the N_n pole in the πN_{n-1} scattering amplitude, all become self-consistent as $n\to\infty$, so that within the confines of the special assumptions made, a situation is achieved in which the "physical nucleon" $N_\infty=\lim_{n\to\infty}N_n$ is treated consistently both as an external particle and as a bound state.

The model can be specified in full detail by writing down the Hamiltonian which is operative at each stage.

$$H_2=-\left[\frac{\nabla_1^2}{2m}+\frac{\nabla_2^2}{2m}\right]+\lambda_2 V(\mathbf{r}_1-\mathbf{r}_2),$$
$$H_3=-\left[\frac{\nabla_1^2}{2m}+\frac{\nabla_2^2}{2m}+\frac{\nabla_3^2}{2m}\right]+\lambda_3 V(\mathbf{r}_1-\mathbf{R}_{23})+\lambda_2 V(\mathbf{r}_2-\mathbf{r}_3),\tag{1}$$
$$H_4=-\left[\frac{\nabla_1^2}{2m}+\frac{\nabla_2^2}{2m}+\frac{\nabla_3^2}{2m}+\frac{\nabla_4^2}{2m}\right]+\lambda_4 V(\mathbf{r}_1-\mathbf{R}_{234})+\lambda_3 V(\mathbf{r}_2-\mathbf{R}_{34})+\lambda_2 V(\mathbf{r}_3-\mathbf{r}_4),$$

etc.

At the nth stage, the coordinate of N_1 is $\mathbf{r}_n$, the coordinates $\mathbf{r}_1, \mathbf{r}_2, \cdots r_{n-1}$ refer to π's, and $\mathbf{R}_{ij\cdots k}$ is the center-of-mass coordinate of the system consisting of particles $i, j, \cdots k$. The n-particle Schrödinger equation separates if instead of the variables $\mathbf{r}_1, \mathbf{r}_2, \cdots \mathbf{r}_n$, the variables $\mathbf{R}_{12\cdots n}$, $\mathbf{R}_{1(2\cdots n)}$, $\mathbf{R}_{2(3\cdots n)}$, $\cdots \mathbf{R}_{n-1(n)}$ are used, where $\mathbf{R}_{i(jk\cdots l)}=\mathbf{r}_i-\mathbf{R}_{jk\cdots l}$. Consider, for example, the case of $n=3$. The Hamiltonian can be written as

$$H_3=-\left[\frac{\nabla_{123}^2}{2M_{123}}+\frac{\nabla_{1(23)}^2}{2M_{1(23)}}+\frac{\nabla_{2(3)}^2}{2M_{2(3)}}\right]+\lambda_3 V(\mathbf{R}_{1(23)})+\lambda_2 V(\mathbf{R}_{2(3)}),\tag{2}$$

where $M_{123}=3m$, $M_{1(23)}=m(2m)/(m+2m)=2m/3$, and $M_{2(3)}=m(m)/(m+m)=\frac{1}{2}m$. In general, $M_{ij\cdots l}$ denotes the total mass of the $(i,j,\cdots l)$ system, and $M_{i(j,\cdots l)}$ denotes the reduced mass of the $i,(j,\cdots l)$ system. Eigenfunctions of H_3 are of the form

$$e^{i\mathbf{P}_3\cdot\mathbf{R}_{123}}\psi_3(\mathbf{R}_{1(23)})\psi_2(\mathbf{R}_{2(3)}),$$

where

$$e^{i\mathbf{P}_2\cdot\mathbf{R}_{12}}\psi_2(\mathbf{R}_{1(2)})$$

is an eigenfunction of $H_2(\mathbf{R}_{12},\mathbf{R}_{1(2)})$. In the case of interest here, namely, scattering of π from the bound state of N_2 of H_2, the eigenfunctions have the form

$$e^{i\mathbf{P}_3\cdot\mathbf{R}_{123}}\psi_3(\mathbf{R}_{1(23)})\psi_B(\mathbf{R}_{2(3)}),$$

where $\psi_B(\mathbf{R}_{2(3)})$ is the wave function of N_2 in the πN_1 system, which is taken to be the ground state. The Schrödinger equation for $\psi_3(\mathbf{R}_{1(23)})$ becomes

$$\left[-\frac{\nabla_{1(23)}^2}{2M_{1(23)}}+\lambda_3 V(\mathbf{R}_{1(23)})\right]\psi_3(\mathbf{R}_{1(23)})=[E-B_2]\psi_3(\mathbf{R}_{1(23)}),\tag{3}$$

where B_2 is the energy of the bound state N_2 in the πN_1 system. As mentioned earlier, the condition of self-consistency is applied in this model by requiring that

[15] The usual relation between the residue and the square of the asymptotic wave function holds here.

the wave function of N_n in the πN_{n-1} system be the same as that of N_{n-1} in the πN_{n-2} system. For the present case of $n=3$, this means $2M_{1(23)}\lambda_3=2M_{2(3)}\lambda_2$ or $\frac{2}{3}\lambda_3=\frac{1}{2}\lambda_2$. For this choice of λ_3, the Hamiltonian H_3 has a bound state N_3 with bound-state wave function $\psi_B(\mathbf{R}_{1(23)})\psi_B(\mathbf{R}_{2(3)})$. The energy of N_3 measured with respect to the πN_2 threshold is

$$B_2\frac{M_{2(3)}}{M_{1(23)}}=B_2\frac{\frac{1}{2}m}{\frac{2}{3}m}=\tfrac{3}{4}B_2.$$

The residue of the N_3 pole in the energy plane of the πN_2 scattering amplitude is

$$\Gamma_2\frac{M_{2(3)}}{M_{1(23)}}=\tfrac{3}{4}\Gamma_2,$$

where Γ_2 is the residue of the N_2 pole in the πN_1 scattering amplitude. From these results for $n=3$, the next step is to generalize to arbitrary values of n. The bound-state wave function of N_n will be $\psi_B(\mathbf{R}_{1(2\cdots n)})\psi_B(\mathbf{R}_{2(3\cdots n)})\cdots\psi_B(\mathbf{R}_{n-1(n)})$. The self-consistency condition on the wave function requires for the coupling constant that

$$\frac{n-1}{n}\lambda_n=\frac{n-2}{n-1}\lambda_{n-1}=\cdots=\tfrac{1}{2}\lambda_2. \tag{4}$$

The recursion relations for the position and residue of the N_n pole in the πN_{n-1} scattering amplitude are of exactly this same form

$$\frac{n-1}{n}B_n=\frac{n-2}{n-1}B_{n-1}=\cdots=\tfrac{1}{2}B_2,$$
$$\frac{n-1}{n}\Gamma_n=\frac{n-2}{n-1}\Gamma_{n-1}=\cdots=\tfrac{1}{2}\Gamma_2. \tag{5}$$

It is clear that in the limit as $n\to\infty$, all three of these quantities approach definite limits. This along with the self-consistency condition on the wave function implies that the particles N_{n-1} and N_n become completely self-consistent as $n\to\infty$.

III. FORM FACTOR OF THE COMPOSITE NUCLEON

The main motivation for the construction of the model of Sec. II was the desire to be able to investigate large momentum-transfer quantities for an infinitely composite particle. In this section, the electromagnetic form factor of the particle $N_\infty\equiv\lim_{n\to\infty}N_n$ is calculated. This is done by first calculating the form factor for finite n and then letting $n\to\infty$. The form factor of N_n can be calculated in an elementary way by noting that the coupling of the electromagnetic field to N_n will be given completely by the coupling to the elementary charged particle N_1. The Born amplitude for scattering of an electron from N_n is given by

$$\delta^3(\mathbf{P}'+\mathbf{k}'-\mathbf{P}-\mathbf{k})f(\mathbf{k}-\mathbf{k}')=-\frac{1}{(2\pi)^4}\frac{(m_e)(nm)}{m_e+nm}\int e^{i(\mathbf{P}-\mathbf{P}')\cdot\mathbf{R}_{12\cdots n}}|\psi_B(\mathbf{R}_{1(2\cdots n)})\cdots\psi_B(\mathbf{R}_{n-1(n)})|^2$$
$$\times\frac{-Ze^2}{|\mathbf{r}_n-\mathbf{r}_e|}e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}_e}d^3r_1\cdots d^3r_nd^3r_e \tag{6}$$
$$=\delta^3(\mathbf{P}'+\mathbf{k}'-\mathbf{P}-\mathbf{k})\frac{Ze^2}{|\mathbf{k}-\mathbf{k}'|^2}\frac{2m_e(nm)}{m_e+nm}F_n(\mathbf{k}-\mathbf{k}')$$

for a charge $+Ze$ on particle N_1. The quantity $F_n(\mathbf{k})$ is by definition the form factor of N_n. Comparing the two terms in the second equality in (6) and changing the variables of integration, $F_n(\mathbf{k})$ can be expressed as

$$F_n(\mathbf{k})=\int e^{i\mathbf{k}\cdot(\mathbf{r}_n-\mathbf{R}_{12\cdots n})}|\psi_B(\mathbf{R}_{1(2\cdots n)})\cdots\psi_B(\mathbf{R}_{n-1(n)})|^2$$
$$\times d^3R_{1(2\cdots n)}\cdots d^3R_{n-1(n)}. \tag{7}$$

For the case of general masses, $\mathbf{r}_n$ is expressed in terms of $\mathbf{R}_{12\cdots n}$, etc. by

$$\mathbf{r}_n=\mathbf{R}_{12\cdots n}-\frac{m_1}{M_{12\cdots n}}\mathbf{R}_{1(2\cdots n)}$$
$$-\frac{m_2}{M_{2\cdots n}}\mathbf{R}_{2(3\cdots n)}\cdots-\frac{m_{n-1}}{M_{n-1,n}}\mathbf{R}_{n-1(n)}. \tag{8}$$

Using this expression for $\mathbf{r}_n$ in (7), the integration breaks up into a product of integrals. In the present case of equal masses, the result for $F_n(\mathbf{k})$ can be written

$$F_n(k)=F_2(k/2)F_2(k/3)\cdots F_2(k/n), \tag{9}$$

where

$$F_2(k)=\int e^{i\mathbf{k}\cdot\mathbf{r}}|\psi_B(\mathbf{r})|^2d^3r \quad\text{and}\quad k=|\mathbf{k}|. \tag{10}$$

Before undertaking a general discussion of the properties of (9) and the formula it implies for $F_\infty(k)$, it is useful to consider a special case which allows a closed expression to be evaluated. Such a special case is provided by the Coulomb potential, $\lambda_2V(r)=-\beta^2/r$, where $\psi_B(\mathbf{r})$ is the wave function of the lowest Bohr level. Then

$$F_2(k)=\left[\frac{(2\alpha)^2}{k^2+(2\alpha)^2}\right]^2, \tag{11}$$

where

$$\alpha^2 = -mB_2 = (m\beta^2/2)^2.$$

Substituting this into (9) and letting $n \to \infty$, $F_\infty(k)$ becomes

$$F_\infty{}^c(k) = \prod_{n=2}^{\infty} \left[\frac{(2\alpha)^2}{(k/n)^2 + (2\alpha)^2}\right]^2. \tag{12}$$

Using the standard infinite product representations for hyperbolic functions, this expression for $F_\infty{}^c(k)$ can be evaluated to give

$$F_\infty{}^c(k) = \left\{\frac{(k\pi/2\alpha)[1+(k/2\alpha)^2]}{\sinh(\pi k/2\alpha)}\right\}^2. \tag{13}$$

For large k, $F_\infty{}^c(k)$ approaches

$$4\pi^2(k/2\alpha)^6 e^{\mp(\pi k/\alpha)},$$

where the upper (lower) sign refers to the right (left) half-plane in k. Thus $F_\infty{}^c(k)$ vanishes exponentially at infinity in every direction in the k plane except the imaginary axis. Regarded as a function of k^2, $F_\infty{}^c$ is analytic in the finite k^2 plane except for a series of double poles on the negative k^2 axis at $k^2 = -4n^2\alpha^2$, $n = 2\cdots\infty$. The appearance of double poles is characteristic of the Coulomb potential and does not occur for Yukawa-like potentials. Aside from this unreasonable feature, the Coulomb potential has the saving grace of providing an example where the strikingly rapid decrease of form factors predicted by the present model can be seen explicitly.

For the potentials which resemble strong interaction forces most closely, namely Yukawas or superpositions thereof, it is not possible to evaluate (9) explicitly. However, it is still possible to give a rather complete discussion of the properties of $F_\infty(k)$. First some properties of $F_2(k)$ are needed: For the case of an s-wave bound state, $F_2(k)$ is analytic in the k^2 plane, except for a cut on the negative k^2 axis which runs from $-4\alpha^2$ to $-\infty$, where $\alpha^2 = -mB_2$. Furthermore, $F_2(k)$ is $O(1/k^4)$ as $k \to \infty$. Both of these properties are easily proven, using the Schrödinger equation for superpositions of Yukawas regular near the origin, i.e.,

$$V(r) = \frac{1}{r}\sum_{n=0}^{\infty} b_n r^n$$

near $r=0$. Using these properties for $F_2(k)$ in (9), it follows that $F_\infty(k)$ is analytic in the cut k^2 plane where the cut runs from $-16\alpha^2$ to $-\infty$.

The asymptotic behavior of $F_\infty(k)$ can also be ascertained. First introduce a modified function $\tilde{F}_2(k)$ which has no zeros.

$$F_2(k) \equiv \prod_{i=1}^{j} \left[1 - \frac{k^2}{k_i{}^2}\right] \tilde{F}_2(k), \tag{14}$$

where $\pm k_i$ are the zeros of $F_2(k)$, which must occur in complex conjugate pairs. $\tilde{F}_2(k)$ has the same domain of regularity and evenness properties as $F_2(k)$ does, but is free from zeros. Rewriting (9) and taking the limit as $n \to \infty$, $F_\infty(k)$ becomes

$$F_\infty(k) = \left\{\prod_{n=2}^{\infty}\prod_{i=1}^{j}\left[1 - \frac{k^2}{n^2 k_i{}^2}\right]\right\} \exp\left[\sum_{n=2}^{\infty} \ln \tilde{F}_2\left(\frac{k}{n}\right)\right]. \tag{15}$$

The asymptotic behavior of $F_\infty(k)$ follows after evaluating the part due to the zeros of $F_2(k)$ explicitly and converting the sum in the argument of the exponential into an integral. The error made in doing so is proven negligible by applying the Euler-Maclaurin sum formula.[16] One obtains

$$F_\infty(k) \xrightarrow[|k|\to\infty,\ \mathrm{Re}k>0]{} \prod_{i=1}^{j} \frac{\sin(\pi k/k_i)}{(\pi k/k_i)[1-(k^2/k_i{}^2)]} e^{-\gamma k}, \tag{16}$$

where γ is a constant defined by

$$\gamma = -\int_0^\infty \ln \tilde{F}_2(k') \frac{dk'}{(k')^2}. \tag{17}$$

The quantity γ is easily seen to be finite and positive since $|F_2(k)| \leq 1$ on the real axis and $F_2(k) = 1 + O(k^2)$ near $k=0$. The corrections to γ are $O(\ln k/k)$ and therefore asymptotically negligible. The asymptotic behavior of $F_\infty(k)$ for $\mathrm{Re}k<0$ is also given by (16) since $F_\infty(k)$ is even in k. Note that only in the case where $F_2(k)$ has no zeros does $F_\infty(k)$ vanish in every direction except the imaginary k axis. However, in every case, there is a finite range of directions in the k plane symmetric about the real k axis in which $F_\infty(k)$ vanishes exponentially.

To summarize, the present model produces form factors for Yukawa-like potentials, including the limiting case of the Coulomb potential, which are analytic in the cut k^2 plane, and fall off extremely rapidly for spacelike momentum transfers. The type of falloff is the same as that conjectured by Wu and Yang,[10] namely, $e^{-\gamma\sqrt{(-t)}}$, where $t = -k^2$. This widely conjectured behavior represents a maximal rate of decrease for a relativistic form factor, either from the viewpoint of axiomatic quantum field theory[17] or the analytic S matrix.[18] Faster decrease would represent a violation of locality from the field-theory viewpoint, or in S-matrix terms would necessarily preclude a finite number of subtractions.

[16] E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis* (Cambridge University Press, New York, 1958), 4th ed.
[17] A. M. Jaffe, Phys. Rev. Letters **17**, 661 (1966).
[18] A. Martin, Nuovo Cimento **37**, 671 (1965).

It is also of interest to consider the distribution of charge in N_∞. Defining $\rho_\infty(\mathbf{R})$ by

$$\rho_\infty(\mathbf{R})=\frac{1}{(2\pi)^3}\int e^{i\mathbf{k}\cdot\mathbf{R}}F_\infty(\mathbf{k})d^3k =\frac{4\pi}{(2\pi)^3}\int_0^\infty \frac{\sin kR}{kR}F_\infty(k)k^2dk\,, \quad (18)$$

it is easily seen using the properties of $F_\infty(k)$ that $\rho_\infty(\mathbf{R})$ is a function only of $R=|\mathbf{R}|$, is even in R, is $O(e^{-4\alpha R})$ as $R\to\infty$, and is analytic in R in $|\mathrm{Im}R|<\gamma$. Furthermore, the fact that the wave function ψ_B is a ground-state wave function and therefore positive implies that $\rho_\infty(\mathbf{R})$ is positive for real R. Thus $\rho_\infty(\mathbf{R})$ is a perfectly acceptable function with no pathologies and describes a charge distribution qualitatively similar to that of an ordinary two-body bound state. There is one crucial difference, however, which is that $\rho_\infty(\mathbf{R})$ is an analytic function of the components of $\mathbf{R}$ in a finite neighborhood surrounding every real point, as can be seen directly from (18), making use of the rapid decrease of $F_\infty(k)$. Analyticity in components of $\mathbf{R}$ is also implied by evenness and analyticity in R. This unusual property is the feature which reflects itself in the k plane in the rapid decrease of $F_\infty(k)$. This is easily seen by the successive application of Green's theorem to the pair of functions $e^{i\mathbf{k}\cdot\mathbf{R}}$, $\rho_\infty(\mathbf{R})$, which gives rise to the following equation for $F_\infty(k)$:

$$F_\infty(k)=\frac{(-1)^n}{(k^2)^n}\int [(\nabla^2)^n e^{i\mathbf{k}\cdot\mathbf{R}}]\rho_\infty(\mathbf{R})d^3R =\frac{(-1)^n}{(k^2)^n}\int e^{i\mathbf{k}\cdot\mathbf{R}}(\nabla^2)^n\rho_\infty(\mathbf{R})d^3R. \quad (19)$$

For analytic $\rho_\infty(\mathbf{R})$, n can be taken as large as desired. Therefore, $F_\infty(k)$ decreases faster than any inverse power of k. The precise rate of decrease is of course controlled by the singularity in R^2 nearest the real R axis.

By way of contrast, consider $F_2(k)$ for the Coulomb case. Here

$$\rho_2(\mathbf{R})=(\alpha^3/\pi)e^{-2\alpha R}.$$

The singularity in R^2 at $R=0$ limits the rate of decrease of $F_2(k)$ to $O(1/k^4)$. The use of Green's theorem for $n>2$ gives rise to terms of the form $(\nabla^2)^{n-2}\delta^3(\mathbf{R})$ in $(\nabla^2)^n\rho(\mathbf{R})$ which prevent successively higher inverse powers of k^2 from being obtained. In general, if a function $\rho(\mathbf{R})$ is analytic in R and even up to terms of the form R^{2n+1}, the operator ∇^2 can be applied at most $n+2$ times without generating terms in $\nabla^2\delta^3(\mathbf{R})$, and therefore $F(k)$ falls no faster than $(1/k^2)^{n+2}$. Thus, barring essential singularities in $\mathbf{R}$, analyticity in the components of $\mathbf{R}$ is a necessary condition for $F_\infty(k)$ to fall faster than any inverse power of k.

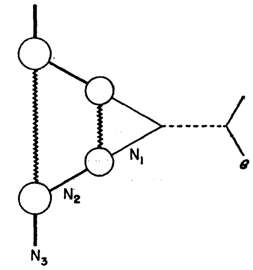

FIG. 2. The scattering process for $n=3$.

An example where all of the properties of $\rho_\infty(\mathbf{R})$ can be seen explicitly is again provided by the Coulomb potential. The integration in (18) can be carried out to yield

$$\rho_\infty{}^c(\mathbf{R})=-\frac{1}{4\pi R}\frac{d^3}{dR^3}\left[1-\frac{1}{4\alpha^2}\frac{d^2}{dR^2}\right]^2[R\coth\alpha R]. \quad (20)$$

IV. GENERALIZATIONS AND CONCLUDING REMARKS

The model of the preceding sections supports the basic contention of this paper, namely, that an infinitely composite particle will give rise to a rapidly decreasing form factor. The purpose of the present section is to argue that this behavior is not a special feature of a particular model but will occur whenever an infinitely composite particle is present, and therefore may provide the explanation for the observed rapid decrease of hadron form factors.

As a preliminary step, it is useful to try to understand qualitatively the rapid decrease found in the model. Consider the formula (9) for $F_n(k)$:

$$F_n(k)=F_2(k/2)F_2(k/3)\cdots F_2(k/n). \quad (9)$$

The factors in (9) can be understood by visualizing the scattering process as a sequence of transfers of momentum, first to N_1, then to the center of mass of N_2, etc., and finally to the center of mass of N_n. (See Fig. 2.) The jth factor in (9) represents the form factor associated with the distribution of N_j in N_{j+1}, $j=1, 2, \cdots n-1$. In these terms then, the $(1/k^4)^{n-1}$ behavior of $F_n(k)$ comes about because the charged particle N_1 is at the beginning of a chain of $n-1$ transfers of momentum, each one of which carries with it a form factor which behaves as $1/k^4$ as $k\to\infty$.

The question that must be answered is the extent to which the $(1/k^4)^{n-1}$ behavior of $F_n(k)$ is a consequence of the assumption of a center-of-mass potential, this clearly being the most objectionable feature of the model. To answer this question, consider models of the same general character, in which a sequence of more and more composite particles N_n is generated and in which nonrelativistic kinematics is used, but in which the n-body dynamics is treated in as realistic a way as possible. The simplest way to do this, conforming with the general features of strong interactions, is to have potentials of Yukawa type[19] act between all pairs of the elementary particles in the problem. Then at the nth stage, the bound state N_n will be described by a solution of the full n-body Schrödinger equation. Exact solutions in closed form are now out of the question. In addition, little is known about the analytic properties of multibody bound-state wave functions. The arguments that follow therefore make little pretense at rigor. Nevertheless, it is strongly suggested that the asymptotic behavior of $F_n(k)$ will continue to be $(1/k^4)^{n-1}$, in other words that the assumption of a center-of-mass potential is in no way crucial. Consider first the expression for $F_n(k)$, written in more symmetric variables and allowing any number of charged particles in N_n.

$$F_n(k)=\sum_{i=1}^{n}\int|\psi_n(\mathbf{r}_1,\cdots\mathbf{r}_n)|^2 \times Z_i e^{i\mathbf{k}\cdot\mathbf{r}_i}\delta^3(\mathbf{R}_{12\cdots n})d^3r_1\cdots d^3r_n. \quad (21)$$

Reasoning similar to that of the Riemann-Lebesgue lemma would suggest that $F_n(k)$ should fall at an increasingly rapid rate as $k\to\infty$ for successively greater values of n. Before a more precise statement can be made, some knowledge of the smoothness properties of $\psi_n(\mathbf{r}_1,\cdots\mathbf{r}_n)$ is needed. If use is made of known results for two-body wave functions to make plausible assumptions about the smoothness properties of $\psi_n(\mathbf{r}_1,\cdots\mathbf{r}_n)$, the result conjectured above is obtained. As a specific case, consider a three-body bound state with zero total angular momentum. Translational and rotational invariance allow $\psi_3(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3)$ to be written as $\psi_3(r_{12},r_{23},r_{31})$ where $r_{12}=|\mathbf{r}_1-\mathbf{r}_2|$, etc. If such a wave function behaves in a similar fashion to a two-body wave function when any pair of particles gets close together, then a wave function of the form

$$\psi_3(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3)=Ne^{-\frac{1}{2}[\alpha r_{12}+\beta r_{23}+\gamma r_{31}]}$$

should provide a form factor with the same asymptotic behavior as a real three-body wave function.[20] Substituting in (21) and representing $\psi_3(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3)$ by its Fourier transform, the following formula is obtained:

$$F_3(k)=2^6N^2\alpha\beta\gamma\int\frac{1}{[(\mathbf{k}')^2+\alpha^2]^2[(\mathbf{k}'+\frac{1}{3}\mathbf{k})^2+\beta^2]^2[(\mathbf{k}'-\frac{1}{3}\mathbf{k})^2+\gamma^2]^2}d^3k', \quad (22)$$

where, for simplicity, particle 3 is assumed to be the only particle with charge and the masses are taken to be equal. This clearly exhibits the conjectured $(1/k^4)^2$ behavior as $k\to\infty$. The generalization of this result to the four-body case is provided by the wave function

$$\psi_4(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3,\mathbf{r}_4)=N\exp(-\tfrac{1}{2}\sum_{i<j}\alpha_{ij}r_{ij}),\quad i,j=1,2,3,4 \quad (23)$$

and its form factor

$$F_4(k)=\left[\frac{8^3N^2}{\pi^3}\prod_{i<j}\alpha_{ij}\right] \times\int\frac{d^3k_1d^3k_2d^3k_3}{[(\frac{1}{4}\mathbf{k}-\mathbf{k}_2-\mathbf{k}_1)^2+\alpha_{14}^2]^2[(\frac{1}{4}\mathbf{k}+\mathbf{k}_1-\mathbf{k}_3)^2+\alpha_{24}^2]^2[(\frac{1}{4}\mathbf{k}+\mathbf{k}_2+\mathbf{k}_3)^2+\alpha_{34}^2]^2(\mathbf{k}_1^2+\alpha_{12}^2)^2(\mathbf{k}_3^2+\alpha_{23}^2)^2} \times\frac{1}{(\mathbf{k}_2^2+\alpha_{13}^2)^2}, \quad (24)$$

which approaches $(1/k^4)^3$ as $k\to\infty$, where again for simplicity all masses are equal and only particle 4 is charged. Corresponding results are easily obtained for higher values of n. Thus if multibody wave functions are no more singular than two-body wave functions when pairs of particles move close together, their form factors will behave as $(1/k^4)^{n-1}$, when $k\to\infty$.[21] This would imply that the result of the model studied earlier is a special case of a much more general result, and that visualizing the scattering process as a sequence of $n-1$ transfers of momentum provides a sound intuitive basis for understanding the asymptotic behavior of $F_n(k)$, even for a realistic n-body system.

[19] Here, as in Sec. III, we restrict the discussion to potentials such that $rV(r)$ can be expanded around the origin.

[20] It is really only the behavior of ψ_3 in the limit $r_{ij}\to 0$ which is relevant. Since the potentials are analytic functions of $\mathbf{r}_i-\mathbf{r}_j$ everywhere except $r_{ij}=0$, and the Schrödinger equation is an elliptic equation, analyticity of $\psi_3(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3)$ everywhere except $r_{ij}=0$ is guaranteed by general theorems on the analytic properties of the solutions of elliptic equations. See, e.g., R. Courant and D. Hilbert, *Methods of Mathematical Physics* (Interscience Publishers, Inc., New York, 1953), Vol. II.

[21] The fact that it is $1/k^4$ that appears is a direct consequence of the $1/r_{ij}$ behavior of the potentials when $r_{ij}\to 0$. More general potentials can easily be incorporated, but for simplicity are not considered here.

Assuming that the above results are correct and taking the limit $n \rightarrow \infty$, it is clear that $F_\infty(k)$ will fall faster than any inverse power of k. Exponential decrease will be guaranteed if $\rho_\infty(\mathbf{R})$ is analytic as a function of the components of $\mathbf{R}$, as mentioned earlier. As can be seen from (18), analyticity at all real points except $\mathbf{R}=0$ is guaranteed by the cut-plane analyticity of $F_\infty(k)$. The question therefore hinges on the behavior of $\rho_\infty(\mathbf{R})$ at $\mathbf{R}=0$. In the model of Secs. II and III the singularities of the two-body wave functions $\psi_B(\mathbf{R}_{i(j\cdots l)})$ at $|\mathbf{R}_{i(j\cdots l)}|=0$ become increasingly smoothed out in their effects on $\rho_n(\mathbf{R})$ at $\mathbf{R}=0$, giving rise in the limit $n\rightarrow\infty$ to a function $\rho_\infty(\mathbf{R})$ which is analytic at $\mathbf{R}=0$. For a realistic n-body system, the singularities of $\psi_n(\mathbf{r}_1,\cdots \mathbf{r}_n)$ lie at the points $r_{ij}=0$ rather than $|\mathbf{R}_{i,(j\cdots l)}|=0$. However, the problem of the resultant behavior of $\rho_n(\mathbf{R})$ at $\mathbf{R}=0$ is mathematically similar for the two cases, since both involve deducing the effects of singularities at translationally invariant points on a distribution referred to the center of mass. The smoothing will be the same in the two cases if the behavior of multibody wave functions near $r_{ij}=0$ is governed by the behavior at the origin of two-body wave functions as conjectured above. This suggests that $\rho_\infty(\mathbf{R})$ will be an analytic function of $\mathbf{R}$ here also and therefore that the exponential decrease found in the model does not depend on the assumption of a center-of-mass potential but only on the presence of an infinitely composite particle.

For application to bootstrap theories of hadrons, treatment of the dynamics in a fully relativistic way is a necessity. The dynamical framework which stays closest to the spirit of the preceding nonrelativistic discussion is provided by the Bethe-Salpeter equation and its multibody generalizations.[22] For forces which are not too singular, it is plausible that a generalization of the above nonrelativistic results can be achieved, since the result depends only on the behavior of relativistic wave functions in the bound-state region, where it is known that the Bethe-Salpeter equation is no more pathological than the Schrödinger equation.[23] Direct relativistic extensions of the nonrelativistic models considered so far would of course be subject to the same objection that they contain some particles which are not composite. Nevertheless, if results in the limit $n\rightarrow\infty$ could be achieved, they would give valuable insight into realistic bootstrap theories through the presence of at least one infinitely composite particle.

To conclude, it is suggested that the collection of hadrons form a set of infinitely composite particles, and that this property provides the natural framework for understanding the asymptotic behavior of form factors. In particular, if the results of the present work really are capable of relativistic generalization, a criterion for a bootstrap theory should be exponentially decreasing form factors for all hadrons.

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[22] On-the-mass-shell methods are unlikely to be of much use for understanding asymptotic behavior of form factors. The usual approximations always lead to constant asymptotic behavior. S. Mandelstam [UCRL Report No. 17250, 1966 (unpublished)], has recently shown that this will be true of any approximation which is incapable of treating an arbitrary number of particles in intermediate states in the t channel.

[23] See e.g., C. Schwartz, Phys. Rev. 137, B717 (1965), and references cited therein.