Unitarity Corrections to Current-Algebra Calculations of the s-Wave Pion-Pion Scattering Lengths*

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A method of imposing unitarity on the amplitude obtained from current algebra is examined as a possible extrapolation procedure for the calculation of the s-wave pion-pion scattering lengths. This method consists of two steps: The current-algebra result is incorporated to fix the amplitude at the symmetry point in the unphysical region, and the dispersion relation based on rigorous grounds is used to extrapolate up to the physical threshold. This procedure allows us to derive a set of exact sum rules for the s-wave scattering lengths which enables us to estimate the unitarity corrections to the current-algebra calculations.

I. INTRODUCTION

RECENTLY a number of authors[1–9] have exploited the current-algebra techniques to obtain the s-wave $\pi\pi$ scattering lengths. In all of these calculations, the extrapolation of the amplitude from the consistency region off the mass shell to the physical threshold is an essential and additional ingredient; one must assume that the amplitude does not vary significantly even if the external mass of one or two pions goes to zero as in πN scattering. However, the extrapolation in the $\pi\pi$ case seems more ambiguous than in the πN case, in view of the fact that several results are not consistent with each other. But they all agree in giving scattering lengths that are much smaller than those of the previous S-matrix calculations[10,11] and the semiphenomenological analysis.[12]

In practice, the extrapolation is done by assuming in one way or another a parametrization of the amplitude which maintains crossing symmetry and by determining the arbitrary parameters from information given by current algebra. Weinberg[1] and subsequently Khuri[4] extrapolated the amplitude through a power-series expansion in the variables[13] s, t, u and the external pion mass μ_i^2 $(i=1,2,3,4)$, in which the first few terms are retained. Their amplitudes thus seem necessarily appropriate only for the small scattering lengths, as the authors admit, and clearly are not correct beyond the elastic-unitarity branch point. The investigation by Khuri,[4] who kept the second-order terms in the expansion and reproduced essentially Weinberg's values (within 10%), may be regarded as remarkable evidence against any significant effect due to the unitarity cut; but Sucher and Woo[9] more recently have entertained the possibility of deviations from Weinberg's results and have argued the possible existence of another, larger solution for the scattering lengths by treating a specific and simple example of unitarizing the amplitude at the threshold. Strictly speaking, therefore, current algebra may not exactly prove the scattering lengths to be small enough. On the other hand, we do know of some successful current-algebra predictions on, for example, the $K_{3\pi}$ and K_{l4} form factors,[14] which are predicted on the smallness of the $\pi\pi$ scattering lengths.

Thus, it is interesting to investigate another possibility of extrapolation[15] which at the same time accommodates elastic unitarity in the amplitude. The purpose of this article is to explore this possibility. We *assume* the Weinberg-Khuri amplitude, not in order to determine the scattering lengths directly, but to fix the amplitude at the symmetry point $(s=t=u=\frac{4}{3})$ on the mass shell,[16] and we use the dispersion relations based on the rigorous results of Froissart[17] and of Jin and Martin[18] to estimate the corrections due to the on-mass-shell extrapolation from the symmetry point to the physical threshold. This procedure may be reasonable in that the Weinberg-Khuri amplitude is real and the power series can be safely used in some region which includes the symmetry point and the consistency region

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$0 \leq s,t,u \leq 1$ of current algebra as long as the off-mass-shell amplitude has no singularity of s, t, u, and μ_i^2 there.[19] The symmetry point is the farthest point from all the unitarity branch points, and the power-series expansion of the amplitude is guaranteed to converge in some neighborhood of the symmetry point which is not too far from $s=t=u=1$. In other words, what we assume is that the extrapolation along the path defined by $s=t=u=\frac{1}{3}\sum_i \mu_i^2$, $0 \leq \mu_1^2 \leq 1$, and $\mu_i^2=1$ $(i=2,3,4)$ is smooth. Although this may not be as reliable an extrapolation as in the on-mass-shell case, the arguments of Khuri[4] and Meiere[8] seem to suggest that the effect is not very serious.

In Sec. II, we shall derive sum rules for the s-wave scattering lengths which have a rather firm foundation and contain the familiar parameters λ and λ_1 introduced by Chew and Mandelstam.[10] Section III contains the numerical results of the scattering lengths corresponding to the various parametrizations for the s-wave absorptive parts. Some concluding remarks are also given in Sec. III.

II. SUM RULES FOR THE SCATTERING LENGTHS

We use the following invariant amplitudes:

$$\begin{aligned}
F_1(s,t,u) &= A^{I=0}(s,t,u)+2A^{I=2}(s,t,u)\,,\\
F_2(s,t,u) &= A^{I=1}(s,t,u)+A^{I=2}(s,t,u)\,,\\
F_4(s,t,u) &= 4F_3(s,t,u)/(s-u)\\
&= 4\{2A^{I=0}(s,t,u)+3A^{I=1}(s,t,u)\\
&\qquad -5A^{I=2}(s,t,u)\}/(s-u)\,, \qquad (1
\end{aligned}$$

where $A^I(s,t,u)$ is the invariant amplitude with isotopic spin I. $F_1(s,t,u)$ is completely symmetric in the three variables, while $F_2(s,t,u)$ and $F_4(s,t,u)$ are symmetric in the variables s and u. One can easily see that

$$F_1 = 3\langle\pi^0\pi^0|A|\pi^0\pi^0\rangle\,,$$
$$F_2 = 2\langle\pi^+\pi^0|A|\pi^+\pi^0\rangle\,,$$

and

$$F_3 = 6\{\langle\pi^+\pi^-|A|\pi^+\pi^-\rangle - \langle\pi^+\pi^+|A|\pi^+\pi^+\rangle\}\,.$$

Under the crossing $u \leftrightarrow t$, $s \leftrightarrow s$, the amplitude $F_i (i=1,2,3)$ transforms as

$$\begin{aligned}
F_1 &\to F_1\\
F_2 &\to \tfrac{1}{3}F_1 - \tfrac{1}{2}F_2 - \tfrac{1}{6}F_3\\
F_3 &\to F_1 - \tfrac{9}{2}F_2 + \tfrac{1}{2}F_3\,.
\end{aligned} \qquad (2)$$

We observe that at the threshold,

$$\begin{aligned}
F_1 &= a_0+2a_2\,,\\
F_2 &= a_2\,,\\
F_4 &= 2a_0-5a_2\,,
\end{aligned} \qquad (3)$$

where a_I is the s-wave scattering length with isospin I; while at the symmetry point these amplitudes are related to the familiar Chew-Mandelstam parameters[10] λ and λ_1 by $F_1=-9\lambda$, $F_2=-2\lambda$, and $F_4=6\lambda_1$. These parameters are defined by

$$\begin{aligned}
-\lambda &= \tfrac{1}{5}A^{I=0}(s=t=u=\tfrac{4}{3}) = \tfrac{1}{2}A^{I=2}(s=t=u=\tfrac{4}{3})\,,\\
\lambda_1 &= \tfrac{1}{6}F_4(s=t=u=\tfrac{4}{3})\\
&= \left\{\frac{\partial}{\partial\cos\theta}\left[\frac{A^{I=1}(\nu,\cos\theta)}{\nu}\right]\right\}_{\nu=-\frac{2}{3},\cos\theta=0}
\end{aligned} \qquad (4)$$

and the Weinberg amplitude gives

$$\lambda = -\tfrac{1}{12}L\,, \quad \lambda_1 = L\,, \qquad (5)$$

where

$$L = \frac{1}{2\pi}\left(\frac{g_v}{F}\right)^2 \simeq 0.116\,.$$

It can be shown[20] that the $s-u$-symmetric amplitudes F_i $(i=1,2)$ are concave for fixed t in the triangle bounded by $s=4$, $t=4$, and $u=4$. Furthermore, the completely crossing-symmetric amplitude $F_1(s,t,u)$ has an absolute minimum value at the symmetry point. We note that the concavity is entirely absent in Weinberg's amplitude. This clearly indicates a deviation from the exact amplitude.

We can express $F_i(4,0,0)-F_i(8/3,0,4/3)$ $(i=1,2,4)$ by the forward dispersion integrals, and $F_i(8/3,0,4/3) - F_i(4/3,4/3,4/3)$ by the fixed-momentum-transfer dispersion integrals at $t=4/3$ after using the $t \leftrightarrow u$ crossing symmetric properties (2) of $F_i (i=1,2,3)$. Because of the $s \leftrightarrow u$ crossing symmetry of F_i $(i=1,2,4)$, these dispersion integrals have a twice-subtracted energy denominator, and convergence is guaranteed rigorously according to Froissart[17] and Jin and Martin.[18] Thus we obtain exact sum rules for the s-wave scattering lengths,

$$\begin{aligned}
a_0+2a_2 &= -9\lambda+I_1+J_1\,,\\
a_2 &= -2\lambda-\tfrac{2}{3}\lambda_1+I_2+\tfrac{1}{3}J_1-\tfrac{1}{2}J_2-\tfrac{1}{9}J_4\,,\\
2a_0-5a_2 &= 6\lambda_1+I_4+3J_1-(27/2)J_2+J_4\,,
\end{aligned} \qquad (6)$$

where the integrals I_i and J_i are given by

$$I_i = \frac{2}{9\pi}\int_0^\infty d\nu \frac{(2\nu+1)\,\mathrm{Im}F_i(\nu,\cos\theta=1)}{\nu(\nu+\frac{1}{3})(\nu+\frac{2}{3})(\nu+1)}\,, \qquad (7a)$$

$$J_i = \frac{2}{9\pi}\int_0^\infty d\nu \frac{\mathrm{Im}F_i(\nu,\cos\theta=1+2/3\nu)}{(\nu+\frac{1}{3})(\nu+\frac{2}{3})(\nu+1)}\,. \qquad (7b)$$

Since the existence of the integrals (7) can be shown under quite general assumptions,[18] our sum rules (6) not only are exact, but also have a rather firm foundation.

The third sum rule in (6), for $2a_0-5a_2$, should be equal to this combination as obtained from the first and

[19] Indeed, the existence of such a domain, needed to justify the power series expansion around $\zeta=0$ when $\mu_1^2=\mu_3^2=\zeta$, is shown by Y. S. Jin, Nuovo Cimento 50, 256 (1967).

[20] Y. S. Jin and A. Martin, Phys. Rev. 135, B1369 (1964).

second sum rules. As long as only s and p waves are kept in the absorptive parts in the integrals (7), the two integrals are identical. But when d and higher partial waves are included, they are not. Thus, this inclusion would result in new superconvergence-type sum rules. To see this, let us rewrite (6) as

$$a_0=-5\lambda+\tfrac{4}{3}\lambda_1+I(a_0)+(14/9)J(a_0)+(10/9)J(a_2)-2P+\delta_0\,, \quad (8a)$$

$$a_2=-2\lambda-\tfrac{2}{3}\lambda_1+(2/9)J(a_0)+I(a_2)+(13/9)J(a_2)+P+\delta_2\,, \quad (8b)$$

$$2a_0-5a_2=6\lambda_1+2[I(a_0)+J(a_0)]-5[I(a_2)+J(a_2)]-9P+\delta\,, \quad (8c)$$

where

$$I(a_I)=\frac{4}{9\pi}\int_0^\infty d\nu\frac{\mathrm{Im}A_{l=0}{}^I(\nu)}{\nu(\nu+\frac{2}{3})^2(\nu+1)}\,, \quad (9a)$$

$$J(a_I)=\frac{1}{3\pi}\int_0^\infty d\nu\frac{\mathrm{Im}A_{l=0}{}^I(\nu)}{(\nu+\frac{2}{3})^2(\nu+1)}\,, \quad (9b)$$

$$P=\frac{1}{\pi}\int_0^\infty d\nu\frac{\mathrm{Im}A_{l=1}{}^{I=1}(\nu)}{\nu(\nu+\frac{2}{3})(\nu+1)}\,, \quad (9c)$$

and δ_0, δ_2, and δ represent the contribution from the higher partial waves other than $l=0$ and 1. Since $\delta-(2\delta_0-5\delta_2)$ should be zero, we obtain a sum rule

$$\frac{4}{9\pi}\int_0^\infty d\nu\frac{2\Delta A^{I=0}-9\Delta A^{I=1}-5\Delta A^{I=2}}{(\nu+\frac{1}{3})(\nu+\frac{2}{3})(\nu+1)}=0\,, \quad (10)$$

where

$$\Delta A^{I=0,2}=\sum_{l=2,\mathrm{even}}(2l+1)\times\left[P_l\left(1+\frac{2}{3\nu}\right)-1\right]A_l{}^{I=0,2}(\nu)\,, \quad (11a)$$

$$\Delta A^{I=1}=\sum_{l=3,\mathrm{odd}}(2l+1)\times\left[P_l\left(1+\frac{2}{3\nu}\right)-1-\frac{2}{3\nu}\right]A_l{}^{I=1}(\nu)\,. \quad (11b)$$

We feel that the sum rule (10) would be useful in estimating the d-wave contributions in the low-energy region. But since we are primarily concerned with the s-wave scattering lengths, we will not discuss the consequences of the sum rule (10) here.

However, we point out that a sum rule very similar to (8c) for $2a_0-5a_2$ has been considered by Meiere and Sugawara.[5] Indeed, if we insert the Goldberger-Treiman relation[21] for F from (5), the sum rule (8c) gives $2(G_\beta/G_A)^2$ in terms of $2a_0-5a_2$ and an integral over the total $\pi^+\pi^\pm$ cross sections, as in Ref. 5, except for the different subtraction point in the integral. While their subtraction is made at $s=u=2$ and $t=0$, ours is at the symmetry point. They noted that the integral in this once-subtracted Adler-Weisberger-type sum rule[22] is much less sensitive to the high-energy behavior and to the parametrizations of the s waves than is the integral in the unsubtracted one. We shall also observe this in the next section.

III. NUMERICAL RESULTS AND CONCLUSIONS

Owing to the large energy denominators in the dispersion integrals of (7) and (9), the higher partial waves are much suppressed, and the contribution from the higher resonances becomes negligible. For instance, if we estimate δ_0, δ_2, and δ from the established f_0 meson ($m_{f_0}=1254$ MeV, $\Gamma_{f_0}=117$ MeV)[23] by the narrow-width approximation, we find that $\delta_0=6\times10^{-4}$, $\delta_2=9\times10^{-5}$, and $\delta=9\times10^{-4}$, so that $\delta-(2\delta_0-5\delta_2)=10^{-4}$. On the other hand, the contribution of the ρ meson ($m_\rho=778$ MeV, $\Gamma_\rho=160$ MeV)[23] to the integral (9c) in the narrow-width approximation is $P=4\times10^{-3}$. Thus we can safely neglect the contributions from $l\geq2$ in (8). We mention again that in this case (8a) and (8b) are identical to (8c).

For given λ and λ_1, one could calculate a_I by solving (8a) and (8b) simultaneously. This approach would then give values quite analogous to those from the two-parameter calculations of low-energy $\pi\pi$ scattering by the S-matrix method.[24,25] To see this from (8), we have to somehow estimate (9a) and (9b). We consider three different parametrizations for the s-wave absorptive parts: the Chew-Mandelstam approximation, the scattering-length approximation, and the nonrelativistic effective-range formula.

A. Chew-Mandelstam Approximation

In this case, the s-wave absorptive part is parametrized by

$$\mathrm{Im}A_{l=0}{}^I(\nu)=(\nu/\nu+1)^{1/2}\times\{[(1/a_I)+h(\nu)]^2+(\nu/\nu+1)\}^{-1}\,, \quad (12)$$

where

$$h(\nu)=2/\pi(\nu/\nu+1)^{1/2}\ln[\nu^{1/2}+(\nu+1)^{1/2}]\,.$$

Then for $a_I>0$, the integrals of (9a) and (9b) can be written in analytic form

$$I(a_I)=a_I+\frac{3}{1/a_I+h(-\frac{2}{3})}-\frac{4}{1/a_I+h(-1)}-\frac{9}{2}\frac{h(-\frac{2}{3})-(4/3\pi)}{[1/a_I+h(-\frac{2}{3})]^2}\,, \quad (13a)$$

[21] M. L. Goldberger and S. B. Treiman, Phys. Rev. **110**, 1178 (1958). In our definition of L, F is given by $-2m_NG_A/G_{\pi NN}K(0)$.

[22] S. L. Adler, Phys. Rev. Letters **14**, 1051 (1965); Phys. Rev. **140**, B736 (1965); W. I. Weisberger, Phys. Rev. Letters **14**, 1047 (1965); Phys. Rev. **143**, 1302 (1966).

[23] A. H. Rosenfeld *et al.*, Rev. Mod. Phys. **39**, 1 (1967).

[24] J. S. Ball and D. Y. Wong, Phys. Rev. Letters **7**, 390 (1961).

[25] B. R. Desai, Phys. Rev. Letters **6**, 497 (1961).

$$J(a_I)=-\frac{3}{1/a_I+h(-\frac{2}{3})}+\frac{3}{1/a_I+h(-1)}+\frac{9}{4}\frac{h(-\frac{2}{3})-(4/3\pi)}{[1/a_I+h(-\frac{2}{3})]^2}. \qquad (13b)$$

B. Scattering-Length Approximation

By this, we mean the parametrization

$$\mathrm{Im}A_{l=0}{}^{I}(\nu)=\left(\frac{\nu}{\nu+1}\right)^{1/2}\left[\frac{1}{a_I{}^2}+\frac{\nu}{\nu+1}\right]^{-1} \qquad (14)$$

and (9a) and (9b) give

$$I(a_I)=\frac{8}{\pi}\left[1-\frac{6a_I{}^2}{2a_I{}^2-1}+9\left(\frac{a_I{}^2}{2a_I{}^2-1}\right)^2\right]a_I\tan^{-1}a_I+\frac{a_I{}^2}{2a_I{}^2-1}\left(\frac{15}{2}-\frac{18a_I{}^2}{2a_I{}^2-1}\right)h(-\tfrac{2}{3})-\frac{6}{\pi}\frac{a_I{}^2}{2a_I{}^2-1}, \qquad (15a)$$

$$J(a_I)=-\frac{6}{\pi}\left[1-\frac{5a_I{}^2}{2a_I{}^2-1}+6\left(\frac{a_I{}^2}{2a_I{}^2-1}\right)^2\right]a_I\tan^{-1}a_I-\frac{a_I{}^2}{2a_I{}^2-1}\left(\frac{21}{4}-\frac{9a_I{}^2}{2a_I{}^2-1}\right)h(-\tfrac{2}{3})+\frac{3}{\pi}\frac{a_I{}^2}{2a_I{}^2-1}. \qquad (15b)$$

C. Nonrelativistic Effective-Range Formula

Here, the s-wave absorptive part is parametrized by

$$\mathrm{Im}A_{l=0}{}^{I}(\nu)=(\sqrt{\nu})\bigg/\left[\left(\frac{1}{a_I}+\tfrac{1}{2}r_I\nu\right)^2+\nu\right], \qquad (16)$$

and we get for instance for $a_I>0$ and $r_I<0$ that

$$J(a_I)=a_I+3\left(\frac{1}{a_I}+\tfrac{1}{3}|r_I|+\sqrt{\tfrac{2}{3}}\right)^{-1}-4\left(\frac{1}{a_I}+\tfrac{1}{2}|r_I|+1\right)^{-1}-[(\sqrt{\tfrac{3}{2}})+|r_I|]\left(\frac{1}{a_I}+\tfrac{1}{3}|r_I|+\sqrt{\tfrac{2}{3}}\right)^{-2}, \qquad (17a)$$

$$J(a_I)=-3\left(\frac{1}{a_I}+\tfrac{1}{3}|r_I|+\sqrt{\tfrac{2}{3}}\right)^{-1}+3\left(\frac{1}{a_I}+\tfrac{1}{2}|r_I|+1\right)^{-1}+\tfrac{1}{2}[(\sqrt{\tfrac{3}{2}})+|r_I|]\left(\frac{1}{a_I}+\tfrac{1}{3}|r_I|+\sqrt{\tfrac{2}{3}}\right)^{-2}. \qquad (17b)$$

One can also evaluate these integrals in analytic form for the case $a_Ir_I>0$. In any case, we consider the effective-range parameter r_I for $|r_I|\leq 1$.

Fortunately, the results are not sensitively dependent on a particular parametrization. For instance, the combination $I(a_0)+(14/9)J(a_0)\equiv g(a_0)$ which appears on the right-hand side of (8a) behaves almost the same way for both small and large values of a_0. For small a_0, we see that $g(a_0)\approx 0.37a_0{}^2$ for the parametrizations (A) and (B), and $g(a_0)\approx 0.42a_0{}^2$ for (C); while as $a_0\to\infty$, $g(a)\approx a_0-2.4$ for (A), $g(a_0)\approx a_0-1.6$ for (B), and $g(a_0)\approx a_0-1.8$ or $a_0-1.4$, depending on whether $r_0=0$ or $r_0=-1$ for (C).

To compare the results from (8) with the two-parameter solutions[24,25] by the S-matrix method, we have plotted in Fig. 1 the function $f(a_0)\equiv a_0-g(a_0)$ versus a_0. We also include there $f(a_0)=a_0-0.42a_0{}^2$, which corresponds to the parametrization (C) but with a threshold approximation made in addition, so that $\mathrm{Im}A_{l=0}{}^{I}(\nu)=a_I{}^2\sqrt{\nu}$. On the other hand, $f(a_0)$ is related to λ and λ_1 by $-5\lambda+(4/3)\lambda_1+(10/9)J(a_2)-2P$ from (8a), so that it can be fixed by fixing λ, λ_1, and a_2. For example, for $\lambda=-0.1$, $\lambda_1=0.15$, and $a_2=0.1$ [for which $J(a_2)$ can be safely estimated by the scattering-length approximation] we get $f(a_0)=0.7$, which gives (from Fig. 1) $a_0=0.96$–1.28, to be compared with $a_0=1.1$ obtained by Ball and Wong.[24] A similar procedure applied to (8b) for this range of a_0 reveals $a_2=0.11$. The agreement convinces us that there is some usefulness in our sum rules (8).

Next, we notice from (5) that Weinberg's calculation gives $\lambda=-0.01$ and $\lambda_1=0.116$. His values for the scattering lengths, $a_0=0.2$ and $a_2=-0.06$, can be exactly reproduced if all the dispersion-integral terms are neglected in (8), so that

$$a_0=-5\lambda+\tfrac{4}{3}\lambda_1,\qquad a_2=-2\lambda-\tfrac{2}{3}\lambda_1. \qquad (18)$$

Clearly, his solution for the s-wave scattering lengths contains some contributions from the p wave, as is evidenced by the presence of λ_1 in (18). With these values of λ and λ_1, we observe from (8) that the unitarity correction increases a_0 merely by about 5%, and decreases the magnitude of a_2 by about 10%, for all three parametrizations considered of the s-wave absorptive parts. It should be remarked that our sum rules (8) do not allow a second solution with large a_0 for all the parametrizations. However, if we further make a threshold approximation to the nonrelativistic effective-range formula (iii) to estimate the dispersion integrals (9a) and (9b), then we obtain in addition a larger solution $a_0=2.1$, as in the example of Sucher and Woo.[9,26] We feel, however, that this larger solution is

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[26] Interpreting the parameters a, b, c, and x of Sucher and Woo in terms of λ, λ_1, and a_0, one gets $a_0=-5\lambda+\frac{4}{3}\lambda_1+\frac{4}{3}[(\sqrt{\frac{2}{3}})-\frac{1}{2}]a_0{}^2$, which agrees with the sum rule (8a) if $(10/9)J(a_2)-2P+\delta_0$ is dropped and $I(a_0)+(14/9)J(a_0)$ is estimated by using $\mathrm{Im}A_{l=0}{}^{I=0}(\nu)=a_0{}^2\sqrt{\nu}$.

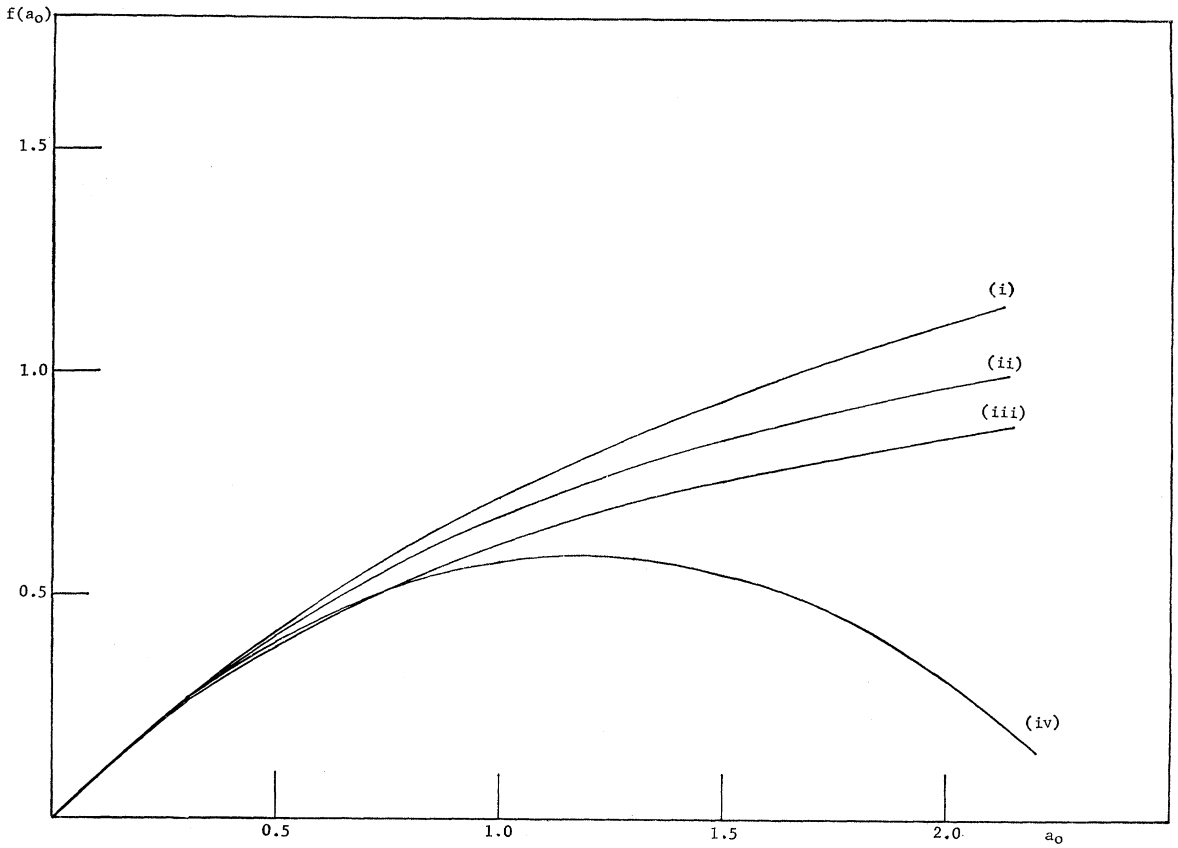

FIG. 1. $f(a_0)$ versus a_0 for the s-wave parametrizations by (i) the Chew-Mandelstam approximation, (ii) the scattering-length approximation, and (iii) the nonrelativistic effective-range approximation with $r_0=-1$. The curve (iv) corresponds to (iii) for which a threshold approximation is further made.

not physically favorable, for two reasons: The threshold approximation is good only for a small scattering length; and if a_0 is indeed as large as 2, then the amplitude should have a rapid variation from the consistency region to the symmetry point, which is clearly not consistent with the smoothness assumption of the hypothesis of partially conserved axial-vector current.

If a σ meson ($M_\sigma=400$ MeV, $\Gamma_\sigma=95$ MeV)[27] is assumed in order to estimate (9a) and (9b), then in Eq. (8) a_0 can be increased by 50% and a_2 by 15% over the values obtained by Weinberg for $\lambda=-0.01$ and $\lambda_1=0.116$. A σ with a higher mass gives a less significant change in the scattering lengths. Thus we find in any case that a_0 can hardly be larger than half a pion Compton wavelength once the behavior of the amplitude near the symmetry point is determined by the results of current algebra.

[27] L. M. Brown and P. Singer, Phys. Rev. Letters **8**, 460 (1962); Phys. Rev. **133**, B812 (1964); L. M. Brown, Phys. Rev. Letters **14**, 836 (1965).

Finally, we remark that the small scattering lengths from current algebra may be attributed to the smallness of λ, which in turn seems to be a natural consequence of Adler's self-consistency condition[28] that the amplitude should vanish at $s=t=u=1$. If experiments indeed confirm a_0 to be as large as unity, then this will be difficult to reconcile with the consistency condition of current algebra, at least for $\pi\pi$ scattering, where no mass scale is available.

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