We use the term quasiprojection operator because of the particular order of the indices in (A2). Thus, we do not require, for instance,

$$\theta_{\beta,\alpha_1\cdots\alpha_l}{}^{\nu,\mu_1\cdots\mu_l}\theta_{\mu_i,\mu_1\cdots\mu_{i-1}\nu\mu_{i+1}\cdots\mu_l}{}^{\sigma,\rho_1\cdots\rho_l}$$

to have a simple form.

It is readily checked that our propagator (5) indeed obeys 1−3. Its uniqueness is then established by constructing

$$\theta_{\beta,\alpha_1\cdots\alpha_l}{}^{\beta,\alpha_1\cdots\alpha_l}(p)\,,$$

$$p'^{\beta}p'^{\alpha_1}\cdots p'^{\alpha_l}p_\nu'p_{\mu_i}'\cdots p_{\mu_l}'\theta_{\beta,\alpha_1\cdots\alpha_l}{}^{\nu,\mu_1\cdots\mu_l}(p)\,,$$

with

$$p\cdot p'=0\,,$$

and

$$\delta^{\beta\alpha_1}\delta_{\nu\mu_1}\theta_{\beta,\alpha_1\cdots\alpha_l}{}^{\nu,\mu_1\cdots\mu_l}$$

from Eq. (5) on the one hand and from Eq. (A1) on the other and by equating these expressions.

A completely similar argument can be used to derive the oscillatorlike propagator Eq. (12). The main differences in this case are the nonvalidity of the tracelessness condition of point 2, and the further reducibility of Eq. (A1).

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Superconvergent Dispersion Relations and Pion-Nucleon Sum Rules*

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Both the Adler-Weisberger sum rule and the spin-flip sum rule for pion-nucleon scattering have been derived from superconvergent dispersion relations for weak amplitudes. Our basic assumption is that the weak axial-vector–nucleon scattering amplitude $T_{\mu\nu}{}^A$ approaches the weak vector-nucleon scattering amplitude $T_{\mu\nu}{}^V$ at high energies. This allows us to write down superconvergent dispersion relations for certain invariant amplitudes in the decomposition of $T_{\mu\nu}=T_{\mu\nu}{}^A-T_{\mu\nu}{}^V$. We then use the hypotheses of partially conserved axial-vector current and of conserved vector current to obtain pion-nucleon scattering sum rules while avoiding the ambiguities of the $q\rightarrow 0$ limit which is usually used in the current-algebra approach. We also discuss sum rules for $G_A(q^2)$ away from $q^2=0$.

1. INTRODUCTION

THE chiral current algebra of Gell-Mann and the hypothesis of partial conservation of axial-vector current (PCAC) has been intensively used to obtain sum rules of interest in strong-interaction physics. The most celebrated of these is the Adler-Weisberger (AW) sum rule,[1] connecting the axial neutron β-decay constant G_A to an integral over π-N total cross sections. This sum rule is regarded as the direct confirmation of the validity of chiral current algebra and the PCAC hypothesis. It is interesting to see whether it is possible to obtain the AW sum rule without explicitly using the current commutation relations. The purpose of this paper is to show that this is indeed possible.

Our derivation of pion-nucleon sum rules is based on the following basic assumption: At high energy the weak axial-vector–nucleon scattering amplitude $T_{\mu\nu}{}^A$ approaches the weak vector-nucleon scattering amplitude $T_{\mu\nu}{}^V$, i.e., $T_{\mu\nu}(\nu)=T_{\mu\nu}{}^A(\nu)-T_{\mu\nu}{}^V(\nu)\rightarrow 0$ as $\nu\rightarrow\infty$.[2] This allows us to write superconvergent dispersion relations for certain invariant amplitudes appearing in the decomposition of $T_{\mu\nu}$. This gives sum rules connecting the weak axial-vector form factor $G_A(q^2)$ with vector form factors $F_1(q^2),F_2(q^2)$ and an integral over certain weak vector and axial-vector–nucleon scattering amplitudes. One of these sum rules at $q^2=0$ together with the PCAC hypothesis and conservation of vector current (CVC) gives the AW sum rule. In addition, we obtain the pion-nucleon spin-flip sum rule which has been very recently derived by Gerstein[3] and Maiani and Preparata.[4] We also discuss the relation between the axial-vector form factor $G_A(q^2)$ and vector form factors $F_1(q^2)$ and $F_2(q^2)$ at finite q^2.

It must be emphasized that we do not explicitly use the current commutation relations, i.e., we do not postulate a current algebra. However, our basic assumption would follow if $SU(2)\otimes SU(2)$ were a good asymptotic symmetry.[2] We feel that this is a weaker assumption than explicitly postulating the current commutation relations.

In Sec. II we give the derivation of the sum rules. In Sec. III we discuss the sum rules for $G_A(q^2)$ away from $q^2=0$, and the conclusions are in Sec. IV.

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[2] T. Das, V. Mathur, and S. Okubo, Phys. Rev. Letters **18**, 761 (1967); S. Gasiorowicz, Phys. Rev. **146**, 1067 (1966).

[3] I. S. Gerstein, Phys. Rev. **161**, 1631 (1967).

[4] L. Maiani and G. Preparata, Report No. ISS 67/6, Istituto Superiore di Sanita, Rome (unpublished).

2. DERIVATION OF SUM RULES

We define the weak axial-vector and vector-nucleon scattering amplitudes $T_{\mu\nu}{}^{A,ij}$ and $T_{\mu\nu}{}^{V,ij}$ to be

$$T_{\mu\nu}{}^{A,ij}=i\int d^4x\, e^{-iq_2\cdot x}\theta(x_0)\times\langle p_2|[A_\mu{}^i(x),A_\nu{}^j(0)]|p_1\rangle, \quad (2.1)$$

$$T_{\mu\nu}{}^{V,ij}=i\int d^4x\, e^{-iq_2\cdot x}\theta(x_0)\times\langle p_2|[V_\mu{}^i(x),V_\nu{}^j(0)]|p_1\rangle. \quad (2.2)$$

In Eqs. (2.1) and (2.2), i and j denote isospin. We also define

$$q_2=p_1+q_1-p_2,\quad P=\tfrac{1}{2}(p_1+p_2),$$
$$\nu=-q_1\cdot P/m=-q_2\cdot P/m,\quad t=-(q_1-q_2)^2.$$

In order to simplify the calculations, we shall confine ourselves to the forward direction only, so that $q_1=q_2=q$, $p_1=p_2=p$, $\nu=-q\cdot p/m$, and $t=0$. Now our basic assumption implies that

$$\lim_{\nu\to\infty} T_{\mu\nu}{}^{ij}(\nu,q^2)=\lim_{\nu\to\infty}[T_{\mu\nu}{}^{A,ij}(\nu,q^2)-T_{\mu\nu}{}^{V,ij}(\nu,q^2)]\to 0. \quad (2.3)$$

Following Gerstein's[3] careful analysis, we expand the amplitude $T_{\mu\nu}$ in a complete set of independent covariants as follows:

$$T_{\mu\nu}=p_\mu p_\nu[A_1+\bar{A}_1 i\gamma\cdot q]+p_\mu q_\nu(A_2+\bar{A}_2 i\gamma\cdot q)+q_\mu p_\nu(A_3+\bar{A}_3 i\gamma\cdot q)+q_\mu q_\nu(A_4+\bar{A}_4 i\gamma\cdot q)$$
$$+\delta_{\mu\nu}(A_5+\bar{A}_5 i\gamma\cdot q)+i\sigma_{\mu\nu}D_1+\tfrac{1}{2}i(\gamma_\mu\gamma\cdot q\gamma_\nu-\gamma_\nu\gamma\cdot q\gamma_\mu)D_2+ip_\mu\gamma_\nu B_1+ip_\nu\gamma_\mu B_2+iq_\mu\gamma_\nu B_3+iq_\nu\gamma_\mu B_4$$
$$+i(p_\mu\sigma_{\nu\lambda}q_\lambda+p_\nu\sigma_{\mu\lambda}q_\lambda)C_1+i(q_\mu\sigma_{\nu\lambda}q_\lambda+q_\nu\sigma_{\mu\lambda}q_\lambda)C_2. \quad (2.4)$$

Let $t_{\mu\nu}$ denote the imaginary part of $T_{\mu\nu}$, so that $t_{\mu\nu}$ can be expanded as

$$t_{\mu\nu}=p_\mu p_\nu[a_1+\bar{a}_1 i\gamma\cdot q]+\cdots. \quad (2.5)$$

In the isospin space, $T_{\mu\nu}{}^{ij}$ can be decomposed as

$$T_{\mu\nu}{}^{ij}=\delta_{ij}T_{\mu\nu}{}^{(+)}+\tfrac{1}{2}[\tau_i,\tau_j]T_{\mu\nu}{}^{(-)}. \quad (2.6)$$

The properties of these amplitudes under crossing and time reversal are given by Gerstein.[3] For the amplitudes of interest to us, we write down the crossing properties:

$$\begin{aligned}A_1{}^{(-)}(\nu,q^2)&=-A_1{}^{(-)}(-\nu,q^2),\\ \bar{A}_1{}^{(-)}(\nu,q^2)&=\bar{A}_1{}^{(-)}(-\nu,q^2),\\ B_i{}^{(-)}(\nu,q^2)&=-B_i{}^{(-)}(-\nu,q^2),\quad i=1,2\\ D_1{}^{(-)}(\nu,q^2)&=D_1{}^{(-)}(-\nu,q^2).\end{aligned}\quad (2.7)$$

As we shall work throughout with $T_{\mu\nu}{}^{(-)}$, we shall drop the superscript $(-)$ in what follows.

If the spins of the nucleons are summed, then we get from Eq. (2.4)

$$\begin{aligned}\tilde{T}_{\mu\nu}&=p_\mu p_\nu[A_1-\nu\bar{A}_1+(1/m)(B_1+B_2)]+\cdots,\\ \tilde{t}_{\mu\nu}&=p_\mu p_\nu[a_1-\nu\bar{a}_1+(1/m)(b_1+b_2)]+\cdots.\end{aligned}\quad (2.8)$$

From Eq. (2.3), it follows that

$$\nu R(\nu,q^2)\to 0\quad\text{as}\quad \nu\to\infty,$$

where

$$R=[A_1-\nu\bar{A}_1+(1/m)(B_1+B_2)].$$

Hence R satisfies a superconvergent dispersion relation. Separating the Born term and noting the crossing relations given in Eq. (2.7), we get the sum rule

$$G_A{}^2(q^2)+(4m^2/\pi)\int[r^A(\nu,q^2)-r^V(\nu,q^2)]d\nu=F_1{}^2(q^2)+(q^2/4m^2)F_2{}^2(q^2), \quad (2.9)$$

where $r^A-r^V=r$ is the absorptive part of R.

At $q^2=0$, we have from Eq. (2.9),

$$G_A{}^2(0)+(4m^2/\pi)\int[r^A(\nu)-r^V(\nu)]d\nu=F_1{}^2(0). \quad (2.10)$$

We now show that $r^V(\nu)=0$. Applying the CVC hypothesis above the physical threshold to $\tilde{t}_{\mu\nu}{}^V$, we get

$$q_\mu\tilde{t}_{\mu\nu}{}^V=0,$$

which at $q^2=0$ gives

$$r^V(\nu)=0\quad\text{at}\quad \nu\neq 0. \quad (2.11)$$

Similarly, applying the PCAC condition

$$\partial_\mu A_\mu{}^i=-(f_\pi/\sqrt{2})m_\pi{}^2\pi^i \quad (2.12)$$

to $\tilde{t}_{\mu\nu}{}^A$ above the physical threshold, we get

$$q_\mu\tilde{t}_{\mu\nu}{}^A q_\nu=\tfrac{1}{2}f_\pi{}^2\frac{m_\pi{}^4}{(q^2+m_\pi{}^2)^2}\,\mathrm{Im}(A-\nu B), \quad (2.13)$$

where the pion-nucleon scattering amplitude is defined as

$$T^{ij}=i\int d^4x\, e^{-iq\cdot x}\theta(x_0)\langle p|[j_\pi{}^i,j_\pi{}^j]|p\rangle=A^{ij}(\nu,q^2)+i\gamma\cdot qB^{ij}(\nu,q^2).$$

At $q^2=0$, we get

$$\begin{aligned}\nu^2r^A(\nu)&=\tfrac{1}{2}f_\pi{}^2\,\mathrm{Im}(A-\nu B)\\ &=\tfrac{1}{2}f_\pi{}^2\,\mathrm{Im}T.\end{aligned}\quad (2.14)$$

Hence we get from Eqs. (2.10), (2.11), and (2.14) the sum rule

$$G_A{}^2(0)+\frac{2}{\pi}f_\pi{}^2\int\frac{\mathrm{Im}T^{(-)}}{\nu^2}d\nu=F_1{}^2(0). \quad (2.15)$$

If the spins of the nucleons are not summed, we note from the condition (2.3) and Eqs. (2.4) and (2.7), that A_1, B_1, and B_2 separately satisfy superconvergent dispersion relations which at $q^2=0$ are given by

$$\tfrac{1}{4}G_A{}^2(0)+(2m/\pi)\int b_i(\nu)d\nu=\tfrac{1}{4}F_1(0)\times[F_1(0)+F_2(0)],\quad i=1,2 \tag{2.16}$$

$$(2m/\pi)\int a_1(\nu)d\nu=-(1/2m)F_1(0)F_2(0). \tag{2.17}$$

Here, $b_i=b_i{}^A-b_i{}^V$ and $a_1=a_1{}^A-a_1{}^V$. Again applying CVC to $t_{\mu\nu}{}^V$, we get $b_i{}^V(\nu)=0$ and $a_1{}^V(\nu)=0$ at $q^2=0$, for $\nu\neq0$, so that we get from Eqs. (2.16) and (2.17),

$$\tfrac{1}{4}G_A{}^2(0)+(2m/\pi)\int b_i{}^A(\nu)d\nu=\tfrac{1}{4}F_1(0)\times[F_1(0)+F_2(0)], \tag{2.18}$$

$$(4m^2/\pi)\int a_1{}^A(\nu)d\nu=-F_1(0)F_2(0). \tag{2.19}$$

Noting that $r^V(\nu)=0$ at $\nu\neq0$, we rewrite Eq. (2.10):

$$G_A{}^2(0)+(4m^2/\pi)\int r^A(\nu)d\nu=F_1{}^2(0). \tag{2.10$'$}$$

Subtracting Eq. (2.19) from Eq. (2.10′), we get

$$G_A{}^2(0)+(4m/\pi)\int[-m\nu\bar{a}_1{}^A(\nu)+b_1{}^A(\nu)+b_2{}^A(\nu)]d\nu=F_1(0)[F_1(0)+F_2(0)]. \tag{2.20}$$

Now applying PCAC to $t_{\mu\nu}{}^A$, we get

$$-(1/m\nu)\,\mathrm{Im}B(\nu)=(2/f_\pi{}^2)\times[-m\nu a_1{}^A+b_1{}^A+b_2{}^A-(2/\nu)d_1{}^A].$$

Hence we get from Eq. (2.20)

$$G_A{}^2(0)-2f_\pi{}^2(1/\pi)\int(1/\nu)\,\mathrm{Im}B^{(-)}(\nu)d\nu=F_1(0)[F_1(0)+F_2(0)]-(4m/\pi)\times\int(2/\nu)d_1{}^A(\nu)d\nu. \tag{2.21}$$

Our normalization implies

$$F_1(0)=1$$

and

$$F_1(0)+F_2(0)=\mu_p-\mu_n=4.7\,,$$

so that Eq. (2.15) is immediately recognized as the Adler-Weisberger sum rule, whereas Eq. (2.21) is the spin-flip sum rule derived by Gerstein.[3]

3. SUM RULE FOR $G_A(q^2)$

In this section we discuss our sum rule (2.9) at finite momentum transfer q^2. If we neglect the contribution from the integral in Eq. (2.9), we get

$$G_A{}^2(q^2)=F_1{}^2(q^2)+\frac{q^2}{4m^2}F_2{}^2(q^2)=\left(1+\frac{q^2}{4m^2}\right)^{-1}\left[G_E{}^2(q^2)+\frac{q^2}{4m^2}G_M{}^2(q^2)\right], \tag{3.1}$$

where G_E and G_M are the Sachs form factors. This approximation of neglecting the contribution of the integral is certainly not justified at small q^2 but at high q^2 this approximation may not be bad. Therefore we examine the consequences of this approximation at high q^2. From Eq. (3.1), it is clear that for large q^2 ($q^2\gg4m^2$),

$$G_A{}^2(q^2)\simeq G_M{}^2(q^2). \tag{3.2}$$

Empirically,[5] the following relation appears to hold:

$$G_M(q^2)=(\mu_p-\mu_n)[1+(q^2/m_V{}^2)]^{-2}, \tag{3.3}$$

with $m_V{}^2=0.71$ BeV^2 for q^2 up to 10 BeV^2. If Eq. (3.3) holds for $q^2\gg4m^2$, we get

$$G_A(q^2)\simeq(\mu_p-\mu_n)\left(1+\frac{q^2}{m_V{}^2}\right)^{-2}. \tag{3.4}$$

If further we assume the "double-pole" fit also for $G_A(q^2)/G_A$,

$$\frac{G_A(q^2)}{G_A}=\left(1+\frac{q^2}{m_A{}^2}\right)^{-2}, \tag{3.5}$$

then we get the relation[6]

$$G_A=\left(\frac{m_V}{m_A}\right)^4(\mu_p-\mu_n). \tag{3.6}$$

This is a surprisingly good expression if we take m_V and m_A to be the physical masses of the vector and axial-vector mesons as pointed out by Schechter and Venturi.[6] If, on the other hand, we saturate the absorptive part

[5] W. Albrecht, H. J. Behrend, H. Dorner, W. Flauger, and H. Hultschig, Phys. Rev. Letters **18**, 1014 (1967); M. Goitein *et al.*, *ibid.* **18**, 1016 (1967).

[6] J. Schechter and G. Venturi, Phys. Rev. Letters **19**, 276 (1967).

in the integral of Eq. (2.9), by the N^* resonance we get

$$F_1^2(q^2)+\frac{q^2}{4m^2}F_2^2(q^2)-\frac{4}{9}q^2\left(1+\frac{q^2}{(m+m^*)^2}\right)^{-1}[D_1^V(q^2)]^2$$
$$=G_A^2(q^2)-\frac{2}{9_{m^*}{}^2}\Big\{[(m+m^*)^2+q^2][D_1^A(q^2)]^2+2\frac{D_1^A(q^2)D_2^A(q^2)}{(m+m^*)}[m^*q^2+(m+m^*)(m^{*2}-m^2-q^2)]$$
$$+2D_1^A(q^2)D_3^A(q^2)(m+m^*)^{-2}(m^{*2}-m^2-q^2)[(m+m^*)^2+q^2]+2D_2^A(q^2)D_3^A(q^2)(m+m^*)^{-2}$$
$$\times[4m^{*2}q^2+(m^{*2}-m^2-q^2)^2]+[D_3^A(q^2)]^2(m+m^*)^{-4}[(m+m^*)^2+q^2][4m^{*2}q^2+(m^{*2}-m^2-q^2)^2]$$
$$+[D_2^A(q^2)]^2(m+m^*)^{-2}[4m^{*2}q^2+(m^{*2}-m^2-q^2)^2]\Big\}. \quad (3.7)$$

Here we have used only the $M1$ coupling for vector current to N^*N,

$$\langle N^*(p')|V_\mu|N(p)\rangle=D_1^V(q^2)\times\left(1+\frac{q^2}{(m+m^*)^2}\right)^{-1}\epsilon_{\mu\nu\kappa\lambda}P_\nu q_\kappa\bar{u}_\lambda(p')u(p)\,,$$

and for the N^*NA vertex we have used the following[7]:

$$\langle N^*(p')|A_\alpha|N(p)\rangle=\bar{u}_\beta(p')[D_1^A(q^2)\delta_{\alpha\beta}+D_2^A(q^2)\times(m+m^*)^{-1}\gamma_\alpha q_\beta+D_3^A(q^2)(m+m^*)^{-2}P_\alpha q_\beta+D_4^A(q^2)(m+m^*)^{-2}q_\alpha q_\beta]u(p)\,,$$

with $P=p+p'$, $q=p-p'$.

At $q^2=0$ we get

$$F_1^2(0)=G_A^2(0)-\frac{2}{9}\frac{(m+m^*)^2}{m^{*2}}\times\left\{D_1^A(0)+\frac{m^*-m}{m^*+m}[D_2^A(0)+D_3^A(0)]\right\}^2. \quad (3.8)$$

[7] R. Oehme, Phys. Rev. **143**, 1138 (1966); C. H. Albright and L. S. Liu, *ibid.* **140** B748 (1965); R. Oehme, Phys. Letters **19**, 518 (1965); S. M. Berman and M. Veltman, Nuovo Cimento **38**, 993 (1965).

As $F_1^2(0)=1$, this is precisely the same sum rule as obtained by Oehme[7] from current algebras. This equation is known to give too high a value of $G_A(0)$.[7]

Due to lack of experimental knowledge of the axial form factors appearing in Eq. (3.7), we do not know whether this is a good relation away from $q^2=0$. At $M=M^*$, Eq. (3.8) becomes the familiar $SU(6)$ relation

$$1=g_A^2-(8/9)[D_1^A(0)]^2.$$

4. CONCLUSIONS

We feel that our technique of deriving the AW sum rule has certain advantages over the usual current algebra approach. Our Eq. (2.9) was obtained without using the PCAC condition. We get the AW relation from it by subsequently using PCAC. In this way we have avoided the ambiguous limit $q\to 0$ which is required in the usual derivation of the AW relation from the algebra of currents. Also, (2.9) gives us information about $G_A(q^2)$ away from $q^2=0$.

We have also been able to obtain the πN spin-flip sum rule in a much simpler manner without having to postulate moment sum rules, etc.[3,4]

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