Nonforward Superconvergence Relations, Higher Symmetries, and Mass Spectra*

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Superconvergence relations for the reactions $PV \to PV$, $PV \to VV$, and $VV \to VV$ between nonets of pseudoscalar and vector mesons can be saturated at zero momentum transfer with sets of particles corresponding to the representations $(6,\bar{6};l)$ of the rest-symmetry group $U(6)\times U(6)\times O(3)$. Every single one of these representations saturates the relations, provided the vertices are invariant under the collinear $U(6)\times O(2)$ group. The infinite sequence of representations $(6,\bar{6};l)$, $l=0,1,\ldots$ is then used in order to saturate the nonforward superconvergence relations for the reactions $PV \to PV$. For mass spectra m_l with accumulation points $m_\infty^2 > 4m_0^2$, it is found that the resulting equations have *no* nontrivial solution for the coupling constants. This result remains unchanged for an oscillator-like spectrum. The possibility is discussed that mass splitting within the multiplets $(6,\bar{6}0;l)$ and/or symmetry breaking at the vertices can in principle make a saturation possible. It is argued that an approximate saturation of the amplitudes and a few of their derivatives at $t=0$ with a finite number of resonances may well be reasonable. For higher derivatives the saturation is expected to depend sensitively upon the absorptive parts at higher energies, which are more reasonably described by Regge terms than by direct-channel resonances. The formal saturation of superconvergence relations with mass-degenerate multiplets is discussed briefly.

1. INTRODUCTION

SUPERCONVERGENCE relations for amplitudes with collinear kinematics generally have been found[1] to allow a consistent saturation with sets of particles grouped into representations of the rest-symmetry groups $U(6)\times U(6)$ or $U(6)\times U(6)\times O(3)$. These groups are closely associated with the quark model of mesons and baryons and in particular the latter group has the sequence of representations $(6,\bar{6};l)$, which correspond to the orbital excitations of the quark-antiquark system. The sets of equations resulting from such saturation schemes have solutions for the vertices which correspond to collinear $U(6)$,[2] $U(6)\times O(2)$,[3] or to larger symmetries into which these groups can be embedded.

Of special interest are the superconvergence relations for the reactions $PV \to PV$, $PV \to VV$ and $VV \to VV$ between nonets of pseudoscalar and vector mesons. These relations can be satisfied in the collinear case by inserting intermediate states corresponding to any representation $(6,\bar{6};l)$ of $U(6)\times U(6)\times O(3)$ with mass m_l, provided the vertices are invariant under the collinear $U(6)\times O(2)$ group. This saturation will work no matter what the value of l or m_l. Hence we have here an infinite sequence of sets of particles with different masses m_l which separately saturate the forward relations.

It is the purpose of this paper to consider the saturation of the corresponding nonforward superconvergence relations using this infinite sequence $(6,\bar{6};l)$, $l=0, 1, 2, \cdots$ of orbital excitations of the quark-antiquark system. Any such saturation with an infinite sequence of single particle states is an extreme idealization. We know that there exists continua with quantum numbers other than those of the particles we take into account. It is assumed, *a priori*, that these continua are disconnected from the particles we consider, although in the case of infinite sequences of particles this may well be a rather drastic assumption.

A further aspect of this problem is that, as always in saturation schemes, one treats higher (and this in our case means very much higher) states in the zero-width approximation.

Another problem is connected with the fact that complete saturation of superconvergence relations for a finite neighborhood in the momentum transfer t is generally only possible if the spin of the particles is not limited. Regge trajectories which go to infinity create difficulties with the polynomial bound of the amplitudes in s and t simultaneously. However, for mass spectra or trajectories of limited increase, we may at least have sufficient boundedness in one variable for restricted values of the other variable in order to define a *unique* analytic interpolation of the partial wave amplitudes.[4]

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[1] R. Oehme, Phys. Letters **21**, 567 (1966); **22**, 207 (1966); Phys. Rev. **154**, 1358 (1967); V. de Alfaro, G. Furlan, and G. Rossetti, Phys. Letters **21**, 576 (1966); R. Oehme and G. Venturi, Phys. Rev. **159**, 1283 (1967), K. Bardakci and G. Segrè, *ibid.* **159**, 1263 (1967); G. Venturi, *ibid.* this issue **163**, 1826 (1967); V. A. Matveev, L. D. Soloviev, B. V. Struminsky, A. N. Tavkhelidze and V. P. Shelest, Dubna Report No. P2-3118 (unpublished); P. G. O. Freund, R. Oehme and P. Rotelli; Nuovo Cimento **51**, 217 (1967).

[2] K. J. Barnes, P. Carruthers, and F. von Hippel, Phys. Rev. Letters **14**, 81 (1965); R. Oehme, *ibid.* **14**, 664, 866 (1965), and in *Preludes in Theoretical Physics*, edited by A. de-Shalit, H. Feshbach, and L. Van Hove (North-Holland Publishing Company, Amsterdam, 1966), pp. 143–153; H. J. Lipkin and S. Meshkov, Phys. Rev. Letters **14**, 670 (1965); Phys. Rev. **143**, 1269 (1966); K. J. Barnes, Phys. Rev. Letters **14**, 798 (1965); Phys. Rev. **139**, B947 (1965); P. G. O. Freund, Phys. Rev. Letters **14**, 803 (1965); P. G. O. Freund and R. Oehme, *ibid.* **14**, 1085 (1965).

[3] P. G. O. Freund, Phys. Rev. Letters **16**, 291 (1966); P. G. O. Freund, A. Maheshwari, and E. Schonberg, Phys. Rev. **159**, 1232 (1967); H. J. Lipkin, Phys. Rev. **159**, 1303 (1967).

[4] See, for example, R. Oehme, *Strong Interactions and High Energy Physics*, edited by R. G. Moorhouse (Oliver and Boyd, Edinburgh, 1964), pp. 129–222, especially p. 139.

All these problems should be kept in mind when discussing our results, which are as follows: with an infinite sequence of one-particle intermediate states from $(6,\bar{6};l)$-type multiplets, and $U(6)\times O(2)$ invariant vertices, it is not possible to saturate the superconvergence relations for the double helicity-flip $PV\to PV$ amplitudes for a finite interval in the momentum transfer variable t. This result holds for any reasonable mass spectrum. The only mass spectra which are not directly excluded are those with an accumulation point below the threshold $s=4m_0^2$ for the reactions $PV\to PV$; however such spectra are highly unrealistic in view of existing experiments. The meaning and the implications of our result will be discussed in Sec. 7. We note that a virtue of our approach is that we consider hadron spectra and vertex symmetries with roots both in the quark model and in experiment.

2. THE SUPERCONVERGENCE RELATIONS

In order to have a definite example, we concentrate on the superconvergence relations for the amplitudes $PV\to PV$ which are selected by taking the infinite momentum limit for the pseudoscalar meson; they correspond to a helicity flip $\Delta h=2$. Using the crossing properties of these amplitudes, we can write our relations in the form

$$\int_0^\infty d\nu\, a_{[i,j]}(\nu,t)=0\,, \qquad (1)$$

for values of the momentum transfer variable t in the neighborhood of $t=0$. Here we have used the variable $\nu=\frac{1}{4}(s-u)$, and the indices i, j describe the unitary spin of the vector mesons; and by $[i,j]$ we indicate the part of the amplitude which is odd under $i\leftrightarrow j$. Other quantum numbers have been suppressed.

For reason of comparison, we will later also consider an integral similar to Eq. (1) for the corresponding absorptive parts $b_{[i,j]}(\nu,t)$ of the amplitudes for the reactions $PP\to PP$. These amplitudes are *not* superconvergent. Hence we expect to have only sum rules of the form[5]

$$\int_0^{\nu_L} d\nu\, b_{[i,j]}(\nu,t)=\sum_a \beta_a(t)\nu_L^{\alpha_a(t)+1}[\alpha_a(t)+1]^{-1}, \qquad (1a)$$

where $\alpha_a(t)$ are the dominant Regge trajectories, and where ν_L is large compared to the external masses.

[5] This relation is essentially the condition that the continued negative signature partial wave amplitude in the crossed t channel does not have a pole at $l=-1$. Generally there are singularities in the l plane at $l=\alpha_i(t)$ with $\mathrm{Re}\,\alpha_i(t)\geq -1$, and these give rise to those terms on the right-hand side of Eq. (1a) which do not vanish for $\nu_L\to\infty$. See Ref. 4, p. 146 ff; also A. Logunov, A. L. Soloviev and A. N. Tavkhelidze, Phys. Letters **24B**, 181 (1967); L. A. P. Balázs and J. M. Cornwall, Phys. Rev. **160**, 1313 (1967); D. Horn and C. Schmid, Caltech (to be published).

3. MULTIPLETS AND VERTICES

We describe the pseudoscalar and the vector mesons by the matrix

$$M_0(p)=[1+(i\gamma\cdot p/m_0)](\gamma_5 P+\gamma_\alpha V_\alpha)\,, \qquad (2)$$

where P and V_α ($p_\alpha V_\alpha=0$) are the usual $U(3)$ matrices for the respective nonets. The quantity $M_0(p)$ corresponds to the representation $(6,\bar{6};l=0)$ of the rest-symmetry group $U(6)\times U(6)\times O(3)$. As intermediate states we want to introduce the full sequence of representations $(6,\bar{6};l)$, $l=0, 1, \cdots$. For a given l, we write

$$(M_l)_{\alpha_1\cdots\alpha_l}(p) =\left(1+\frac{i\gamma\cdot p}{m_l}\right)\{\gamma_5 B_{\alpha_1\cdots\alpha_l}+\gamma_\beta C_{\beta,\alpha_1\cdots\alpha_l}\}\,, \qquad (3)$$

where $B_{\alpha_1\cdots\alpha_l}$ and $C_{\beta_1\alpha_1\cdots\alpha_l}$ are symmetric, divergenceless and traceless in the indices $\alpha_1\cdots\alpha_l$, and m_l is the "central" mass of the $(6,\bar{6};l)$ multiplet. The $U(3)$ matrix $B_{\alpha_1\cdots\alpha_l}$ describes a nonet of mesons with spin $J=l$, $P=(-1)^{l+1}$ and $C=(-1)^l$. It is a singlet as far as the quark spin is concerned. Correspondingly, the tensor $C_{\beta,\alpha_1\cdots\alpha_l}$, which is divergenceless also in the index β, describes the triplet states consisting of three nonets with spins $J=l-1$, l, and $l+1$, $P=(-1)^l$ and $C=(-1)^{l+1}$; altogether, these amount to $3\cdot(2l+1)$ individual states. We can decompose (see Appendix) the tensor C with respect to the $U(3)$ matrices of the three nonets taking account of the normalization condition that in the "propagator" or "mass term"

$$\sum_{\text{polarizations}} \mathrm{Tr}(M_l)_{\alpha_1\cdots\alpha_l}(-p)(M_l)_{\alpha_1\cdots\alpha_l}(p) \qquad (4)$$

every one of the $4(2l+1)$ components appears with the same weight. However, as long as the masses within a given multiplet $(6,\bar{6};l)$ remain degenerate, we have no need to write out this decomposition. Since the group $O(3)$ appears as a factor in the direct product $U(6)\times U(6)\times O(3)$, the propagator of the spin triplet field $C_{\beta,\alpha_1\cdots\alpha_l}$ can be written in the simple factorized form (see Appendix)

$$\Theta_{\beta'}{}^{\beta}(p)\theta_{\alpha_1'\cdots\alpha_l'}{}^{\alpha_1\cdots\alpha_l}(p)\equiv\Theta_{\beta',\alpha_1'\cdots\alpha_l'}{}^{\beta,\alpha_1\cdots\alpha_l}(p)\,, \qquad (5)$$

where

$$\theta_\beta{}^\alpha(p)=\delta_{\alpha\beta}+(p_\alpha p_\beta/m_l^2)\,, \qquad (6)$$

and where the second factor is the usual propagator for a particle of spin l and mass m_l.

We are interested in the saturation of the superconvergence relations (1) under the assumption that the vertices $M_0M_0M_l$ are invariant under the collinear $U(6)\times O(2)$ group which is the maximal natural dynamical symmetry of vertex functions in view of our choice of the rest symmetry as $U(6)\times(6)\times O(3)$. The coupling constants are defined by writing the most general invariant vertex in the form

$$ig_{00l}(2m_l)^{-l}\,\mathrm{Tr}\{(M_l)_{\alpha_1\cdots\alpha_l}(p_3)\times[M_0(p_1)M_0(p_2)+(-1)^lM_0(p_2)M_0(p_1)]\}\times(p_1-p_2)_{\alpha_1}\cdots(p_1-p_2)_{\alpha_l},\quad(7)$$

where $p_1+p_2+p_3=0$, $p_1^2=p_2^2=-m_0^2$, and $p_3^2=-m_l^2$.

4. SATURATION WITH ORBITAL EXCITATIONS

Let us now saturate the superconvergence relations (1) with the states of the sequence of representations $(6,\bar{6};0)$, $(6,\bar{6};1)$, $(6,\bar{6};2)$, $\cdots$, assuming the vertices (7). This may be written in the Regge notation $(6,\bar{6};l=\alpha(t))$. Such a spectrum would be predicted by the noncompact group $U(6)\times U(6)\times O(3,1)$. It is useful to introduce the notations S(even) and S(odd) in the form

$$S(\text{even, odd})\equiv\sum_{l=\text{even,odd}}g_{00l}^2\left(1+\frac{2m_0}{m_l}\right)^2 2^{-2l}\times\left(1-\frac{4m_0^2}{m_l^2}\right)^l\frac{l!}{(2l-1)!!}P_l\left(1+\frac{2t}{m_l^2-4m_0^2}\right).\quad(8)$$

Upon saturation the superconvergence relation (1) then takes the form

$$[m_0^2S(\text{even})+(-m_0^2+\tfrac{1}{2}t)S(\text{odd})]\times\sum_n(f_{ain}f_{njb}-f_{ajn}f_{nib})+[m_0^2S(\text{odd})+(-m_0^2+\tfrac{1}{2}t)\times S(\text{even})]\sum_n(d_{ain}d_{njb}-d_{ajn}d_{nib})=0,\quad(9)$$

where f_{ain}, d_{ain}, etc., are the familiar $U(3)$ constants; with i, j and a, b being the $U(3)$ indices of the vector and pseudoscalar mesons, respectively.

Observing that

$$\sum_{n=0}^{8}(f_{ain}f_{njb}-f_{ajn}f_{nib})-\sum_{n=0}^{8}(d_{ain}d_{njb}-d_{ajn}d_{nib})\equiv0,$$

it is a simple matter to verify that the Eqs. (9) are all satisfied provided we have

$$t\{S(\text{even})+S(\text{odd})\}=0.\quad(10)$$

First of all, we see from Eqs. (8)–(10) that the forward ($t=0$) superconvergence relations are satisfied separately for every individual multiplet $(6,\bar{6};l)$; no restrictions are obtained in this case for the coupling constants g_{00l} and the masses m_l.[6] As far as $t\neq0$ is concerned, we consider Eq. (10) in the neighborhood of $t=0$ by taking all derivatives evaluated at this point.

We obtain the unlimited set of equations

$$t^{n+1}\sum_{l=n}^{\infty}g_{00l}^2\left(1+\frac{2m_0}{m_l}\right)^2\left(1-\frac{4m_0^2}{m_l^2}\right)^{l-n}\times\frac{1}{2^{2l}m_l^{2n}}\frac{l!}{(2l-1)!!}\frac{(l+n)!}{(l-n)!}\frac{1}{n!}=0.\quad(11)$$

If there exists an l_0 such that for $l\geq l_0$, $m_l^2>4m_0^2$, then all terms in Eq. (11) with $n>l_0$ will be positive definite.

Therefore, Eq. (11) has no nontrivial ($g_{00l}\neq0$) solution with real coupling constants for mass spectra with accumulation points m_∞^2 satisfying $m_\infty^2>4m_0^2$.

There may or may not be a solution for mass spectra with accumulation points $m_\infty^2<4m_0^2$, but spectra of this type are not very reasonable within the dispersion theoretical framework underlying our superconvergence relations, nor do they seem to correspond to reality.

5. SATURATION WITH AN OSCILLATOR-TYPE SPECTRUM

So far, we have considered as intermediate states only rotational excitations of the basic $U(6)\times U(6)$ structure of the quark-antiquark system. We may include "radial" excitations as well, but as long as the collinear $U(6)\times O(2)$ invariance of the vertices is not broken, our conclusions concerning the mass spectra will not be changed. A very simple example with "radial" excitations is the oscillator-like model which is obtained by retaining all traces in the tensors (3). This would be the sequence $l=0$; $l=0$, 2; $l=1$, 3; $l=0$, 2, 4; $\cdots$ as required by the noncompact group $U(6)\times U(6)\times U(3,1)$. The propagator $\theta_{\alpha_1'\cdots\alpha_l'}{}^{\alpha_1\cdots\alpha_l}$ in Eq. (5) is then replaced by (see Appendix)

$$P_{\alpha_1'\cdots\alpha_l'}{}^{\alpha_1\cdots\alpha_l}(p)=\frac{1}{(l!)^2}\sum_{P(\alpha),P(\alpha')}\prod_{i=1}^{l}\theta_{\alpha_i'}{}^{\alpha_i}(p),\quad(12)$$

and the sums S in Eq. (9) become

$$S^{(0)}(\text{even,odd})=\sum_{l=\text{even,odd}}^{\infty}g_{00l}^2\left(1+\frac{2m_0}{m_l}\right)^2 2^{-2l}\times\left(1-\frac{4m_0^2}{m_l^2}\right)^l\left(1+\frac{2t}{m_l^2-4m_0^2}\right)^l.\quad(13)$$

In place of the Eq. (11) we now have the set

$$t^{n+1}\sum_{l=n}^{\infty}g_{00l}^2\left(1+\frac{2m_0}{m_l}\right)^2\left(1-\frac{4m_0^2}{m_l^2}\right)^{l-n}\times\frac{2^n}{2^{2l}m_l^{2n}}\frac{l!}{(l-n)!}=0,\quad(14)$$

which has exactly the same characteristic features as Eq. (11) of Sec. 4. Thus, even though the set of inter-

[6] Using somewhat different methods, a proof for this result has also been given by R. Amann, University of Chicago (unpublished).

mediate states is much richer in this case, superconvergence is still impossible for all resonable meson mass spectra.

6. DEGENERATE MASSES

It is of interest to compare our superconvergence condition (10) for the $\Delta h=2\ PV\to PV$ amplitudes with the corresponding but formal expression we would obtain for the reactions $PP\to PP$. These amplitudes are not superconvergent, but they may satisfy sum rules of the type (1a).

Formal saturation of the integral over the absorptive part $b_{[i,j]}$ with the sets of states corresponding to the representations $(6,\bar{6};l)$ gives an expression very similar to Eqs. (8)–(10), namely

$$\sum_{l=0}^{\infty} g_{00l}^2\left(1+\frac{2m_0}{m_l}\right)^2\frac{m_l^2}{m_0^2}\left(1-\frac{m_l^2+2t}{4m_0^2}\right)\left(1-\frac{4m_0^2}{m_l^2}\right)^l \times 2^{-2l}\frac{l!}{(2l-1)!!}P_l\left(1+\frac{2t}{m_l^2-4m_0^2}\right). \quad (15)$$

Note that, for large values of m_l^2, the expression (15) has two more powers of m_l^2 than Eqs. (8)–(10).

In the case of degenerate masses $m_l\equiv m$, the sum (15) reduces to

$$\left(\frac{3}{4}-\frac{t}{2m^2}\right)\sum_l g_{00l}^2(-3)^l 2^{-2l}\frac{l!}{(2l-1)!!}P_l\left(1-\frac{2t}{3m_l^2}\right), \quad (16)$$

and the corresponding expression (10) for the $PV\to PV$ amplitudes with $\Delta h=0$ becomes

$$t\sum_l g_{00l}^2(-3)^l 2^{-2l}\frac{l!}{(2l-1)!!}P_l\left(1-\frac{2t}{3m_l^2}\right), \quad (17)$$

which is the same as Eq. (13) except for an over-all factor. We see that, for degenerate masses, the condition (10) no longer has any direct relation to the superconvergent character of the amplitude. Partial and approximate saturation of superconvergence relations with mass degenerate multiplets presumably is only sensible for finite sets of particles and in connection with continuum contributions.

In certain situations, formal superconvergence of helicity-flip amplitudes can be achieved[7] by using a complete set of intermediate states corresponding to an infinite dimensional, unitary representation of $O(3,1)$ [*not* $U(6)\times U(6)\times O(3,1)$]. The superconvergence condition, which in this mass-degenerate case corresponds to the identical vanishing of the helicity-flip amplitudes themselves, is then reduced to the completeness relation of the corresponding unitary representation. Since the masses are all degenerate, the asymptotic properties of the amplitudes are not relevent and the saturation has little to do with the actual problem of superconvergence. Nevertheless, it may be instructive to discuss in some detail the difference between our saturation scheme in the limit of degenerate masses and other schemes which give rise to completeness relations. From Eq. (11), we see that our "superconvergence condition" (10) has only the trivial solution (all couplings zero) for $m_l\equiv m$ and $t\neq 0$; it is then an orthogonal expansion. On the other hand, if one considers the mesons as the particles associated with the usual ladder representation of $O(3,1)$, it has been shown by Fubini[7] that the helicity-flip amplitudes for the scattering of these mesons by a superscaler can be made to vanish by a proper choice of the vertices. Hence, in this model, we have a solution of the superconvergence condition which is nontrivial in the sense that the couplings do not vanish identically. The mesons used in this scheme[7] can essentially be viewed as the ground state and the orbital excitations of a system consisting of two spin-zero particles. Then the spin of the vector meson, for example, is completely of orbital origin, and all orbital excitations are controlled by the group $O(3,1)$. On the other hand, in our case, part of the mesonic angular momentum is due to the spin of the constituent quarks, and the remainder comes from orbital excitations. We also "explode" the orbital angular momentum, but we combine it with the total "quark-spin" of the quark-antiquark system, which is not exploded. In particular, the vector mesons in our scheme are spin-triplet s-wave states. The analogous states in Fubini's model would be 0^+ states, and there is, of course, no superconvergence condition for 0^+-0^+ scattering. In our scheme too, superconvergence relations for say 0^-2^+ scattering (M_0M_1 scattering) could be saturated without difficulties related to positive definiteness. It is only the vector mesons which by their nature as $q\bar{q}$ S states cause a problem, but a nonnegligible problem at that.

Finally, we note that, in the case of saturation schemes with degenerate masses and noncompact groups, some care must be taken to specify what one means by the "superconvergence condition." We consider the equal mass limit in the actual *saturation* of a superconvertence condition like

$$\int_0^{\alpha} ds\, a(s,t)=0. \quad (18)$$

The sum

$$a(s,t)=\sum_{n=0}^{\infty} a^{(n)}(t)\delta(s-m_n^2)$$

is reduced to a single term

$$a(s,t)=\left[\sum_n a^{(n)}(t)\right]\delta(s-m^2),$$

[7] S. Fubini, Report at the IVth Coral Cables Conference on Symmetry Principles at High Energies, January, 1967 (unpublished). See also V. de Alfaro, Torino Report (unpublished).

and the condition (18) implies that the absorptive part itself vanishes.

In some models using noncompact groups, one obtains expressions for helicity-flip amplitudes which are formally superconvergent.[7,8] However, in these cases superconvergence is achieved at the cost of unphysical singularities which are not allowed in conventional dispersion theory.

7. CONCLUSIONS AND DISCUSSIONS

The undesired positive definiteness of the terms in the superconvergence condition (10) for $m_l^2 > 4m_0^2$ is clearly a direct consequence of the same feature of collinear $U(6)\times O(2)$-invariant vertices which also gives rise to the satisfactory term by term cancellation for $t=0$. By introducing mass splittings within the multiplets $(6,\bar{6};l)$, we can obtain factors in the sum (11) which change sign for at least some terms for every value of n. In this way, one could possibly construct a nontrivial solution for a given set of superconvergence conditions.[9] But the main problem is to find a general scheme for the symmetry breaking which allows a nontrivial solution for the complete set of all relevant superconvergence relations. Although one may think that it is possible in this way to obtain strong restrictions on the mass spectrum and the symmetry breaking, it is perhaps somewhat implausible that the (presumably rather small) mass splittings within the $U(6)\times U(6)$ multiplets or the $U(6)\times O(2)$-symmetry breaking at the vertices should salvage the whole edifice of superconvergence, but we cannot exclude this possibility.

There are other and perhaps more reasonable possibilities, especially in view of the dispersion-theoretical difficulties connected with rising trajectories.

Let us mention here only one other possible view concerning the saturation of superconvergence relations: Even though there are some indications from present experiments for linearly rising Regge trajectories, such a pattern may well not continue indefinitely. But if trajectories turn downward at finite energies, we have only resonances with limited spin values, and we know that we cannot saturate the full set of nonforward superconvergence relations with these states alone. Rather, we must assume that at higher energies the continuum, which includes quantum numbers other than those of the resonances, takes over. Of course, this continum could also be considered as a superposition of very wide resonances, but it appears that, at higher energies, it is more sensible to characterize the amplitudes by the singularities in the *crossed* channel which appear in the form of Regge poles, moving branch points, and possibly other singularities in the complex angular momentum plane of this channel.

Once we are willing to include this continuum at least at high energies, the question arises how these continuum contributions collaborate with the nonet resonances at lower energies in order to saturate the superconvergence relations. We expect that in superconvergent amplitudes the continuum contribution at high energies is rather small in comparison with the *absolute value* of the resonance contribution at lower energies. Hence we may expect also that an approximate cancellation of the lower energy resonances in superconvergence integrals like Eq. (1), is a reasonable condition for the general validity of the equation, at least for $t=0$ and a few derivatives at $t=0$. Of course, superconvergence conditions with higher powers of t eventually require the help of the continuum at higher energies.[10] We must remember that these higher-order terms also determine in principle the amplitude as an analytic function of t, and hence they also characterize the singularities in that channel.

APPENDIX

Here we shall establish the factorized forms (5) and (12) of the spin triplet propagators.

We first decompose the spin triplet part $C_{\beta,\alpha_1\cdots\alpha_l}$ of the $(6,\bar{6};l)$ supermultiplet into its spin $(l+1)$, l, and $(l-1)$ components $X_{\beta,\alpha_1\cdots\alpha_l}$, $Y_{\alpha_1\cdots\alpha_l}$, and $Z_{\alpha_2\cdots\alpha_l}$ (all totally symmetric and traceless). In detail, we have $[P(\alpha_1\cdots\alpha_l)=$ permutation of indices $\alpha_1\cdots\alpha_l]$

$$\begin{aligned}C_{\beta,\alpha_1\cdots\alpha_l}(p)&=X_{\beta,\alpha_1\cdots\alpha_l}(p)+\frac{1}{[l(l+1)]^{1/2}}\sum_{P(\alpha_1\cdots\alpha_l)}\epsilon_{\beta\alpha_1\mu\nu}p_\mu\\&\times Y_{\nu\alpha_2\cdots\alpha_l}(p)+\frac{1}{l}\left(\frac{2l-1}{2l+1}\right)^{1/2}\left\{\sum_{P(\alpha_1\cdots\alpha_l)}\left(\delta_{\beta\alpha_1}+\frac{p_\beta p_{\alpha_1}}{m_l^2}\right)\right.\\&\times Z_{\alpha_2\cdots\alpha_l}(p)-\frac{2}{2l-1}\sum_{P(\alpha_1\cdots\alpha_l)}\left(\delta_{\alpha_1\alpha_2}+\frac{p_{\alpha_1}p_{\alpha_2}}{m_l^2}\right)\\&\left.\times Z_{\beta\alpha_3\cdots\alpha_l}(p)\right\}.\end{aligned}\tag{A1}$$

We observe that the propagator $\theta_{\beta,\alpha_1\cdots\alpha_l}{}^{\nu,\mu_1\cdots\mu_l}$ of C has to be

(1) divergenceless in *all* its indices,

(2) symmetric and traceless in any pair of indices $\alpha_1\cdots\alpha_l$ and in any pair of indices $\mu_1\cdots\mu_l$,

(3) a "quasiprojection operator," namely

$$\theta_{\beta,\alpha_1\cdots\alpha_l}{}^{\nu,\mu_1\cdots\mu_l}\theta_{\nu,\mu_1\cdots\mu_l}{}^{\sigma,\rho_1\cdots\rho_l}=\theta_{\beta,\alpha_1\cdots\alpha_l}{}^{\sigma,\rho_1\cdots\rho_l}.\tag{A2}$$

[8] G. Cocho, C. Fronsdal, I. T. Grodsky, and R. White, Phys. Rev. **162**, 1662 (1967).

[9] See, in this connection, S. Klein, Phys. Rev. Letters **18**, 1074 (1967).

[10] Our preference for the neighborhood of $t=0$ is due, to some extent, to the successful saturation of the superconvergence relations at this point with the help of collinear groups. Certainly, this success is a consquence of the additional collinear symmetry of the amplitudes at $t=0$. It may well be that the approximate saturation with finite sets of resonances works somewhat better at *fixed* negative values of t, because there the suppression of the high-energy continuum is improved.

We use the term quasiprojection operator because of the particular order of the indices in (A2). Thus, we do not require, for instance,

$$\theta_{\beta,\alpha_1\cdots\alpha_l}{}^{\nu,\mu_1\cdots\mu_l}\theta_{\mu_i,\mu_1\cdots\mu_{i-1}\nu\mu_{i+1}\cdots\mu_l}{}^{\sigma,\rho_1\cdots\rho_l}$$

to have a simple form.

It is readily checked that our propagator (5) indeed obeys 1—3. Its uniqueness is then established by constructing

$$\theta_{\beta,\alpha_1\cdots\alpha_l}{}^{\beta,\alpha_1\cdots\alpha_l}(p)\,,$$

$$p'^{\beta}p'^{\alpha_1}\cdots p'^{\alpha_l}p_\nu'p_{\mu_i}'\cdots p_{\mu_l}'\theta_{\beta,\alpha_1\cdots\alpha_l}{}^{\nu,\mu_1\cdots\mu_l}(p)\,,$$

with

$$p\cdot p'=0\,,$$

and

$$\delta^{\beta\alpha_1}\delta_{\nu\mu_1}\theta_{\beta,\alpha_1\cdots\alpha_l}{}^{\nu,\mu_1\cdots\mu_l}$$

from Eq. (5) on the one hand and from Eq. (A1) on the other and by equating these expressions.

A completely similar argument can be used to derive the oscillatorlike propagator Eq. (12). The main differences in this case are the nonvalidity of the tracelessness condition of point 2, and the further reducibility of Eq. (A1).

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Superconvergent Dispersion Relations and Pion-Nucleon Sum Rules*

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Both the Adler-Weisberger sum rule and the spin-flip sum rule for pion-nucleon scattering have been derived from superconvergent dispersion relations for weak amplitudes. Our basic assumption is that the weak axial-vector–nucleon scattering amplitude $T_{\mu\nu}{}^A$ approaches the weak vector-nucleon scattering amplitude $T_{\mu\nu}{}^V$ at high energies. This allows us to write down superconvergent dispersion relations for certain invariant amplitudes in the decomposition of $T_{\mu\nu}=T_{\mu\nu}{}^A-T_{\mu\nu}{}^V$. We then use the hypotheses of partially conserved axial-vector current and of conserved vector current to obtain pion-nucleon scattering sum rules while avoiding the ambiguities of the $q\rightarrow 0$ limit which is usually used in the current-algebra approach. We also discuss sum rules for $G_A(q^2)$ away from $q^2=0$.

1. INTRODUCTION

THE chiral current algebra of Gell-Mann and the hypothesis of partial conservation of axial-vector current (PCAC) has been intensively used to obtain sum rules of interest in strong-interaction physics. The most celebrated of these is the Adler-Weisberger (AW) sum rule,[1] connecting the axial neutron β-decay constant G_A to an integral over π-N total cross sections. This sum rule is regarded as the direct confirmation of the validity of chiral current algebra and the PCAC hypothesis. It is interesting to see whether it is possible to obtain the AW sum rule without explicitly using the current commutation relations. The purpose of this paper is to show that this is indeed possible.

Our derivation of pion-nucleon sum rules is based on the following basic assumption: At high energy the weak axial-vector–nucleon scattering amplitude $T_{\mu\nu}{}^A$ approaches the weak vector-nucleon scattering amplitude $T_{\mu\nu}{}^V$, i.e., $T_{\mu\nu}(\nu)=T_{\mu\nu}{}^A(\nu)-T_{\mu\nu}{}^V(\nu)\rightarrow 0$ as $\nu\rightarrow\infty$.[2] This allows us to write superconvergent dispersion relations for certain invariant amplitudes appearing in the decomposition of $T_{\mu\nu}$. This gives sum rules connecting the weak axial-vector form factor $G_A(q^2)$ with vector form factors $F_1(q^2)$,$F_2(q^2)$ and an integral over certain weak vector and axial-vector–nucleon scattering amplitudes. One of these sum rules at $q^2=0$ together with the PCAC hypothesis and conservation of vector current (CVC) gives the AW sum rule. In addition, we obtain the pion-nucleon spin-flip sum rule which has been very recently derived by Gerstein[3] and Maiani and Preparata.[4] We also discuss the relation between the axial-vector form factor $G_A(q^2)$ and vector form factors $F_1(q^2)$ and $F_2(q^2)$ at finite q^2.

It must be emphasized that we do not explicitly use the current commutation relations, i.e., we do not postulate a current algebra. However, our basic assumption would follow if $SU(2)\otimes SU(2)$ were a good asymptotic symmetry.[2] We feel that this is a weaker assumption than explicitly postulating the current commutation relations.

In Sec. II we give the derivation of the sum rules. In Sec. III we discuss the sum rules for $G_A(q^2)$ away from $q^2=0$, and the conclusions are in Sec. IV.

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