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# Formal Reconstruction of Theory from Experiment*

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In large classes of models, we show formally how to reconstruct canonical field theories of the strong interaction in terms of electromagnetic, weak, and gravitational scattering data (the currents and the stress-energy-momentum tensor).

## I. INTRODUCTION

In this paper, we wish to call attention to the fact that, in a very large class of models, the (canonical) field theory of a strongly interacting system can be formally reconstructed through electromagnetic, weak, and gravitational probing. That is to say, one can reconstruct the fields, interaction Hamiltonian density, etc., as functionals of the matrix elements of observables like the electromagnetic and weak currents, and the stress-energy-momentum tensor. In fact, one can in general (formally) construct an equivalence class of canonical field theories from the data, at most one of which in general may be "simple" or "ordinary" in the usual sense that it has no derivative coupling, or some not-too-complicated derivative coupling. Our ideas are closely related, and complementary to the ideas of Dashen and Sharp[1] that observables like the currents may form a complete set of coordinates for the strong interactions.

The basic idea of the reconstruction procedure is simple and common to all our models. If one can measure an irreducible set[2] of observables, say, at fixed time, and if one can guess a representation of the algebra of these observables which is unitarily equivalent to the data, then the transformation function between the data and this representation is determined (by Schur's lemma for infinite matrices). If one has chosen the representation wisely, one finds that the transformation function is very close to the "theory" itself. For example, our first model is potential theory; in this case we imagine the determination, through electric dipole radiation experiments, of the matrix elements of $p$, $x$ between states of definite energy. A suitable representation space for the algebra of these observables is the coordinate representation. Because $p$, $x$ form an irreducible set, the transformation function between the energy representation (data) and the coordinate representation is determined. The transformation function is, of course, the wave function. This is the subject of Sec. II.

In Sec. III we begin discussion of field theory. As an introduction, we discuss in this section the simpler subproblem of constructing the field from an irreducible set of explicit bilinear functions (currents) of the field. In practice, facing a set of data, one does not know the explicit field dependence of the currents; this added complication will be discussed in Sec. IV, where we propose a much more general method than that of Sec. III. In particular, Sec. III discusses the reconstruction of the pseudoscalar field $\phi$ from the irreducible set of scalar bilinears $S=\phi^2$, $\dot{S}=[\phi,\dot{\phi}]_+$ at fixed time. The procedure is very similar to that used in potential theory. We represent the "data," matrix elements of $S$, $\dot{S}$, on a space whose basis is the set of eigenvectors of $S$. The transformation functions from the data to this representation are thus determined. Then we write the field in terms of the transformation functions.

In Sec. IV, we propose a more general and more formal method of reconstruction. Illustrating our remarks in a universe of charged scalar mesons, we propose representation of the algebra of the irreducible set of observables on the space of eigenstates of the free Hamiltonian. In this case, the transformation function between the data and the representation is the familiar operator $U(0,-\infty)$—assuming temporarily the absence of bound states. Having $U(0,-\infty)$, one can easily reconstruct the interacting field from the free field. One does not know *a priori* the explicit field dependence of the currents, and each representation of the algebra in terms of canonical fields leads to a different $U(0,-\infty)$ and different fields. For example, if one can represent the algebra in terms of bilinear forms, one reconstructs a field theory in which the currents are bilinear. If one represents the data with quartic currents, one reconstructs a field theory with such currents. However, the set of all field theories obtainable in this way is a unitary equivalence class, in the sense that each theory is unitarily equivalent to the data. Although the matrix elements of the currents are invariant under the unitary transformation that characterizes the equivalence class, the matrix elements of the fields come out differently in each theory (but related by explicitly unitary transformations). One could search through the equivalence class for the "simplest" or "garden-variety" field theory, if such existed, on the basis of nonderivative coupling, or the simplest derivative

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[1] R. F. Dashen and D. Sharp (to be published).

[2] A set is irreducible if anything which commutes with all members of the set is a multiple of unity. In our field-theoretic models, this will be the case for appropriate bilinear sets (including weak and electromagnetic currents, stress-energy tensor etc.) at fixed time.

coupling. There is likely to be at most one of these "simple" theories under the data. Toward the end of the section, we speculate on a possible approach to the reconstruction in the presence of bound states. Finally, we shall also mention the possible relation between our class of field theories and what one might obtain as the set of all canonical extrapolations of the $S$-matrix off the mass shell, i.e., a more conventional inverse problem. Several points need to be made before beginning the exposition.

The most obvious problem in principle with our speculations is definition of the various operators and matrix elements used. For example, Haag's theorem[3] is relevant to Sec. IV. Although $U(0,-\infty)$ could be reconstructed with our methods in potential theory and in various static models, Haag's theorem guarantees us that no such operator exists in a translation-invariant relativistic field theory with a bare vacuum. Thus we must imagine *doing all reconstructions in a box*, say with the experimental volume as a parameter. Presumably the reconstructed $U(0,-\infty)$ would diverge as the volume became infinite, although one might hope that the reconstructed Heisenberg fields etc. could remain finite. Alternatively, one might try working in a translational-invariant universe with the so-called inequivalent representations[3] (no bare vacuum), and this possibility will be mentioned from time to time during the discussion.

Another problem which may be one of principle is ultraviolet divergence. $U(0,-\infty)$ diverges in the perturbation expansion of known field theories[4] for much the same reason as do the wave-function renormalizations, etc. There is some hope that $U(0,-\infty)$ is finite in some nonperturbative sense, of course, and certainly our reconstruction of the strong interactions will in no sense be close to perturbation theory. It would be interesting in this connection to define a reconstruction procedure with the data cutoff in some manner above certain energies, etc., and to study the infinte cutoff limit. Again, because of vacuum and ultraviolet divergences, we must imagine doing the functional derivatives of Sec. II on *a three-dimensional grid*. Thus we can forget about normal ordering products of operators in this section. About the box and/or grid limit, we really have nothing to say.

By virtue of our use of grid and/or box at appropriate places, and because we work at fixed time, we have not really said anything about reconstruction of fields in theories of local rings of observables,[5,6] although we hope our arguments may perhaps suggest a more rigorous approach to the problem. One small thing should be mentioned here about rigor. We know that our use of Schur's lemma for infinite matrices depends on the operators being bounded. At such points, one can avoid the use of grids by forming bounded functions of the currents, say by exponentation.

Moreover, we stress that to completely determine an underlying theory, one needs all the matrix elements of an irreducible set, including the highly inelastic ones. We are under no illusions about the practicality of such experiments, especially ones involving gravitational scattering. In spite of this, we give ourselves all these matrix elements, as one feels they can be determined in principle.

Having coped with these matters of principle and practice, we can ask about the possibility of the scheme working in reality. The basic problem say in our approach of Sec. IV is threefold. First, can one measure an irreducible set? In our models, the answer to this question is yes. Moreover, it can be shown to be the case in much larger classes of field theories. The question of irreducibility of the currents and the stress-energy tensor in the quark model will be discussed elsewhere.[1] In fact, we know of no realistic field theories in which such an irreducible set cannot be found. On the basis of this, it seems reasonable to conjecture that an irreducible set can be measured in reality. The second question to ask in the reconstruction is whether a (canonical) field-theoretical representation of the algebra of the observables can be found. For example, if all the hadrons are composite, then a field-theoretical representation of the algebra will be difficult if not impossible. Finally, the representation must be unitarily equivalent to the data. This means that, if the reconstruction is attempted in a translation-invariant system, one may be forced to use representations of the algebra which are unitarily inequivalent to any Fock-space representation. Another problem of this nature would arise if the algebra turned out to be that of the quark model, but no quarks are observed; then only a quarkless representation of the quark algebra could be unitarily equivalent to the data. It may turn out then that, for one reason or another, no such field-theoretic (etc.) representation could be found in reality, in which case, there being no underlying canonical field theory, one would have to be satisfied with a noncanonical theory, or an $S$-matrix theory, or perhaps a theory based on the currents themselves,[1] as described elsewhere.

## II. RECONSTRUCTION OF POTENTIAL THEORY FROM AN IRREDUCIBLE SET OF OBSERVABLES

For simplicity, we shall discuss only the case of a single charged particle of unit mass moving in a one-

[3] R. Haag, Kgl. Danske Videnskab. Selskab, Mat.-Fys. Medd. **29**, 12 (1955); D. Hall and A. S. Wightman, Kgl. Danske Videnskab. Selskab, Mat.-Fys. Skrifter **31**, 5 (1957); A. S. Wightman, lecture notes at Cargese summer school, July 1964 (unpublished); M. Guenin, Commun. Math. Phys. **3**, 120 (1966).

[4] E. C. G. Stückelberg, Phys. Rev. **81**, 130 (1951).

[5] A good referencing of papers in local-ring theory is found in D. W. Robinson, *Lectures in the 1965 Brandeis Summer Institute in Theoretical Physics* (Prentice-Hall Inc., Englewood Cliffs, New Jersey, 1965).

[6] H. J. Borchers, Commun. Math. Phys. **1**, 281 (1965). In more specific models, the problem of constructing fields from bilinears has also been considered by J. Langerholc and B. Schroer, DESY report (unpublished); and E. Prugovecki, Nuovo Cimento **45**, 327 (1966).

dimensional potential. The reconstruction is easily extended to many particles in three dimensions. Moreover, we shall assume at first that the potential is velocity-independent, i.e., that $p$, the momentum conjugate to $x$, is exactly the velocity $v$. After the reconstruction with this proviso, we shall return to include the case of velocity-dependent potentials.

In electric dipole radiation experiments, one can in principle measure the matrix elements of the position operator between states of definite energy (say, at zero time):

$$\langle E_m | x(0) | E_n \rangle = [x]_{mn}. \tag{2.1}$$

Without arguing the attainability of such quantities in detail, we shall assume we have them. From these data, we can construct the matrix elements of the velocity operator as

$$\langle E_m | v(0) | E_n \rangle = i(E_m - E_n)\langle E_m | x(0) | E_n \rangle. \tag{2.2}$$

The fact that the potential is velocity-independent will be manifest in the data, i.e., one will observe that

$$\langle E_m | [x(0),v(0)] | E_n \rangle = i\langle E_m | E_n \rangle. \tag{2.3}$$

Hence we can identify the matrix elements of the velocity with those of the canonical momentum $[p]_{mn}$. Because the matrices (2.1) and (2.2) form an irreducible set (say $\mathcal{O}$), we can recover the wave functions uniquely: The wave functions are the transformation functions between the energy and coordinate representations

$$\psi_m(x) = \langle x | E_m \rangle. \tag{2.4}$$

Hence

$$\langle x | \mathcal{O} | x' \rangle = \sum_{m,n} \psi_m(x)[\mathcal{O}]_{mn}\psi_n^*(x'). \tag{2.5}$$

The matrices on the left of Eq. (2.5) are certainly known,

$$\begin{aligned} \langle x' | x | x'' \rangle &= x'\delta(x'-x''), \\ \langle x' | p | x'' \rangle &= \frac{1}{i}\frac{d}{dx'}\delta(x'-x''). \end{aligned} \tag{2.6}$$

Eq. (2.5) states simply that $\psi$ is the transformation function between two explicitly known representations of the irreducible set $\mathcal{O}$. By Schur's lemma, then, Eq. (2.5) determines $\psi_m(x)$ up to an energy- and coordinate-independent constant. The normalization requirement reduces this ambiguity to a phase. In practice, one would probably use Eq. (2.5) in the form

$$x\psi_n(x) = \sum_m \psi_m(x)[x]_{mn}, \tag{2.7a}$$

$$\frac{1}{i}\frac{d}{dx}\psi_n(x) = \sum_m \psi_m(x)[p]_{mn}. \tag{2.7b}$$

A possible method of solution of Eqs. (2.7) is the following: Assume some particular $\psi_m(x)$ is known, say $\psi_0(x)$, and solve Eq. (2.7a) for all other $\psi$'s in terms of $\psi_0$. Then Eq. (2.7b) becomes an equation for $\psi_0$ alone.

**Formal Reconstruction of the Wave Function**

It is of interest to see in more detail how one might, at least formally, go about actually reconstructing $\psi(x)$: By multiplying together the matrix elements of $x$, we can construct the matrix elements of higher powers of $x$. The $n$th power of $x$ is the $n$th moment of the charge density—whose operator form is a $\delta$ function at a particular point. Using all the moments, we can reconstruct the matrix elements of the charge density itself

$$\begin{aligned} \langle E_m | \rho_\sigma(x) | E_n \rangle &= \langle E_m | \delta(x-\sigma) | E_n \rangle \\ &= \lim_{\epsilon \to 0} \frac{1}{\sqrt{(\pi\epsilon)}} \langle E_m | \exp[-(x-\sigma)^2/\epsilon] | E_n \rangle. \end{aligned} \tag{2.8}$$

The convergence of this limit is only the statement that the charge density is well behaved. In the same way, one can construct

$$\langle E_m | \delta(x-\sigma)p | E_n \rangle. \tag{2.9}$$

In the coordinate representation, we have that

$$\begin{aligned} \langle E_m | \delta(x-\sigma) | E_n \rangle &= \psi_m^*(\sigma)\psi_n(\sigma), \\ i\langle E_m | \delta(x-\sigma)p | E_n \rangle &= \psi_m^*(\sigma)\frac{d}{d\sigma}\psi_n(\sigma). \end{aligned} \tag{2.10}$$

Having (2.10), we can immediately solve for $\psi_n(x)$:

$$\psi_n(x') = \psi_n(x_0)\exp\left\{\int_{x_0}^{x'} d\sigma \frac{i\langle E_m | \delta(x-\sigma)p | E_n \rangle}{\langle E_m | \delta(x-\sigma) | E_n \rangle}\right\}. \tag{2.11}$$

The constant of integration can be fixed by the normalization requirement. In terms of the $\psi_n$, we can reconstruct the Hamiltonian

$$H(x,x') = \sum_n \psi_n(x)E_n\psi_n^*(x'). \tag{2.12}$$

This generates, of course, the Schrödinger equation of motion

$$\int dx' H(x,x')\psi_n(x') = E_n\psi_n(x). \tag{2.13}$$

The potential can be obtained if desired by constructing

$$V(x,x') = H(x,x') - (d^2/dx^2)\delta(x-x'), \tag{2.14}$$

or alternately, by operating with $(d^2/dx^2 + E_n)$ on $\psi_n$ and dividing by $\psi_n$.

**Inclusion of Velocity-Dependent Potentials**

In the presence of a velocity-dependent potential, the momentum conjugate to $x$ is not $\dot{x} = v$. This will be manifest in the data for $x$, $v$ because one will observe that

$$\langle E_m | [x,v] | E_n \rangle \neq i\langle E_m | E_n \rangle. \tag{2.15}$$

In order to proceed with the reconstruction program as outlined above, we need to discover, from the data, the matrix elements of the canonical momentum $p$. A way to go about this is first to search in the data for an irreducible set of observables. Let us assume such a set can be found, and, for simplicity, that the set is just $x$, $v$. If the set is more complicated, the method can be generalized. Now, because a self-adjoint irreducible set is also complete, we can expand the data for the commutator (2.15) in powers of $x$, $v$

$$[x,v]=f(x,v), \tag{2.16}$$

where the functional dependence of $f$ on $x$, $v$ is now assumed known. Toward finding $p$, we consider $v$ as a function of $p$, $x$ and write (2.16) as a differential equation

$$i\frac{d}{dp}v(p,x)=f[x,v(p,x)]. \tag{2.17}$$

The formal solution of this equation is

$$p=i\int\frac{dv(p,x)}{f[x,v(p,x)]}+C(x), \tag{2.18}$$

where $C$ is arbitrary. Equation (2.18) gives us the matrix elements of $p$ in terms of the data.[7] Having $p$ and $x$, we can now proceed to find the wave functions, etc. in the way outlined above.

Evidently the reconstruction procedure is somewhat more difficult in the presence of velocity-dependent potentials. The reason why we have do do more work in this case is simply that the algebra of observables is not so simple as in the case of only velocity-independent potentials: Because we do not in general know representations for complicated algebras, we must search among the polynomials in the observables (here $x$, $v$) for some quantities (here $x$, $p$) which have a simple algebra that we know how to represent (here the coordinate representation). As we shall see, this difficulty is mirrored in the field-theory case when we include derivative coupling.

## III. SIMPLE EXAMPLE OF FIELD RECONSTRUCTION FROM IRREDUCIBLE SET OF BILINEARS

In this section we want to show that methods similar to those discussed for potential theory can be used to reconstruct the field from an irreducible set of bilinears in a simple relativistic field theory. In particular, we set ourselves the task of reconstructing the pseudoscalar field $\phi$ from the matrix elements of the set of scalar bilinears[8]

$$\mathcal{O}=\{S(\mathbf{x})=\phi^2(\mathbf{x}),\ \dot{S}(\mathbf{x})=i[H,S(\mathbf{x})]=[\phi(\mathbf{x}),\dot{\phi}(\mathbf{x})]_+\}, \tag{3.1}$$

where $\phi$, $\dot{\phi}$ are canonically conjugate. It is proved in Appendix A that $\mathcal{O}$ is irreducible in each parity sector. (Because the mesons are pseudoscalar, the data break cleanly into even and odd numbered meson systems, separated by a superselection rule.) By working on a 3-dimensional grid we can take $\phi^2(\mathbf{x})$ as defined without normal ordering. We shall have nothing to say about the limit of fine grid.

### Construction of the Transformation Functions

We begin by giving ourselves the matrix elements of $\mathcal{O}$ between in-states

$${}^{E,0}{}_{\text{in}}\langle b|\,\mathcal{O}(\mathbf{x})\,|a\rangle_{\text{in}}{}^{E,0}, \tag{3.2}$$

where $E$, 0 denotes the even-odd parity subspace. The notation will remind us that $\mathcal{O}(\mathbf{x})$ allows no transitions between the sectors. This set of matrix elements is irreducible in each parity sector, and satisfies the algebra

$$[S(\mathbf{x}),\dot{S}(\mathbf{y})]=4i\delta^{(3)}(\mathbf{x}-\mathbf{y})S(\mathbf{x}). \tag{3.3}$$

Having in hand the matrix elements of an irreducible set, our next step is to guess another representation of the algebra of this set. Because we know there are fields under the data, it is natural to introduce the eigenstates of $S$ with explicitly positive semidefinite eigenvalues[9]

$$S(\mathbf{x})|\chi\rangle^{E,0}=\chi^2(\mathbf{x})|\chi\rangle^{E,0}. \tag{3.4}$$

The representation is done separately in each sector. In this functional representation, we take $\dot{S}$ as

$${}^{E,0}\langle\chi|\dot{S}(\mathbf{x})|\chi'\rangle^{E,0}=\frac{1}{i}\left[\chi(\mathbf{x}),\frac{\delta}{\delta\chi(\mathbf{x})}\right]_+{}^{E,0}\langle\chi|\chi'\rangle^{E,0}. \tag{3.5}$$

In analogy to our treatment of potential theory, the transformation function between the in-state basis and the $\chi$ representation

$$\psi_a[\chi]=\langle\chi|a\rangle_{\text{in}} \tag{3.6}$$

is determined (after normalization) up to an unimportant phase by the relation

$${}^{E,0}\langle\chi|\,\mathcal{O}(\mathbf{x})\,|\chi'\rangle^{E,0}=\sum_{ba}\psi_b{}^{E,0}[\chi]\,{}^{E,0}{}_{\text{in}}\langle b|\,\mathcal{O}(\mathbf{x})\,|a\rangle_{\text{in}}{}^{E,0}\times\psi_b{}^{*\,E,0}[\chi'] \tag{3.7}$$

[7] In practice, the formal integration of (2.18), say, ignoring the noncommutativity of $x$, $v$, results in only a formal $p$, which is in general a very useful guide toward constructing the canonical $p$. An arbitrary function $C(x)$ in $p$ gives rise to an energy-independent phase in the reconstructed wave functions.

[8] Because we cannot probe these particles with electromagnetism, it is not clear that $\dot{S}(\mathbf{x})$ is observable. Our motive in considering such a model is simplicity; all the considerations of this section go through in principle, e.g., in the charged scalar case, where the analogous bilinears, related to the electromagnetic current, are observable.

[9] The $\chi$ representation for a neutral field is, of course, well known. See, e.g., S. S. Schweber, *An Introduction to Relativistic Quantum Field Theory*, (Row, Peterson and Company, Evanston, Illinois, 1961), Sec. 7e.

and the irreducibility of $\Theta$. In analogy to Eqs. (2.7) in the potential case, we can rewrite (3.7) as

$$\chi^2(\mathbf{x})\psi_a{}^{E,0}[\chi]=\sum_b \psi_b{}^{E,0}[\chi]^{E,0}{}_{\rm in}\langle b|S(\mathbf{x})|a\rangle_{\rm in}{}^{E,0}$$

$$\frac{1}{i}\left[\chi(\mathbf{x}),\frac{\delta}{\delta\chi(\mathbf{x})}\right]_+\psi_a{}^{E,0}[\chi]$$

$$=\sum_b \psi_b{}^{E,0}[\chi]^{E,0}{}_{\rm in}\langle b|\dot{S}(\mathbf{x})|a\rangle_{\rm in}{}^{E,0}. \quad (3.8)$$

One can even do a formal functional integration of these equations, constructing first a functional $\delta$ function, in complete analogy with the potential theory case. We shall not go into this here. Having the wave functions or transformation functions $\psi$ in each sector, we can construct a Hamiltonian in each sector

$$H^{E,0}[\chi,\chi']=\sum_a \psi_a{}^{E,0}[\chi]E_a\psi_a{}^{*E,0}[\chi']. \quad (3.9)$$

These Hamiltonians have the property that

$$\int \delta\chi\, H^{E,0}[\chi,\chi']\psi_a{}^{E,0}[\chi']=E_a\psi_a{}^{E,0}[\chi]. \quad (3.10)$$

### Construction of the Field

In terms of the eigenstates of $S$ we define the field $\phi(\mathbf{x})$ as that operator which crosses sectors with eigenvalue $\chi(\mathbf{x})$

$$\phi(\mathbf{x})|\chi\rangle^{E,0}=\chi(\mathbf{x})|\chi\rangle^{0,E}. \quad (3.11)$$

This guarantees that $S=\phi^2$. Similarly, the canonical momentum is defined to be

$${}^{E,0}\langle\chi|\pi(\mathbf{x})|\chi\rangle^{0,E}=\frac{1}{i}\frac{\delta}{\delta\chi(\mathbf{x})}\,{}^{E,0}\langle\chi|\chi\rangle^{E,0}, \quad (3.12)$$

and to be zero between two vectors of the same sector. We can say more about the properties of $\phi$. Its eigenvectors are

$$|\chi\rangle_\pm=\tfrac{1}{2}\sqrt{2}(|\chi\rangle^E\pm|\chi\rangle^0), \quad (3.13)$$

$$\phi(\mathbf{x})|\chi\rangle_\pm=\pm\chi(\mathbf{x})|\chi\rangle_\pm. \quad (3.14)$$

Most important is that we can write $\phi$ explicitly in terms of the transformation functions calculable from the data; e.g., using (3.14), we can write,

$${}^E{}_{\rm in}\langle b|\phi(\mathbf{x})|a\rangle_{\rm in}{}^0=\int \delta\chi\,\psi_b{}^{*E}[\chi]\chi(\mathbf{x})\psi_a{}^0[\chi]. \quad (3.15)$$

Finally, the field at all times is simply

$${}_{\rm in}\langle b|\phi(x)|a\rangle_{\rm in}=e^{i(E_b-E_a)t}\,{}_{\rm in}\langle b|\phi(\mathbf{x})|a\rangle_{\rm in}. \quad (3.16)$$

With this prescription, and its analog for $\pi(x)$, it is easily checked that the canonical commutation relations are invariant under time translation.

## IV. MORE GENERAL AND MORE FORMAL METHOD

The approach through functional representation outlined in Sec. III is not very general. For one thing, it would be difficult, although not impossible, to construct Fermi fields from bosonlike currents in this manner. Moreover, there is the difficulty, as yet undiscussed, that one does not know *a priori* the functional dependence of the currents on the fields. In this section, we want to propose a more general approach, one that should work independently of the quantum numbers involved, and which is flexible enough to allow for not knowing the field dependence of the currents. We propose representing the data on the space of eigenstates of the free Hamiltonian $H_0$. The transformation function between the data and the $H_0$ representation (or interaction picture) is (in the case of no bound states) the familiar operator $U(0,-\infty)$. We shall work out the details of this approach for the case of charged scalar mesons, but it will be clear that the method is much more general. For simplicity we shall also assume the absence of bound states. After the exposition under these conditions, we shall return to include this interesting variant.

### Universe of Charged Scalar Mesons

Let us imagine ourselves in a universe of charged scalar particles (and photons, weak interactions, and gravity). Assuming we know the lowest-order weaker-than-strong interactions, our task is to reconstruct the unknown (strong) interaction between the mesons. Toward reconstructing this theory, we would begin by making what observations we can.

First, one can measure in principle the matrix elements of the electromagnetic current at, say $t=0$,

$${}_{\rm out}\langle b|j^\mu(\mathbf{x})|a\rangle_{\rm in}. \quad (4.1)$$

Now, toward discovering an irreducible set of observables, we begin studying commutators among the data. Suppose the commutator of $j_0$, $j^k$ was found to have the form

$$[j_0(\mathbf{x}),j^k(\mathbf{y})]=-2i\partial_{(x)}{}^k[S(\mathbf{x})\delta^{(3)}(\mathbf{x}-\mathbf{y})], \quad (4.2)$$

where $S(\mathbf{x})$ is defined by (4.2).[10] From the matrix elements of $S$, we can calculate the matrix elements of $\dot{S}$, the time derivative of $S$,

$${}_{\rm out}\langle b|\dot{S}(\mathbf{x})|a\rangle_{\rm in}=i\,{}_{\rm out}\langle b|[H,S(\mathbf{x})]|a\rangle_{\rm in}=i(E_b-E_a)\times{}_{\rm out}\langle b|S(\mathbf{x})|a\rangle_{\rm in}. \quad (4.3)$$

Suppose further that the commutators of $\dot{S}$ back with $S$, $j^k$ were found to be

$$[\dot{S}(\mathbf{x}),S(\mathbf{y})]=-2i\delta^{(3)}(\mathbf{x}-\mathbf{y})S(\mathbf{x}),$$
$$[j^k(\mathbf{x}),\dot{S}(\mathbf{y})]=2ij^k(\mathbf{x})\delta^{(3)}(\mathbf{x}-\mathbf{y}), \quad (4.4)$$

and all other equal-time commutators among

$$J(\mathbf{x})=\{j^\mu(\mathbf{x}),S(\mathbf{x}),\dot{S}(\mathbf{x})\} \quad (4.5)$$

[10] In fact, the inversion of (3.2) is simply

$$s(\mathbf{y})=-\frac{1}{2p^k}e^{-i\mathbf{p}\cdot\mathbf{y}}\int d^3x\, e^{i\mathbf{p}\cdot\mathbf{x}}[\rho(\mathbf{x}),j^k(\mathbf{y})],$$

independent of $k$, $\mathbf{p}$.

were zero. Suppose finally that the set $J$ was observed to be irreducible in each charge sector, in that no nontrivial operator commuted with all members of $J$ (except the charge of the sector).

This particular algebra of the observables would not surprise us. We would recognize it as the algebra of any nonderivative coupling canonical field theory of charged scalar mesons coupled to the electromagnetic field—in which the electromagnetic current, etc., was

$$j^{\mu}(\mathbf{x})=-\tfrac{1}{2}i[\phi^{\dagger}(\mathbf{x}),\overleftrightarrow{\partial^{\mu}}\phi(\mathbf{x})]_{+},$$
$$S(\mathbf{x})=\phi^{\dagger}(\mathbf{x})\phi(\mathbf{x}),\quad \dot{S}(\mathbf{x})=\phi^{\dagger}(\mathbf{x})\dot{\phi}(\mathbf{x})+\dot{\phi}^{\dagger}(\mathbf{x})\phi(\mathbf{x}). \quad (4.6)$$

(In Appendix B we show that, in fact, this set $J$ is irreducible in each charge sector if $\phi$, $\dot{\phi}$, $\phi^{\dagger}$, $\dot{\phi}^{\dagger}$ are assumed irreducible.)

As in Sec. II, III, our next task is to guess a representation of the algebra of observables.

From our field-theoretic experience then, we would first guess this field-theoretic representation (4.6) for the set of data $J$. One expects in general that other representations of the algebra of $J$ exist, the entire set of which forms an equivalence class with (4.6). After the reconstruction with this representation, we shall return to discuss the equivalence class.

### Construction of $U(0,-\infty)$

Having measured an irreducible set of observables, and guessed an explicit representation of the algebra in terms of canonical fields, we are in a position to reconstruct the theory. The first step is to multiply the data for $J$ by the adjoint of the (purely strong) $S$ matrix to construct

$${}_{\rm in}\langle b|J(\mathbf{x})|a\rangle_{\rm in}=\textstyle\sum_{c}\,{}_{\rm in}\langle b|c\rangle_{\rm out}\,{}_{\rm out}\langle c|J(\mathbf{x})|a\rangle_{\rm in}. \quad (4.7)$$

Now introduce the (formally unitary) operator which transforms an eigenstate of the free Hamiltonian $H_0$ (with the physical mass) into an eigenstate of the full Hamiltonian[11]

$$U(0,-\infty)|a\rangle=|a\rangle_{\rm in}. \quad (4.8)$$

The state $|a\rangle$ can also be thought of as an interaction- or Dirac-picture state at infinite time. The $U$ operator relates the data [in the form (4.7)] to analogous matrix elements between free states (i.e., in the interaction picture):

$${}_{\rm in}\langle b|J(\mathbf{x})|a\rangle_{\rm in}=\sum_{cd}(U^{-1})_{bc}\langle c|J(\mathbf{x})|d\rangle U_{da}. \quad (4.9)$$

Taking Heisenberg operators and interaction picture operators equal at $t=0$, we can calculate the matrix elements between the interaction picture states explicitly by hand, assuming the usual Fock representation and the forms (4.6) (i.e., these are just interaction picture operators between bare or infinite-time interaction picture states). These calculated matrix elements form a representation of the algebra of $J$. Hence $U$ is a unitary transformation function between two known representations of an irreducible set. Equation (4.9) thus determines $U$ up to an energy-independent phase. Roughly speaking, $U$ is the outer product of eigenvectors of the data for $J$ times the eigenvectors of $J$ between free states.

We have assumed that the Fock representation generates a representation for the current which is unitarily equivalent to the data. In a box, this must be true (if we are to find any underlying theory at all), but in a translationally invariant universe, it may be necessary to take a so-called inequivalent representation[3] for the fields in order to generate a representation of the currents unitarily equivalent to the data (i.e., to get an essentially unique unitary solution for $U$).

It should also be noted that the procedure of solving for $U(0,-\infty)$ is in a sense a test for $J$'s irreducibility. If there are many solutions for $U$, we would know $J$ is not yet irreducible, and we would have to go back and add more matrix elements to $J$.

In practice, one might go about solving Eq. (4.9) in the following manner: write

$$J_{ba}={}_{\rm in}\langle b|J(\mathbf{x})|a\rangle_{\rm in},\quad J_{ba}{}^{0}=\langle b|J(\mathbf{x})|a\rangle, \quad (4.10)$$

and look for $U$ in the explicitly unitary form

$$U=1-i\kappa/1+i\kappa. \quad (4.11)$$

The resulting equation for $\kappa$ is

$$i[\kappa,J^{+}]=J^{-}+\kappa J^{-}\kappa,\quad J^{\pm}=J\pm J^{0}. \quad (4.12)$$

One can imagine trying to solve (4.12) on a computer in the following fashion: In some suitable manner, cut off the data above what is known experimentally, and put the rest on an energy grid. To avoid difficulties with Haag's theorem, the translational invariance of the data should be broken by introducing the experimental volume or size of the apparatus as a parameter. Before passing on, it is amusing to note that the form of (4.12) is ideal for a possible perturbation expansion in some small parameter: If one had reason to believe that $J$ were close to $J^0$, then $\kappa$, $J^-$ would be first order in this difference, while $\kappa J^-\kappa$ would be third order. Thus one might hope to start an expansion around the first approximation to (4.12):

$$i[\kappa,J^{+}]=J^{-}. \quad (4.13)$$

Again, because $J^+$ is irreducible, this first approximation is unique (up to what amounts to a phase in $U$).

### Construction of the Fields, etc.

Having $U(0,-\infty)$, it is straightforward to construct the fields from the matrix elements of the interaction picture field in the interaction picture.

$${}_{\rm in}\langle b|\phi(x)|a\rangle_{\rm in}=e^{i(E_b-E_a)t}\,{}_{\rm in}\langle b|\phi(\mathbf{x})|a\rangle_{\rm in}$$
$$=e^{i(E_b-E_a)t}\sum_{cd}(U^{-1})_{bc}\langle c|\phi(\mathbf{x})|d\rangle U_{da}. \quad (4.14)$$

[11] Notice that, in the case of the vacuum, what we call $U(0,-\infty)$ is really the less singular operator $U(0,-\infty)/\langle 0|U(0,-\infty)|0\rangle$.

These latter matrix elements (in the interaction picture) are of course constructible by hand. From these matrix elements of the (unrenormalized) Heisenberg field, one can construct the mass and wave-function renormalizations etc. in the ordinary way.

Also of interest is the finite time evolution operator $U(t_2,t_1)$,

$$U(t_2,t_1)=e^{iH_{0D}t_2}e^{-iH(t_2-t_1)}e^{-iH_{0D}t_1}, \tag{4.15}$$

where $H_{0D}$ is the unperturbed Hamiltonian in the interaction (Dirac) picture and $H$ is the full Hamiltonian in, say the Heisenberg picture. Taking matrix elements of (4.15) with eigenstates of $H_{0D}$, we have

$$U_{ba}(t_2,t_1)=e^{i(E_bt_2-E_at_1)}\sum_c U_{bc}e^{-iE_c(t_2-t_1)}(U^{-1})_{ca}. \tag{4.16}$$

### Explicit Field Dependence of the Interaction

Once we know $U(0,-\infty)$, we can reconstruct the interaction Hamiltonian as well. The Hamiltonian in the interaction picture is simply

$$\langle b|H_D(t)|a\rangle=e^{i(E_b-E_a)t}\sum_c U_{bc}E_c(U^{-1})_{ca}. \tag{4.17}$$

To reconstruct the interaction part of the Hamiltonian in the interaction picture, we subtract off the (explicitly constructible) usual kinetic energy and mass terms

$$\langle b|H_{ID}(t)|a\rangle=\langle b|H_D(t)|a\rangle-\langle b|\int d\mathbf{x}\{\pi_D^\dagger(x)\pi_D(x)+\nabla\phi_D^\dagger(x)\cdot\nabla\phi_D(x)+\mu^2\phi_D^\dagger(x)\phi_D(x)\}|a\rangle, \tag{4.18}$$

where $\mu$ is the observed mass of the mesons. Because all the matrix elements of $H_{ID}(t)$ are known, we can find its explicit field dependence by doing an expansion of Haag's type[12] with a (Dirac picture) field for each observed particle. Of course, this procedure will only determine the interaction Hamiltonian density up to a spatial divergence.

We can go further and completely determine the interaction Hamiltonian density with gravitational scattering data. Because a gravitational probe "sees" $T_{\mu\nu}$ (stress-energy-momentum tensor), we can in principle use gravitational scattering to determine

$${}_{\rm out}\langle b|T^{00}(\mathbf{x})|a\rangle_{\rm in}={}_{\rm out}\langle b|\mathcal{H}(\mathbf{x})|a\rangle_{\rm in}. \tag{4.19}$$

From these data, we easily reconstruct the Hamiltonian density in the interaction picture:

$$\langle b|\mathcal{H}_D(x)|a\rangle =e^{i(E_b-E_a)t}\sum_{cd} U_{bc}\ {}_{\rm in}\langle c|\mathcal{H}(\mathbf{x},0)|d\rangle_{\rm in}(U^{-1})_{da}. \tag{4.20}$$

The interaction part of the Hamiltonian density is constructed, just as above, by subtracting off the known matrix elements of $\mathcal{H}_{0D}$. Again, its explicit field dependence can be found by an expansion of Haag's type in terms of the interaction picture fields.

### Equivalence Class of Field Theories

Thus far, we have reconstructed the field theory whose current is given as in (4.6). In general one expects that other (canonical) field-theoretic representations of the algebra of observables [Eqs. (4.2) and (4.4)] exist. It is quite easy to write down other representations of the algebra, but one has also to require that the resulting current, etc., has the correct Lorentz transformation properties. For example, the forms (4.6) with

$$\begin{aligned}\phi&\to\phi, & \pi&=\dot\phi^\dagger\to\pi+\phi,\\ \phi^\dagger&\to\phi^\dagger, & \pi^\dagger&=\dot\phi\to\pi^\dagger+\phi^\dagger,\end{aligned} \tag{4.21}$$

form another representation, but the resulting current does not transform as a four-vector (and the resulting fields will not transform as scalars). The problem of finding all representations with the correct Lorentz properties is evidently a very difficult one, and we shall not attempt to solve it in generality here. Specific examples of other four-vector currents with the current algebra Eqs. (4.2) and (4.4) can presumably be generated through a trick recently used by Weinberg[13]: Define new fields $\hat\phi$, $\hat\phi^\dagger$ in terms of the old, by a time-dependent unitary transformation which leaves the fields as scalars, e.g.,

$$\begin{aligned}\phi(x)&=\hat\phi(x)\exp[i\lambda\hat\phi^\dagger(x)\hat\phi(x)],\\ \phi^\dagger(x)&=\hat\phi^\dagger(x)\exp[-i\lambda\hat\phi^\dagger(x)\hat\phi(x)],\end{aligned} \tag{4.22}$$

where $\lambda$ is a constant. Next rewrite, say, the free Lagrangian for $\phi$, $\phi^\dagger$ in terms of $\hat\phi$, $\hat\phi^\dagger$ and vary this Lagrangian with respect to $\hat\phi$, $\hat\phi^\dagger$ to obtain the current as a function of the new fields. According to Weinberg, if the new fields are now taken as canonical, this current will satisfy the old commutation relations. (We only take the functional form of this current, ignoring its origin in such a procedure.) This and other currents generated with other prescriptions like (4.22) will be in general rather complicated functions of the fields, and will imply complicated derivative coupling in the underlying Lagrangian.

In any event, for any such representation that can be found, one could go through the procedure outlined above to calculate $U(0,-\infty)$ and the fields, etc. Both $U(0,-\infty)$ and the matrix elements of the fields will come out differently with this second representation. One thing that can be said about the new matrix elements of the fields is that they are unitarily related (at fixed time) to the matrix elements of the field in the previous theory: Denoting the first and second $U(0,-\infty)$ as $U_1$, $U_2$ etc., we have

$$\begin{aligned}{}_{\rm in}\langle b|\phi_1(\mathbf{x})|a\rangle_{\rm in}&=\sum_{cd}(U_1^{-1})_{bc}\langle c|\phi(\mathbf{x})|d\rangle(U_1)_{da},\\ {}_{\rm in}\langle b|\phi_2(\mathbf{x})|a\rangle_{\rm in}&=\sum_{cd}(U_2^{-1})_{bc}\langle c|\phi(\mathbf{x})|d\rangle(U_2)_{da},\end{aligned} \tag{4.23}$$

[12] R. Haag (Ref. 3); V. Glaser, H. Lehmann, and W. Zimmermann, Nuovo Cimento 6, 1122 (1957).

[13] S. Weinberg, Phys. Rev. Letters 18, 188 (1967). Care should be taken that this trick does not create or destroy Schwinger terms.

so that

$$_{\rm in}\langle b|\phi_2(\mathbf{x})|a\rangle_{\rm in}=\sum_{cd}(U_2^{-1}U_1)_{bc}\times{}_{\rm in}\langle c|\phi_1(\mathbf{x})|d\rangle_{\rm in}(U_1^{-1}U_2)_{da}. \quad (4.24)$$

Of course, a time-dependent unitary transformation is necessary to relate the fields away from zero time.

By finding all these "physical" representations of the algebra, one can reconstruct the entire class of canonical field theories under the data. This class is a unitary equivalence class in the sense that each representation is unitarily equivalent to the data (and hence to each other representation in the class). In general, there will be at most one simple or "garden-variety" theory in the class, in the usual sense that it have no derivative coupling or some not too complicated derivative coupling.[14] Note that, in a box, all the representations in the equivalence class will be equivalent to a Fock representation, whereas, in a translation-invariant system, if one representation in the equivalence class is inequivalent to the Fock representation, then they all are.

Suppose the algebra of observables came out more complicated than Eqs. (4.2) and (4.4); e.g., the algebra of the irreducible set might not be closed (finite). This would indicate the presence of derivative coupling in all the members of the underlying equivalence class. Our task remains basically the same, of course: If we can represent the algebra of an irreducible set in terms of canonical fields (such that the representation is unitarily equivalent to the data), then we can reconstruct the fields. Tricks for representing the complicated algebras by looking for simple subalgebras (as mentioned in Sec. II for the case of velocity-dependent potentials) can be devised, but we shall not go into detail about this here.

### Bound States

In the absence of bound states, our procedure is fairly well defined: One measures observables, hoping to find an irreducible set. One represents the algebra of the observables in terms of canonical fields corresponding to the observed particles. For each representation we can reconstruct another field theory in the (unitary) equivalence class of theories "under" the data. The possibility of bound states, on the other hand, is a very difficult complication; that is to say, having observed a particle, should we assign it an (elementary) field (by using it in the currents)? We really have no solution to this problem, but we offer a few speculations.

For definiteness, suppose we are examining a universe of pions and nucleons, there being some suspicion that the pion is a nucleon-antinucleon bound state. The first thing to do would be to try to represent the algebra of observables in terms of nucleon fields only. Suppose that this worked, and that the equation for the transformation function between the data (including all states with pions as well) and the representation (only nucleon states) had a unique solution. [Notice that the transformation function, still formally unitary, is no longer $U(0,-\infty)$. Nonetheless, it is to be used just as was $U(0,-\infty)$ above in the constructing fields, etc.] In this theory, one would then say that the pion is a nucleon-antinucleon bound state. The question of interest is, assuming that this one theory (with no explicit pion field) underlay the data, whether another representation of the algebra (and another theory) can be found with a pion field in addition to the nucleon field. It seems reasonable to us that, if such a representation could be found, then, on reconstructing the wave-function renormalization of the pion field, it would turn out zero, but we do not wish to get involved here in the well-known problems of the $Z=0$ approach in field theory. Evidently, if all the particles in a system are composite, our approach through field-theoretic representation finds itself uncomfortably deep into these $Z=0$ arguments.

### Comparison with a Conventional Inverse Problem

We have here discussed the problem, at least formally, of reconstructing all the canonical theories underlying the data for an irreducible set of observables (currents, etc.). The conventional "inverse problem" in potential theory[15] and field theory is, on the other hand, to reconstruct the theory from the $S$ matrix. It would be very interesting to compare our equivalence class of theories with the set of all possible canonical interpolating fields for a given $S$ matrix. Unfortunately, essentially nothing is known about this latter set, and we shall have very little to add beyond a few almost obvious remarks.

On the one hand, our scheme evidently feeds in a great deal more information via the currents etc., and the $S$ matrix than is contained in the $S$ matrix alone. That is, certainly the currents are not determined by the $S$ matrix.[16] Thus one feels that our equivalence class of theories, say $(J)$, must be in some sense more restrictive than the class $(S)$ obtained from the $S$ matrix. On the other hand, despite our having introduced more information, we have never really included any information about the off-mass-shell behavior of a strongly interacting particle; thus one feels our class $(J)$ could not be significantly smaller than the set $(S)$. One can say immediately, of course, that the $S$-matrix reconstruction would not yield the fields as functionals of the data in any ordinary sense, because one needs an

[14] On the other hand, in nature one would not be surprised to find no "simple" theories in the class.

[15] For a review of the inverse problem in potential theory, see L. D. Faddeyev, J. Math. Phys. **4**, 72 (1963).

[16] The currents may be calculable from the $S$ matrix if certain assumptions are made about the high-momentum behavior of the currents. See R. F. Dashen and S. C. Frautschi, Phys. Rev. **143**, 1171 (1966); **145**, 1287 (1966).

irreducible set in terms of which to expand the fields (as we have done). Rather, the $S$-matrix approach would be more a matter of guessing field theories and trying them.

Let us for the moment take the most optimistic view of the $S$-matrix approach and assume that, having constructed the class of theories $(S)$ (say in the charged scalar universe), we find it identical to our class $(J)$. There would still remain the problem in the $S$-matrix approach of determining the real or physical currents in each theory of $(S)$. That is, even if one gave oneself the algebra of the currents, there would be in general an infinite set of structures in each theory of $(S)$ which satisfy the algebra. [These would have the functional form of the various field theoretic representations of the current algebra.] One must then choose the current from among an equivalence class of currents. The $S$ matrix alone unfortunately contains no information about this choice, and some additional principle, like minimal electromagnetic coupling would have to be invoked.

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## APPENDIX A: IRREDUCIBLE SET IN NEUTRAL PSEUDOSCALAR THEORY

To see whether $S, \dot{S}$, as defined in (3.1), are irreducible in each parity sector, we inquire as to what dependence on $\phi$, $\dot{\phi}=\pi$ (assumed irreducible) can remain if a functional $f$ of $\phi$, $\dot{\phi}$ is required to commute with both $S$, $\dot{S}$ at fixed time (say $t=0$)

$$[f\{\phi,\dot{\phi}\},S(\mathbf{x})]=[f\{\phi,\dot{\phi}\},\dot{S}(\mathbf{x})]=0. \tag{A1}$$

There are many ways to see that (A1) limits $f$ to be a multiple of the unit operator. Perhaps the most elegant demonstration proceeds in the following manner.

Define "spinors"

$$\psi(\mathbf{x})=\begin{bmatrix}\phi(\mathbf{x})\\ i\dot{\phi}(\mathbf{x})\end{bmatrix},\quad \bar{\psi}=\psi^\dagger\begin{pmatrix}0&1\\1&0\end{pmatrix}. \tag{A2}$$

Because $\phi$, $\dot{\phi}$ satisfy canonical commutation relations, the $\psi$'s satisfy

$$\begin{aligned}&[\psi_\beta(\mathbf{x}),\bar{\psi}_{\beta'}(\mathbf{y})]=\delta_{\beta\beta'}\delta^{(3)}(\mathbf{x}-\mathbf{y}),\\ &[\psi_\beta(\mathbf{x}),\psi_{\beta'}(\mathbf{y})]=[\bar{\psi}_\beta(\mathbf{x}),\bar{\psi}_{\beta'}(\mathbf{y})]=0.\end{aligned} \tag{A3}$$

In this notation

$$S(\mathbf{x})=\bar{\psi}\sigma_-\psi,\qquad \dot{S}=i\bar{\psi}\sigma_3\psi,$$
$$\sigma_-=\tfrac{1}{2}(\sigma_1-i\sigma_2)=\begin{pmatrix}0&0\\1&0\end{pmatrix},\quad \sigma_3=\begin{pmatrix}1&0\\0&-1\end{pmatrix}. \tag{A4}$$

In the spinor space, $S$, $\dot{S}$ are the generators of a local rotation and a local scale change, respectively. That is, the unitary operators

$$\begin{aligned}U(\alpha_-)&=\exp\left\{i\int dx'\alpha_-(x')S(x')\right\},\\ U(\alpha_3)&=\exp\left\{i\int dx'\alpha_3(x')\dot{S}(x')\right\},\end{aligned} \tag{A5}$$

[$\alpha_-(x)$, $\alpha_3(x)$ being arbitrary "smearing" functions] have the effect

$$\begin{aligned}U(\alpha_3)\psi(\mathbf{x})U^{-1}(\alpha_3)&=e^{2\sigma_3\alpha_3(\mathbf{x})}\psi(\mathbf{x}),\\ U(\alpha_3)\bar{\psi}(\mathbf{x})U^{-1}(\alpha_3)&=\bar{\psi}(\mathbf{x})e^{-2\sigma_3\alpha_3(\mathbf{x})},\\ U(\alpha_-)\psi(\mathbf{x})U^{-1}(\alpha_-)&=e^{2i\sigma_-\alpha_-(\mathbf{x})}\psi(\mathbf{x}),\\ U(\alpha_-)\bar{\psi}(\mathbf{x})U^{-1}(\alpha_-)&=\bar{\psi}(\mathbf{x})e^{-2i\sigma_-\alpha_-(\mathbf{x})}.\end{aligned} \tag{A6}$$

The important point is that $\sigma_3$, $\sigma_-$ are an irreducible set in the spinor space; that is, there are no "rotations" which commute with $\sigma_3$, $\sigma_-$ that are not multiples of the null rotation. Thus $f(\phi,\dot{\phi})=\tilde{f}(\psi,\bar{\psi})$ must be a functional at most of $\mathbf{1}$ in the spinor space, which is the commutator of $\phi$, $\dot{\phi}$, and hence a constant. Thus $\{S,\dot{S}\}$ is an irreducible set in each parity sector.

## APPENDIX B: IRREDUCIBLE SET IN CHARGED SCALAR THEORY

The proof of irreducibility of the set $J$ as defined in (4.6) is very much like that presented in the neutral case. Introduce the "spinors"

$$\psi=\begin{bmatrix}\phi\\ i\dot{\phi}\end{bmatrix},\quad \bar{\psi}=[-i\dot{\phi}^\dagger,\phi^\dagger]=\psi^\dagger\begin{pmatrix}0&1\\1&0\end{pmatrix}. \tag{B1}$$

These quantities have the same commutation relations as the analogous "spinors" defined in the neutral case, namely (A3). In this notation

$$\begin{aligned}&j_0(\mathbf{x})=\rho(\mathbf{x})=-\bar{\psi}(\mathbf{x})1\psi(\mathbf{x}),\quad S(\mathbf{x})=\bar{\psi}(\mathbf{x})\sigma_-\psi(\mathbf{x}),\\ &\dot{S}(\mathbf{x})=i\bar{\psi}(\mathbf{x})\sigma_3\psi(\mathbf{x}).\end{aligned} \tag{B2}$$

As in the neutral case, because $\sigma_-$, $\sigma_3$ are irreducible in the spinor space, we can be sure that the requirement

$$\begin{aligned}&[S(\mathbf{x}),f\{\phi,\dot{\phi},\phi^\dagger,\dot{\phi}^\dagger\}]=0,\\ &[\dot{S}(\mathbf{x}),f\{\phi,\dot{\phi},\phi^\dagger,\dot{\phi}^\dagger\}]=0,\end{aligned} \tag{B3}$$

implies

$$f=f[\rho(\mathbf{x})=-\bar{\psi}1\psi]. \tag{B4}$$

This much is identical with the neutral case—except that now $\bar{\psi}1\psi$ is not trivial. We can remove the $\rho$ dependence through the requirement of commutativity with $j^k$. Using (4.2)

$$[f,j^k(\mathbf{y})]=-2iS(\mathbf{y})\partial^{(y)k}\{\delta f[\rho]/\delta\rho(\mathbf{y})\}=0. \tag{B5}$$

Hence, any operator $f$ which commutes with the full set $J(\mathbf{x})$ is a multiple of the unit operator, and $J(\mathbf{x})$ is irreducible in each charge sector.