General Properties of the Electromagnetic Corrections to the Beta Decay of a Physical Nucleon*

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Some general properties of the electromagnetic corrections to the β decay of a physical nucleon, that is, a nucleon in the presence of strong interactions, are discussed. The aim of the paper is to isolate and determine general properties which are independent of the details of the strong interactions and the assumptions about the existence of the intermediate boson. The method used consists essentially in separating out in a finite and gauge-invariant manner all the terms of order $1/k$ in the hadronic covariants, and then examining properties of the other contributions. Under some general and plausible mathematical assumptions, it is shown that all the terms of order α in the correction factor to the electron spectrum which explicitly depend on the electron energy E and the electron mass m can be rigorously computed, in spite of the complications of the strong interactions, provided that contributions of relative order $\alpha q/M$, $\alpha(E/M)\ln(M/E)$, and $\alpha v_N/c$ are neglected. In particular, the electromagnetic correction of order α to the shape of the allowed electron spectrum is given by a single universal function $g(E,E_m,m)$ of E, m, and the end-point energy E_m; this function is independent of the details of the strong interactions and the assumption that the weak interactions are mediated by an intermediate boson. It is furthermore independent of the ratio M_{GT}/M_F, and is physically significant, particularly if applied to nuclear β decays with $E_m/m\gg1$. These propositions do not preclude the existence of model-dependent terms which give contributions to the lifetime. No attempt is made here to evaluate these quantities, but their role on the physical observables is discussed. It is pointed out that, aside from the electron spectrum, there are other observables for which the corrections can be evaluated in a model-independent manner. This assertion is illustrated by giving the expression for the virtual radiative corrections of order α to the longitudinal polarization of the electron in allowed transitions. The contribution of the universal function $g(E,E_m,m)$ to the lifetime is briefly discussed. This paper implicitly assumes the validity of the vector and axial-vector theory of weak interactions.

I. INTRODUCTION

THE radiative corrections to the β decay of a nucleon have been discussed extensively in the past.[1-7] One reason why these corrections are of interest is their relevance to the problem of universality of the weak interactions. There are other practical reasons which, although not so well known, are also of experimental interest: for example, the corrections alter the shape of the allowed electron spectrum in β decay and these effects are significant when the end-point energy of the electron is large in comparison with the electron mass.[3]

An interesting theoretical aspect of the corrections is that they are logarithmically divergent in the $V-A$ theory when evaluated in the case of a bare nucleon, i.e., a nucleon in the absence of strong interactions, while the corresponding quantities for muon decay are finite and well defined. In order to save the principle of universality of the weak interactions, which in the usual formulation relates the bare weak-coupling constants of these processes, two main approaches have been suggested. One, proposed by Lee[8] and also investigated by other authors, assumes that the weak interactions are mediated by an intermediate boson. We recall that one of the main results of these calculations is that the relative renormalization in μ and β decay, which is the physically observable quantity, turns out to be finite. The other approach, discussed by Feynman,[1] by Berman and the present author,[4] and by Källén,[7] suggests that the strong interactions themselves may provide a natural cutoff in the ultraviolet region. Recently, Bjorken,[9] and Abers, Norton, and Dicus,[10] using current-algebra techniques have asserted in very interesting papers that that part of the electromagnetic corrections to the vector decay coupling constant which arises from the vector current contains a divergent part which is independent of the strong interactions and is therefore identical to the corresponding infinity in the calculations of Refs. 2–4. Even if this assertion is correct, it does not give a complete answer to the problem of the relative renormalization to μ and β decay for two reasons: (a) In the first place, as these authors emphasize, there is a contribution from the axial current which may be also divergent and model-dependent, and (b) these calculations follow implicitly the usual assumption that the hadronic and

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[1] R. P. Feynman, in *Proceedings of the 1960 Annual International Conference on High-Energy Physics, Rochester, 1960* (Interscience Publishers, Inc., New York, 1960), p. 501; and *Symmetries in Elementary Particle Physics* (Academic Press Inc., New York, 1965), p. 111.

[2] S. M. Berman, Phys. Rev. 112, 267 (1958).

[3] T. Kinoshita and A. Sirlin, Phys. Rev. 113, 1652 (1959).

[4] S. M. Berman and A. Sirlin, Ann. Phys. (N.Y.) 20, 20 (1962).

[5] L. Durand, L. F. Landovitz, and R. B. Marr, Phys. Rev. 130, 1188 (1963).

[6] A. Sirlin, *Lecture Notes on Weak Interactions* (W. A. Benjamin, Inc., New York, 1963), p. 173. This paper contains further references but, by now, the list is incomplete.

[7] G. Källén, Nucl. Phys. B1, 225 (1967). The author wishes to thank Professor Källén for a very interesting private communication.

[8] T. D. Lee, Phys. Rev. 128, 899 (1962); R. A. Shaffer, *ibid.* 128, 1452 (1962); G. Dorman, Nuovo Cimento 32, 1226 (1964); D. Bailin, Phys. Rev. 135, B166 (1964).

[9] J. D. Bjorken, Phys. Rev. 148, 1467 (1966).

[10] E. S. Abers, R. E. Norton, and D. A. Dicus, Phys. Rev. Letters 18, 676 (1967).

leptonic currents are taken at the same space-time point in the weak-interaction Lagrangian. However, from the calculations with intermediate bosons we know that possible intrinsic structure effects of the weak interactions may alter the convergence properties of the electromagnetic corrections.[11]

The investigation of these convergence properties is clearly a subject of great practical and conceptual interest. The corrections as a whole cannot be lightly dismissed as a subject of study because they are important in providing a detailed understanding of the experiments on the muon decay energy angle distribution[12] and some aspects of β decay. Without these corrections, for example, the very precise experiments available in μ decay would not coincide with the predictions of the $V-A$ theory. One is then faced with the problem that the calculations are necessary in order to understand in detail some experiments while their effect on the problem of universality is far from clear.

In the present paper, the problem of the convergence properties of the electromagnetic corrections in the ultraviolet region is not investigated and is to a large extent bypassed. We will take here the hopeful point of view that somehow the weak, electromagnetic, and strong interactions successfully conspire to give a finite value for the relative renormalization of the β and μ decay coupling constants.

Rather, the aim of the present paper is to try to isolate and determine some general properties of the corrections which are independent of the details of the strong interactions and of the assumption that the weak interactions are mediated by an intermediate boson.[13] We will show, under some general and plausible mathematical assumptions, the validity of the following proposition. "All the terms of order α in the correction factor to the electron spectrum[14] which explicitly depend on the electron energy E and the electron mass m can be rigorously computed in spite of the complications of the strong interactions, provided that contributions of relative order $\alpha q/M$, $\alpha(E/M)\ln(M/E)$, and $\alpha v_N/c$ are neglected.[15] Moreover, such terms modify the Fermi

and Gamow-Teller contributions to the transition probability in the same manner and coincide exactly with the corresponding quantities in the bare nucleon calculations." In particular, we conclude that the "electromagnetic corrections of order α to the shape of the allowed electron spectrum in neutron β decay is given by a single universal function $g(E,E_m,m)$ of E, m and the end-point energy E_m, which is independent of the details of the strong interactions and the assumption that the weak interactions are mediated by an intermediate boson." Furthermore, they can be applied to allowed nuclear β decay using an independent-particle model of the nucleus.[7]

It is important to note that the above propositions do not preclude the existence of terms independent of E and m which are affected by the strong interactions. Such terms give contributions to the β-decay lifetime. We will not attempt to evaluate them in this paper, as they are model-dependent, but rather we will limit ourselves to indicating the contribution of the finite function $g(E,E_m,m)$ to the lifetime. If the relative renormalization of the μ and β decay-coupling constants turns out to be finite, we believe that these contributions will play a significant role, as they contain the large terms of logarithmic order in the electron energy.

The discussion in the paper proceeds according to the following pattern. In Sec. II the virtual radiative corrections of order α are separated into two parts. The first part, which can be evaluated independently of the details of the strong interactions, contains among other contributions all the terms of order $1/k$ in the hadronic covariants (k is the photon four-momentum); it is also finite (in the uv region) and gauge-invariant. This first part, when integrated over the photon four-momentum, gives the usual infrared-divergent terms, the Coulomb term of order α, and various logarithms and Spence functions characteristic of the point-structure calculation. This method of separation is in principle similar to a procedure used by Frautschi, Suura, and Yennie[16] and by Meister and Yennie[17] in their investigations of the infrared divergence in quantum electrodynamics. However, our presentation is quite different, as we wish to emphasize some of the effects of the strong interactions. In the discussion of Sec. II we first consider a nucleon in the presence of strong interactions without assuming that the decay is mediated by an intermediate boson. At the end of Sec. II we then explain the simple changes needed to allow for this possibility. In fact, it seems very likely that the discussion and results of this paper can be extended to more general space-time structures for the weak interactions, such as those investigated, for example, in Ref. 11. However, we will not study this point in the present paper.

The second class of contributions, which do not contain terms of order $1/k$ in the hadronic covariants,

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[11] This intrinsic space-time structure of the weak interactions need not be necessarily associated with intermediate bosons. We may consider more general structures such as discussed, for example, by T. D. Lee and C. N. Yang, Phys. Rev. 108, 1611 (1957).

[12] V. L. Telegdi, University Lectures on Physics, New York University, 1967 (unpublished).

[13] That there are properties of the electromagnetic corrections which can be determined independently of detailed assumptions about the strong interactions was already recognized on the basis of heuristic arguments in Ref. 4 (Sec. VI and Appendix). In that paper this "universality property" was conjectured for the term $3(\alpha/2\pi)\ln(m_p/E_m)$ which plays an important role in the corrections to the lifetime. In the present paper we refine the theoretical arguments and extend the determination of such properties to physical observables.

[14] The spectrum can be expressed as the usual allowed spectrum multiplied by an energy-dependent factor of order α. We refer to the latter as the correction factor.

[15] Here v_N is the nucleon velocity, q is the total momentum transfer to the leptons, and M is a hadronic mass. All noncovariant statements in this paper refer to the rest frame of the decaying system. A more detailed characterization of M is given in Sec. III.

[16] D. R. Yennie, S. C. Frautschi, and H. Suura, Ann. Phys. (N. Y.) 13, 379 (1961).

[17] N. Meister and D. R. Yennie, Phys. Rev. 130, 1210 (1962).

depend on the details of the strong interactions and the assumptions about the intermediate boson. They involve three- and four-point functions. In Sec. III, we discuss these contributions and show, under general and plausible mathematical assumptions, that they consist of terms independent of E and m plus terms of order $\alpha q/M$ and $\alpha(E/M)\ln(M/E)$. We further prove some general properties of the first type of terms, which are necessary for the physical applications.

In Sec. IV we give the general expression for the correction factor to the electron spectrum. We discuss then the role played by the model-dependent and model-independent terms regarding the physical observables. After taking note of the contributions of the universal function $g(E,E_m,m)$ to the lifetime, we emphasize that there are other observables, aside from the electron spectrum, for which the radiative corrections of order α can be evaluated in a model-independent manner. We illustrate this point by giving the expression for the virtual radiative corrections to the longitudinal polarization of the electron in allowed decays.

The discussion and results of this paper implicitly assume the validity of a vector and axial-vector theory of weak interactions.

II. THE SEPARATION

In this section the virtual radiative corrections of order α are separated into two parts. The first part is characterized essentially by the following properties: (i) It is independent of the details of the strong interactions; (ii) it contains all the terms of order $1/k$ in the hadronic covariants; (iii) it is finite (in the uv region) and it is invariant under gauge transformations of the photon propagator. The second part depends on the details of the strong interactions and involves three- and four-point functions.

We consider first the diagrams in which a photon is exchanged between the hadronic and electron lines. Such diagrams can be divided into three classes according to whether the photon is emitted from an electromagnetic vertex attached to the external proton line, or the external neutron line, or from the internal structure of the weak vertex. The contribution of the first class of diagrams [Fig. 1(a)] is given by[18]

$$M^{(p)} = -\frac{G_V^0}{\sqrt{2}} \frac{\alpha}{4\pi^3 i} \int \frac{d^4k\, D_{\nu\mu}(k)[\bar{u}_e(2l_\nu - \gamma_\nu k)O_\lambda v_\nu]}{(k^2 - 2l\cdot k + i\epsilon)}$$
$$\times [\bar{u}_p \tilde{\Gamma}_\mu^{(p)}(p_2, p_2+k)\tilde{S}_F'(p_2+k)W_\lambda(p_2+k, p_1)u_n],$$
(1)

where p_1, p_2, l, and k stand for the neutron, proton, electron, and photon four-momenta; λ_{min} is the photon mass, $O_\lambda \equiv \gamma_\lambda(1+\gamma_5)$; G_V^0 is the bare-coupling constant of the weak-vector current, W_λ and $\tilde{\Gamma}_\mu^{(p)}$ are the fully

[18] In this paper we adopt the same conventions and definitions as R. P. Feynman, *Quantum Electrodynamics* (W. A. Benjamin, Inc., New York, 1962), with the exception of γ_5 which we define as $\gamma_5 = i\gamma_1\gamma_2\gamma_3\gamma_0$.
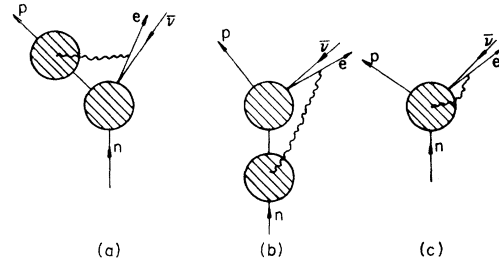

FIG. 1. Diagrams involving photon exchanged between hadronic and electron lines.

dressed renormalized proper vertices of the weak interaction and electromagnetic currents; and $i\tilde{S}_F'$ is the fully dressed renormalized propagator of the proton. It is important to note that W_λ obtains contributions from both the vector and axial-vector currents. The quantity $-iD_{\nu\mu}(k)$ is the photon propagator, which we choose to write in the Landau gauge:

$$D_{\nu\mu}(k) = \frac{(g_{\nu\mu} - k_\nu k_\mu/k^2 + i\epsilon)}{k^2 - \lambda_{min}^2 + i\epsilon}.$$
(2)

We now introduce the "proton electromagnetic form factor" $F_\mu^{(p)}(p_2, p_2+k)$ for the case in which p_2 corresponds to a proton on the mass shell but p_2+k is off the mass shell:

$$\bar{u}_p\tilde{\Gamma}_\mu^{(p)}(p_2, p_2+k)\tilde{S}_F'(p_2+k)$$
$$= \bar{u}_p F_\mu^{(p)}(p_2, p_2+k)[\not{p}_2+\not{k}-m_p]^{-1},$$
(3)

where m_p is the proton mass. This equation may be regarded as the definition of $F_\mu^{(p)}$. Using the generalized Ward identity, one readily checks from Eq. (3) that

$$k_\mu F_\mu^{(p)}(p_2, p_2+k) = \not{k},$$
(4a)

$$F_\mu^{(p)}(p_2,p_2) = \gamma_\mu,$$
(4b)

where it is understood that $F_\mu^{(p)}$ acts on the spinor $\bar{u}_p$ on the left.

Equations (4) suggest writing[19]

$$F_\mu^{(p)}(p_2, p_2+k) = \gamma_\mu + X_\mu^{(p)}(p_2, p_2+k),$$
(5a)

where

$$k_\mu X_\mu^{(p)}(p_2, p_2+k) = 0,$$
(5b)

$$X_\mu^{(p)}(p_2,p_2) = 0.$$
(5c)

We now insert Eqs. (3) and (5a) into Eq. (1) and separate the terms of order $1/k$ in the hadronic covariant in the following manner:

$$H_{\mu\lambda}^{(p)}(p_2,p_1,k) \equiv \bar{u}_p\tilde{\Gamma}_\mu^{(p)}(p_2, p_2+k)\tilde{S}_F'(p_2+k)$$
$$\times W_\lambda(p_2+k, p_1)u_n = (k^2+2p_2\cdot k+i\epsilon)^{-1}$$
$$\times \bar{u}_p[(2p_{2\mu}+k_\mu)W_\lambda(p_2,p_1)+R_{\mu\lambda}^{(p)}(p_2,p_1,k)]u_n,$$
(6a)

[19] A similar decomposition was used for different reasons by B. Zumino, Nuovo Cimento 17, 547 (1960).

where

$$R_{\mu\lambda}^{(p)}(p_2,p_1,k) = (2p_{2\mu}+k_\mu)[W_\lambda(p_2+k,\ p_1) - W_\lambda(p_2,p_1)]$$
$$+ X_\mu^{(p)}(p_2,\ p_2+k)(p_2+k+m_p)W_\lambda(p_2+k,\ p_1)$$
$$+ [\gamma_\mu,k]W_\lambda(p_2+k,\ p_1)/2. \quad (6b)$$

Assuming that $\partial W_\lambda/\partial k_\mu(p_2+k,\ p_1)|_{k_\mu=0}$ exists, we see that $R_{\mu\lambda}^{(p)} \to 0$ linearly with k_μ as $k_\mu \to 0$. Thus, while the first term in Eq. (6a) is obviously singular as $k_\mu \to 0$, we notice that the term proportional to $R_{\mu\lambda}^{(p)}$ $\times (p_2,p_1,k)$ is "regular as $k_\mu \to 0$" in the sense that

$$\lim_{k_0\to 0}\lim_{k_j\to 0}(k^2+2p_2\cdot k)^{-1}R_{\mu\lambda}^{(p)}(p_2,p_1,k)$$

exists. The term $k_\mu W_\lambda(p_2,p_1)$ in Eq. (6a) has been separated together with the term of order $1/k$ for technical reasons that will become apparent later.

It is clear that a similar analysis can be made for the diagrams in which the photon is exchanged between the electron and the electromagnetic vertex attached to the external neutron line [Fig. 1(b)]. The electromagnetic form factor of the neutron $F_\mu^{(n)}(p_1-k,\ p_1)$ can be introduced by means of an equation analogous to Eq. (3). However, because the neutron has zero charge, instead of Eq. (4a) one obtains

$$k_\mu F_\mu^{(n)}(p_1-k,\ p_1) = 0, \quad (7a)$$

$$F_\mu^{(n)}(p_1,p_1) = 0, \quad (7b)$$

where it is understood that $F_\mu^{(n)}$ acts on the spinor u_n on the right. The matrix element for this contribution is analogous to Eq. (3) with the hadronic covariant $H_{\mu\lambda}^{(p)}(p_2,p_1,k)$ of Eq. (6a) replaced by

$$H_{\mu\lambda}^{(n)}(p_2,p_1,k) = [k^2-2p_1\cdot k+i\epsilon]^{-1}$$
$$\times[\bar{u}_p W_\lambda(p_2,\ p_1-k)(p_1-k+m_n)$$
$$\times F_\mu^{(n)}(p_1-k,\ p_1)u_n]. \quad (7c)$$

Clearly, $H_{\mu\lambda}^{(n)}$ is "regular" as $k_\mu \to 0$.

Let us now call $G_{\mu\lambda}(p_2,p_1,k)$ the matrix element for the emission of a photon of four-momentum k and polarization μ from any internal line of the proper vertex of the weak-interaction current [Fig. 1(c)]. By judicious use of the generalized Ward's identity for the electromagnetic vertex functions, one finds the relation:

$$k_\mu G_{\mu\lambda}(p_2,p_1,k) = W_\lambda(p_2,p_1) - W_\lambda(p_2+k,\ p_1). \quad (8a)$$

Expressions analogous to Eq. (8a) are well known in the literature.[20] Perhaps the simplest way of obtaining Eq. (8a) in our case is to consider an arbitrary graph for the process $n \to p+e+\bar{\nu}$ without any photons and then insert a photon of four-momentum k_μ and polarization μ in all the charged lines of the graph, including the external proton and electron lines. Demanding that the sum of the corresponding matrix elements should vanish after multiplication with k_μ and summing over all possible initial graphs one readily obtains Eq. (8a).

[20] B. V. Geshkenbeïn and V. S. Popov, Zh. Eksperim. i Teor. Fiz. **41**, 199 (1961) [English transl.: Soviet Phys.—JETP **14**, 145 (1961)].

A fundamental property of $G_{\mu\lambda}(p_2,p_1,k)$ is that it is "regular" as $k_\mu \to 0$. This can be shown in a number of ways that we briefly summarize:

(a) Assuming that $(\partial W_\lambda/\partial k_\mu)|_{k_\mu=0}$ exists, Eq. (8a) tells us that $k_\mu G_{\mu\lambda}(p_2,p_1,k)$ vanishes linearly with k_μ as $k_\mu \to 0$. This implies that $G_{\mu\lambda}$ cannot have singularities of the form $t_{\mu\lambda}/(p_2\cdot k)$, where $t_{\mu\lambda}$ is a tensor independent of k_μ.

(b) Consider an arbitrary diagram $W_\lambda^i(p_2,p_1)$ contributing to the proper vertex of the weak-interaction current and not involving photons.[21] From perturbation theory, it is known that the matrix element $G_{\mu\lambda}^i(p_2,p_1,0)$ for the emission of a zero-momentum photon from the internal lines of $W_\lambda^i(p_2,p_1)$ can be obtained by differentiation with respect to the proton four-momentum, that is,

$$G_{\mu\lambda}^i(p_2,p_1,0) = -\frac{\partial W_\lambda^i}{\partial p_{2\mu}}(p_2,p_1), \quad (8b)$$

which, of course, is consistent with Eq. (8a). Eq. (8b) can be proved by following the charge line ending in the proton. It is always possible to choose the internal momenta in this line in such a manner that, for example, the propagators of positively and negatively charged fermions are of the form $i(p_2+k_i-m_i)^{-1}$ and $i(-p_2 +k_i-m_i)^{-1}$, respectively, where the m_i and k_i stand for internal masses and momenta. Differentiation with respect to $p_{2\mu}$ inserts a zero-momentum photon in each charged propagator of this line with the correct sign, while, as it is well known, insertions of zero-momentum photons in closed loops give a vanishing contribution. Equation (8b) shows that $G_{\mu\lambda}$ is finite at $k=0$ provided $\partial W_\lambda(p_2,p_1)/\partial p_{2\mu}$ exists.

(c) The matrix element for the emission of a photon of momentum k_μ and polarization μ from any vertex on the hadronic line is given by

$$M_{\mu\lambda} = -i\int d^4x\, e^{ik\cdot x}\langle p|T[j_\mu(x)j_\lambda^w(0)]|n\rangle, \quad (8c)$$

where j_μ and j_μ^w are the electromagnetic and weak currents. Inserting a complete set of intermediate states one sees that only the one-proton state gives rise to a $1/k$ singularity (as $k_\mu \to 0$) and that the residue of this singularity coincides with the first term in the square bracket of Eq. (6a). Thus, the only $1/k$ singularity in $M_{\mu\lambda}$ corresponds to the term already separated in Eq. (6a). Consequently, we see that $G_{\mu\lambda}(p_2,p_1,k)$ is "regular" as $k_\mu \to 0$.

Inserting Eq. (6a) into Eq. (1) and adding the contributions from diagrams in which the photon is emitted from the electromagnetic vertex attached to the external neutron line and from the internal lines of the proper vertex of the weak-interaction current, we obtain

[21] The index i labels the particular diagram under consideration and $\sum_i W_\lambda^i,(p_2,p_1) = W_\lambda(p_2,p_1)$.

the matrix element M_1:

$$M_1 = -\frac{G_V{}^0}{\sqrt{2}}\frac{\alpha}{4\pi^3 i}\int\frac{d^4k\; D_{\nu\mu}(k)[\bar{u}_e(2l_\nu-\gamma_\nu k)O_\lambda v_\nu]}{(k^2-2l\cdot k+i\epsilon)}$$

$$\times\left[\bar{u}_p\left\{\frac{(2p_{2\mu}+k_\mu)W_\lambda(p_2,p_1)}{(k^2+2p_2\cdot k+i\epsilon)}+T_{\mu\lambda}(p_2,p_1,k)\right\}u_n\right],$$

$$\text{(9a)}$$

where

$$T_{\mu\lambda}(p_2,p_1,k)=\frac{R_{\mu\lambda}{}^{(p)}(p_2,p_1,k)}{k^2+2p_2\cdot k+i\epsilon}+G_{\mu\lambda}(p_2,p_1,k)$$

$$+\frac{R_{\mu\lambda}{}^{(n)}(p_2,p_1,k)}{k^2-2p_1\cdot k+i\epsilon},\quad\text{(9b)}$$

$$R_{\mu\lambda}{}^{(n)}(p_2,p_1,k)=W_\lambda(p_2,\;p_1-k)(p_1-k+m_n)$$

$$\times F_\mu{}^{(n)}(p_1-k,\;p_1).\quad\text{(9c)}$$

The quantity M_1 is the matrix element (in the Landau gauge) of all the diagrams in which a photon is exchanged between the hadronic and electron lines [Figs. 1(a), (b), (c)]. The term proportional to $(2p_{2\mu}+k_\mu)W_\lambda(p_2,p_1)$ in Eq. (9a) and the term involving $R_{\mu\lambda}{}^{(p)}$ in $T_{\mu\lambda}$ represent the contribution from graphs in which the photon is emitted from a vertex attached to the external proton line, while $G_{\mu\lambda}$ and the term involving $R_{\mu\lambda}{}^{(n)}$ in $T_{\mu\lambda}$ correspond to the diagrams in which the photon is emitted from the internal lines of the weak vertex and from a vertex attached to the external neutron line, respectively.

The tensor-pseudotensor $T_{\mu\lambda}(p_2,p_1,k)$ has two basic properties: (a) $T_{\mu\lambda}$ is "regular" as $k\to0$; (b) $T_{\mu\lambda}$ is transversal in the sense that

$$k_\mu T_{\mu\lambda}=0.\quad\text{(10)}$$

Equation (10) is obtained immediately by using Eqs. (4a), (7a), (8a) and the definitions (9b) and (9c). We also note that the contribution of the term proportional to $(2p_{2\mu}+k_\mu)W_\lambda(p_2,p_1)$ in Eq. (9a) can be computed explicitly: one readily finds that the corresponding k integration is finite in the uv region [in the Landau gauge used in Eq. (9a)], although it is infrared divergent (i.e., it depends on $\ln\lambda_{min}$).

Thus, in Eq. (9a) we have achieved a separation of the diagrams depicted in Figs. 1(a)–(c) into two parts: the first part is finite (in the Landau gauge used here), contains all the terms of order $1/k$ in the hadronic covariant and can be evaluated independently of the details of the strong interactions; the second part depends on the details of the strong interactions through $T_{\mu\lambda}$ and is regular as $k_\mu\to0$ in the manner described above.

Equation (9a), by itself, is not invariant under gauge transformations of the photon propagator. As is well known, in order to obtain matrix elements which are invariant under such transformations, it is necessary to add to Eq. (9a) the diagrams in which the photon is emitted and absorbed by the electron line (electron
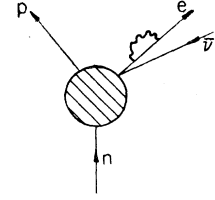
FIG. 2. Wave function renormalization of electron.

wave function renormalization) and also the diagrams in which the photon is emitted and absorbed by the hadronic line.

In the Landau gauge, after mass renormalization, the electron wave function renormalization (Fig. 2) to order α is given by

$$M_2=-\frac{\alpha}{8\pi^3 i}\frac{G_V{}^0}{\sqrt{2}}[\bar{u}_pW_\lambda(p_2,p_1)u_n]$$

$$\times\int\frac{d^4k\; D_{\nu\mu}(k)[\bar{u}_e(2l_\mu-\gamma_\mu k)l(2l_\nu-k\gamma_\nu)(l+m)O_\lambda v_\nu]}{(k^2-2l\cdot k+i\epsilon)^2 2m^2}.$$

$$\text{(11)}$$

One readily verifies that the integral in Eq. (11) is finite in the uv region, a well-known property of the electron wave function renormalization in the Landau gauge. Moreover, M_2 is proportional to the zeroth-order matrix element

$$M_0=\frac{G_V{}^0}{\sqrt{2}}[\bar{u}_eO_\lambda v_\nu][\bar{u}_pW_\lambda(p_2,p_1)u_n].\quad\text{(12)}$$

From Lorentz covariance, it is clear that the matrix element for the diagrams in which a photon is emitted and absorbed by the hadronic line (Fig. 3) must have the following structure:

$$M_3=\frac{\alpha}{2\pi}[\bar{u}_eO_\lambda v_\nu][V_\lambda+A_\lambda],\quad\text{(13a)}$$

where V_λ and A_λ are hadronic covariants of the form

$$V_\lambda=\bar{u}_p[f_1(q^2)\gamma_\lambda+if_2(q^2)\sigma_{\lambda\nu}q_\nu+f_3(q^2)q_\lambda]u_n,\quad\text{(13b)}$$

$$A_\lambda=\bar{u}_p[g_1(q^2)\gamma_\lambda+ig_2(q^2)\sigma_{\lambda\nu}q_\nu+g_3(q^2)q_\lambda]\gamma_5 u_n,\quad\text{(13c)}$$

and we remember that q is the total-momentum transfer to the leptons. If we neglect the terms proportional to αq and the q^2 dependence of the form factors, which most likely are excellent approximations, only the terms $f_1(0)\gamma_\lambda$ and $g_1(0)\gamma_\lambda\gamma_5$ survive in Eqs. (13). The effect of contributions of this type on the physical observables is discussed in Sec. IV.

As it is well known, all the infrared terms in the contributions of Fig. 3 arise from graphs in which the photon is emitted and absorbed by the outgoing proton and are entirely contained in the following quantity[22]:

$$M_3{}^{(c)}=-\frac{\alpha}{8\pi^3 i}M_0\int\frac{d^4k\; D_{\nu\mu}(k)(2p_2-k)_\mu(2p_2-k)_\nu}{(k^2-2p_2\cdot k+i\epsilon)^2},\quad\text{(14)}$$

[22] These contributions must be included in the model-independent part of the corrections to guarantee the gauge invariance of the separation and the cancellation of the infrared divergences.
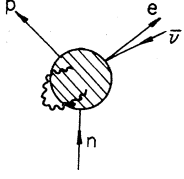
FIG. 3. Photon emitted and absorbed by hadronic lines. We include symbolically in Fig. 3 the diagrams with one or two ends of the photon line attached to the external hadrons.

where we have introduced the superscript c to indicate that $M_3^{(c)}$ is a convection-convection contribution in the classification introduced in Ref. 17. The fact that Eq. (14) contains all the infrared terms in the contributions of Fig. 3 can be discussed by methods similar to those explained after Eq. (8a) and is consistent with statements made in Refs. 16 and 17.

Finally, let us consider the contribution of Eqs. (11), (14), and the first term in Eq. (9a): each of these terms is finite in the uv region and their sum is invariant under gauge transformations of the photon propagator. The last statement can be checked immediately by making the replacement $D_{\nu\mu}(k) \rightarrow k_\nu k_\mu c(k^2)$ (where c is arbitrary) and observing that the sum vanishes. The explicit contribution of these terms and the inner bremsstrahlung diagrams is given in Sec. IV; we anticipate that their dependence on E and m is identical to that obtained in the bare nucleon calculation.

Thus far, we have considered a nucleon in the presence of the strong interactions, but we have not taken into account the possibility that the decay occurs via an intermediate boson. However, it is clear that the separation and all equations discussed in this section can be extended immediately to the case of an intermediate boson by making the following changes: W_λ is replaced everywhere by $W_\rho D_{\rho\lambda}{}^B$, where $D_{\rho\lambda}{}^B$ is the propagator of the intermediate boson and $G_{\mu\lambda}$ is redefined so as to include the emission of a photon by the intermediate boson. Graphs in which a photon is exchanged between the boson and the electron are included in the contribution involving $T_{\mu\lambda}$ in Eq. (9a), while all other graphs in which a photon is attached to the intermediate boson are included in Eq. (13a). In other words, the role of the intermediate boson in this separation is simply to "enlarge" the structure of the weak vertex. As it was mentioned in the Introduction, it is very likely that these arguments can be extended to more general space-time structures for the weak interactions.

III. ON SOME GENERAL PROPERTIES OF THE MODEL-DEPENDENT CONTRIBUTIONS

In this section we discuss some general properties of the term involving $T_{\mu\lambda}$ in Eq. (9a). Aside from constant factors, it is given by the integral

$$I = -\frac{\alpha}{2\pi}\int \frac{d^4k[\bar{u}_e(2l_\mu - \gamma_\mu k)O_\lambda v_\nu][\bar{u}_p T_{\mu\lambda}(p_2,p_1,k)u_n]}{(k^2+i\epsilon)(k^2-2l\cdot k+i\epsilon)},$$
(15)

where we have used the transversality relation of Eq.

(10) and have set $\lambda_{\min}=0$ because the integral is infrared-convergent. Equation (15) depends on the details of the strong interactions and the assumptions about the existence of an intermediate boson via the tensor-pseudotensor $T_{\mu\lambda}$. In particular, the convergence of I in the uv region depends on the asymptotic properties of $T_{\mu\lambda}$ as $k \rightarrow \infty$, and these are model-dependent. We will assume that Eq. (15) is either finite in the uv region or that otherwise it has been regularized so as to have mathematical meaning.

Let us first consider the contribution K to Eq. (15) involving the factor $\gamma_\mu k$ in the leptonic covariant:

$$K = \frac{\alpha}{2\pi}[\bar{u}_e\gamma_\mu\gamma_\rho O_\lambda v_\nu]\int \frac{d^4k\, k_\rho[\bar{u}_p T_{\mu\lambda}(p_2,p_1,k)u_n]}{(k^2+i\epsilon)(k^2-2l\cdot k+i\epsilon)}.$$
(16)

Using the separation

$$[k^2-2l\cdot k+i\epsilon]^{-1}=[k^2+i\epsilon]^{-1}$$
$$+2l\cdot k[k^2+i\epsilon]^{-1}[k^2-2l\cdot k+i\epsilon]^{-1},\quad (17a)$$

K can be written as:

$$K = K_1 + K_2,$$
(17b)

$$K_1 = \frac{\alpha}{2\pi}[\bar{u}_e\gamma_\mu\gamma_\rho O_\lambda v_\nu]\int \frac{d^4k\, k_\rho[\bar{u}_p T_{\mu\lambda}(p_2,p_1,k)u_n]}{(k^2+i\epsilon)^2},\quad (17c)$$

$$K_2 = \frac{\alpha}{\pi}l_\sigma[\bar{u}_e\gamma_\mu\gamma_\rho O_\lambda v_\nu]\int \frac{d^4k\, k_\sigma k_\rho[\bar{u}_p T_{\mu\lambda}(p_2,p_1,k)u_n]}{(k^2+i\epsilon)^2(k^2-2l\cdot k+i\epsilon)}.$$
(17d)

Because $T_{\mu\lambda}(p_2,p_1,k)$ is "regular" as $k_\mu \rightarrow 0$, K_1 and K_2 converge in the infrared region.

The integral in Eq. (17c) does not involve the electron four-momentum l_μ explicitly. If the total-momentum transfer to the leptons q is neglected in $T_{\mu\lambda}$ so that $T_{\mu\lambda}(p_2,p_1,k)$ is approximated by $T_{\mu\lambda}(p_1,p_1,k)$, it is easy to see after performing the k integration that K_1 is of the form (see Appendix A):

$$K_1 = (\alpha/2\pi)[\bar{u}_e O_\lambda v_\nu][\bar{u}_p(a\gamma_\lambda + b\gamma_\lambda\gamma_5)u_n],\quad (17e)$$

where a and b are constants independent of l_μ. The precise values of a and b depend on $T_{\mu\lambda}$ and, therefore, on the details of the strong interactions and the assumptions about the existence of an intermediate boson. The effect of contributions of the form (17e) on the physical observables is discussed in Sec. IV.

We now consider the contribution K_2 of Eq. (17d) and discuss its dependence on the four-vector l. We will illustrate our procedure by considering the contribution to K_2 from the term involving $R_{\mu\lambda}{}^{(p)}(p_2,p_1,k)$ in the definition of $T_{\mu\lambda}$ [Eq. (9b)]. The other two terms can be discussed in a similar manner.

Using the fact that $R_{\mu\lambda}{}^{(p)}(p_2,p_1,k) \rightarrow 0$ as $k \rightarrow 0$ we expand this quantity as follows:

$$R_{\mu\lambda}{}^{(p)}(p_2,p_1,k) = k_\alpha S_{\alpha\mu\lambda}{}^{(a)}(p_2,p_1)$$
$$+ k_\alpha k_\beta S_{\alpha\beta\mu\lambda}{}^{(b)}(p_2,p_1,k),\quad (18a)$$

where $S_{\alpha\mu\lambda}{}^{(a)}(p_2,p_1)$ is independent of k and $S_{\alpha\beta\mu\lambda}{}^{(b)}\times(p_2,p_1,k)$ is regular as $k\to 0$. Using Eq. (18a), and denoting by $K_2{}^{(p)}$ the contribution from the $R_{\mu\lambda}{}^{(p)}$ term in Eq. (9b), one obtains:

$$K_2{}^{(p)}=(\alpha/\pi)l_\sigma[\bar{u}_e\gamma_\mu\gamma_\rho O_\lambda v_\nu]\int\frac{d^4k\; k_\sigma k_\rho[\bar{u}_p(k_\alpha S_{\alpha\mu\lambda}{}^{(a)}(p_2,p_1)+k_\alpha k_\beta S_{\alpha\beta\mu\lambda}{}^{(b)}(p_2,p_1,k))u_n]}{[k^2+i\epsilon]^2[k^2-2l\cdot k+i\epsilon][k^2+2p_2\cdot k+i\epsilon]}.\qquad(18b)$$

The integral involving $S_{\alpha\mu\lambda}{}^{(a)}(p_2,p_1)$ can be evaluated by standard methods and one finds, as intuitively expected, that it contains terms of logarithmic order in E and m but it does not contain terms of order $1/l$. As this particular integral is multiplied by l_σ, we see that the first term in Eq. (18b) gives contributions of order $\alpha(E/m_p)\ln(m_p/E)$, $\alpha(E/m_p)$, and smaller. We refer to these contributions as "terms of higher order in the lepton four-momentum"; they are affected by the strong interactions but very likely they are quite small, as they involve two small factors.

The integral involving $S_{\alpha\beta\mu\lambda}{}^{(b)}(p_2,p_1,k)$ cannot be evaluated explicitly in our method, as the exact dependence of this function on k has not been determined. For the purpose of further discussion, let us define the expression

$$L^{(b)}(\zeta)=\int\frac{d^4k\; k_\sigma k_\rho k_\alpha k_\beta S_{\alpha\beta\mu\lambda}{}^{(b)}(p_2,p_1,k)}{[k^2+i\epsilon]^2[k^2-2\zeta l\cdot k+i\epsilon][k^2+2p_2\cdot k+i\epsilon]},\qquad(18c)$$

where we have suppressed the tensor indexes in L and ζ is a numerical variable. We note that $L^{(b)}(1)$ is the integral involving $S_{\alpha\beta\mu\lambda}{}^{(b)}$ in Eq. (18b). For $\zeta=0$, $L^{(b)}(0)$ is well defined. In fact, because $S_{\alpha\beta\mu\lambda}{}^{(b)}$ is regular as $k_\mu\to 0$, $L^{(b)}(0)$ is convergent in the infrared region, and it possesses the same convergence properties in the ultraviolet as $L^{(b)}(1)$. We will assume in this paper that $L^{(b)}(\zeta)$ is a continuous function of ζ in the interval $-1\leq\zeta\leq 1$ or that, at least, $L^{(b)}(\zeta)$ remains bounded in this region. The continuity assumption seems to be a very plausible mathematical hypothesis and it is illustrated in Appendix B by choosing a particular form for $S_{\alpha\beta\mu\lambda}{}^{(b)}(p_2,p_1,k)$. In this Appendix we discuss also the continuity properties of $L^{(b)}$ regarded as a function of the four variables l_μ ($\mu=1,2,3,0$), in connection with the problem of mass singularities. Under the above mentioned assumptions we see that the second term in Eq. (18b) is also of "higher order in l," as the integral $L^{(b)}(1)$ is multiplied by l_σ. Therefore, the same is true for $K_2{}^{(p)}$. More precisely, the second term in Eq. (18b) is of the order $\alpha E/M$, where M may be the proton mass m_p, the mass of the intermediate boson, or the mass of the hadrons which characterize the behavior of the structure function $S^{(b)}(p_2,p_1,k)$. A similar analysis shows that the term involving l_μ in the leptonic covariant of Eq. (15) is also of higher order in the lepton momentum.

In summary, we conclude that the contributions of the term $T_{\mu\lambda}$ in Eq. (9a), which depends on the details of the strong interactions and the assumptions about

the existence of the intermediate boson, are of the form (17e), provided that terms of order $\alpha(E/M)\ln(M/E)$ and $\alpha q/M$ are neglected.

IV. GENERAL RESULTS

In Sec. I we have separated the contributions from the diagrams of Fig. 1 into two parts discussed in detail in Eq. (9a) and subsequent paragraphs. We pointed out that the first part, when combined with the electron wave function renormalization (Fig. 2) and the convection-convection contribution to the diagrams of Fig. 3, [Eq. (11)], is invariant under gauge transformations of the photon propagator, is finite in the uv region, and is independent of the details of the strong interactions and the assumptions about the existence of the intermediate boson. These particular contributions are expressed in terms of well-defined integrals and can be evaluated by standard methods. Aside from these, we encounter contributions which do depend on the details of the strong interactions and the assumptions about the intermediate boson, namely the term involving $T_{\mu\lambda}$ in Eq. (9a) and the contributions to Fig. 3 not contained in Eq. (11). In Sec. III and in the discussion in Sec. II after Eq. (13a) we have shown, under some general and plausible assumptions, that these two contributions are of the form given in Eq. (17e), provided we neglect terms of order $\alpha(E/M)\ln(M/E)$ and $\alpha q/M$.

Thus, we can write all these "model-dependent terms" in the form

$$S=\frac{G_V{}^0}{\sqrt{2}}\frac{\alpha}{2\pi}[\bar{u}_e O_\lambda v_\nu][\bar{u}_p(c\gamma_\lambda+\rho d\gamma_\lambda\gamma_5)u_n],\qquad(19)$$

$$\rho=-G_A{}^0/G_V{}^0,\qquad(19a)$$

where c and d are constants independent of E and m. Equation (19) obviously contributes to the corrections to the lifetime. However, if we neglect terms of order α^2, it is clear that these terms can be formally "absorbed" in the zeroth-order matrix elements by redefining the vector and axial-vector coupling constants. Therefore, up to terms of order α the contribution of Eq. (19) will not alter any observable which to zero order in α is independent of the values of $G_V{}^0$ and $G_A{}^0$. If the initial neutron is unpolarized, there are two such observables: (i) the electron spectrum when the neutrino direction and lepton polarizations are undetected and (ii) the longitudinal polarization of the electron when the neutrino direction and polarization are undetected.

Combining the contribution of the first term of Eq.

(9a) with Eqs. (11), (14), and (19), and the inner bremsstrahlung diagrams, we obtain the following expression for the decay probability of a neutron after averaging over the initial nucleon polarization, integrating over the neutrino direction and summing over the lepton polarizations:

$$P\,dp=\frac{\xi[E_m-E]^2 p^2}{2\pi^3}dp\left[1+\frac{\alpha}{2\pi}g(E,E_m,m)\right],\quad(20a)$$

where

$$g(E,E_m,m)=3\ln\left(\frac{m_p}{m}\right)-\tfrac{3}{4}+4\left[\frac{\tanh^{-1}\beta}{\beta}-1\right]$$

$$\times\left[\frac{(E_m-E)}{3E}-\tfrac{3}{2}+\ln\left\{\frac{2(E_m-E)}{m}\right\}\right]+\frac{4}{\beta}L\left(\frac{2\beta}{1+\beta}\right)$$

$$+\frac{1}{\beta}\tanh^{-1}\beta\left[2(1+\beta^2)+\frac{(E_m-E)^2}{6E^2}-4\tanh^{-1}\beta\right],\quad(20b)$$

and

$$\xi=|G_V{}^0|^2\left(1+\frac{\alpha}{\pi}\operatorname{Re}c\right)|M_F|^2$$

$$+|G_A{}^0|^2\left(1+\frac{\alpha}{\pi}\operatorname{Re}d\right)|M_{GT}|^2.\quad(20c)$$

In these equations E_m is the end-point energy of the electron, p is the electron momentum, $\beta=p/E$, M_F and M_{GT} are the Fermi and Gamow-Teller matrix elements, and $L(x)$ is the Spence function:

$$L(x)=\int_0^x\frac{dt}{t}\ln(1-t).\quad(20d)$$

In the case of the decay of a free neutron, as it is well known, $|M_F|^2=1$, $|M_{GT}|^2=3$. We have used the more general notation of Eq. (20c) to facilitate further discussion.

The model-dependent constants c and d, which remain undetermined in this approach, have been absorbed in the definition of the factor ξ. Equations (20) show quite clearly that these terms give contributions to the lifetime but do not alter the shape of the electron spectrum (at least up to terms of order α). Following the usual convention[3] we have not included in Eq. (20a) the Coulomb term of order α, although it arises from one of the integrals in Eq. (9a). The reason is that such contribution is already incorporated in the Fermi function $F(Z,E)$. We further recall that terms of relative order $\alpha(E/M)\ln(M/E)$, $\alpha q/M$, and smaller have been neglected in the square bracket of Eq. (20a).

Equations (20) can be applied to allowed nuclear β decay by using an independent particle model of the nucleus.[7] The function $g(E,E_m,m)$ is independent of the details of the strong interactions and the assumptions about the intermediate boson, it is independent of the value of M_{GT}/M_F, and it is the same for electron and positron decay. Thus, $g(E,E_m,m)$ is a universal function

which describes the deviations from the allowed electron spectrum arising from the radiative corrections of order α. It differs from the corrections computed in the past in a numerical constant, independent of E, E_e, and m.[23] Therefore, it gives the same corrections to the shape of the electron spectrum as the "bare-particle" calculations. The difference, of course, is that in the present approach we have concluded that this function is not altered by "structure effects." As was mentioned in Sec. 4 of Ref. 3, the corrections to the spectral shape arising from these effects is significant, particularly in allowed nuclear β decays with $E_m\gg mc^2$.

For given values of the coupling constants $G_V{}^0$ and $G_A{}^0$, the function g increases the decay probability for neutron β decay by 1.5% and that of O^{14} by 1.3%. Clearly, this is not a complete calculation of the corrections to the lifetime because we have not considered the model-dependent contributions of Eq. (20c). However, the effect of g on the lifetime is rather large as it gives rise to the term $3(\alpha/2\pi)\ln(m_p/E_m)$, which was discussed in detail in Ref. 4.

Aside from the electron spectrum, there are other observables for which the radiative corrections can be computed in a "model-independent manner," provided that the terms of order $\alpha q/M$ and $\alpha(E/M)\ln(M/E)$ are neglected.[24] Although at present this seems to be only of academic interest, we illustrate the point for the case of the longitudinal polarization of the electron when the initial nucleus is unpolarized and the neutrino direction is undetected. Most of the terms in the radiative corrections turn out to be proportional to the uncorrected matrix M_0 of Eq. (12). However in Eq. (9a) there are some induced terms of a different tensor character.[25] For allowed decays the interference with the zeroth-order terms can be rigorously computed and we find for the longitudinal polarization the expression

$$P=-\beta\left[1+\frac{\alpha}{2\pi}\frac{(1-\beta^2)}{\beta}\ln\left(\frac{1+\beta}{1-\beta}\right)\right].\quad(21)$$

In Eq. (21) we have only included the contribution of

[23] The function $g(E,E_m,m)$ differs from the expression between curly brackets in Eq. (4.1) of Ref. 3 by a constant term $6\ln(\lambda/m_p)$ $+9/4$, where λ is the uv cutoff in the bare-particle calculation. We have simplified the structure of this function by using the identity

$2L(\beta)-2L(-\beta)+L((1-\beta)/2)-L((1+\beta)/2)$
$\qquad=2L(2\beta/(1+\beta))-\ln((1+\beta)/2)\ln((1+\beta)/(1-\beta))$.

[24] The number of such observables is particularly large for pure Gamow-Teller or pure Fermi transitions.

[25] In the allowed approximation these terms may be written as scalar and tensor contributions of order α multiplied by a function of E. Their interference with M_0 gives a contribution of order α to the Fierz interference coefficient b. The origin of Eq. (21) can then be understood in terms of the conventional theory of β decay by observing Eqs. (9), (A5), and (A10) of the paper of J. D. Jackson, S. B. Treiman, and H. W. Wyld, Jr. [Phys. Rev. 106, 517 (1957)]. These interference terms give also a contribution to the electron spectrum, which have been included in the function $g(E,E_m,m)$. The expression of the terms not proportional to M_0 in terms of induced scalar and tensor terms is not necessary to perform the calculations but it is mentioned here for orientational purposes.

the virtual radiative corrections.[26] This result is valid for an arbitrary value of M_{GT}/M_F. The absence of the model-dependent terms c and d indicates again that Eq. (21) is independent of the details of the strong interactions and the assumptions about the existence of the intermediate boson.

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APPENDIX A

In Eq. (17c) we write $O_\lambda \equiv \gamma_\lambda(1+\gamma_5)$ and use the identity

$$\gamma_\mu\gamma_\rho\gamma_\lambda = g_{\mu\rho}\gamma_\lambda - g_{\mu\lambda}\gamma_\rho + g_{\rho\lambda}\gamma_\mu - i\epsilon_{\mu\rho\lambda\alpha}\gamma_\alpha\gamma_5, \quad (A1)$$

where we adopt the convention $\epsilon_{0123}=1$. The contribution of the first term in Eq. (A1) to K_1 is

$$\frac{\alpha}{2\pi}[\bar{u}_e\gamma_\lambda(1+\gamma_5)v_\nu]\int\frac{d^4k[\bar{u}_p k_\mu T_{\mu\lambda}(p_2,p_1,k)u_n]}{(k^2+i\epsilon)^2}. \quad (A2)$$

Neglecting the momentum transfer to the leptons $q = p_1 - p_2$ so that $T_{\mu\lambda}(p_2,p_1,k) \approx T_{\mu\lambda}(p_1,p_1,k)$, remembering that $\bar{u}_p k_\mu T_{\mu\lambda}(p_1,p_1,k)u_n$ is a combination of vector and axial-vector terms, and noting that after integration over k we have only at our disposal the four-vectors $p_{2\lambda}$, γ_λ, and $\gamma_\lambda\gamma_5$, we see that (A2) reduces to an expression of the form (17e), provided that terms of order q are neglected. Identical conclusions hold for the other contributions of (A1).

APPENDIX B

In this Appendix we discuss continuity properties of some Feynman integrals with reference to Eq. (18c) and the discussion thereafter. As an illustration we consider the following special form for $S_{\alpha\beta\mu\lambda}{}^{(b)}$:

$$S_{\alpha\beta\mu\lambda}{}^{(b)} \rightarrow g_{\alpha\beta}t_{\mu\lambda}\Lambda^2/(\Lambda^2-k^2), \quad (B1)$$

where $t_{\mu\lambda}$ is independent of k and Λ is a constant such that $\Lambda \gg E$. Equation (B1) satisfies the condition that $S^{(b)}$ is finite as $k_\mu \rightarrow 0$. Its particular structure has been

chosen in order to simplify the algebra. In such case the integral $L^{(b)}(\zeta)$ in Eq. (18c) is proportional to the expression

$$J_{3\sigma\tau}(\zeta) = 8i(2\pi)^{-2}$$
$$\times \int\frac{d^4k \; k_\sigma k_\tau[\Lambda^2/(\Lambda^2-k^2)]}{[k^2+i\epsilon][k^2-2\zeta l\cdot k+i\epsilon][k^2+2p_2\cdot k+i\epsilon]}. \quad (B2)$$

We recall that p_2 and l are the proton and electron four-momenta; m_p and m are the corresponding masses. This integral can be evaluated by standard methods. One finds that $J_{3\sigma\tau}$ is in fact a continuous function of ζ in the interval $-1 \le \zeta \le 1$. If $\Lambda \gg m_p$ and terms of order m_p/Λ are neglected,

$$\lim_{\zeta\rightarrow 0}J_{3\sigma\tau}(\zeta) = J_{3\sigma\tau}(0) = \frac{p_{2\sigma}p_{2\tau}}{m_p{}^2} - g_{\sigma\tau}\left[\frac{3}{4}+\ln\left(\frac{\Lambda}{m_p}\right)\right]. \quad (B3)$$

This is in agreement with the general discussion after Eq. (18c).

It is interesting to note, however, that $J_{3\sigma\tau}$ is not a continuous function of the four variables l_μ ($\mu=1,2,3,0$) in the neighborhood of $l_\mu=0$. The reason is that if one regards $J_{3\sigma\tau}$ as a function of these four variables l_μ, the cone $l^2=0$ (we might call it "the light cone of the massless electron") is a surface of logarithmic singularities. These are the mass singularities discussed in Ref. 3 and, in greater depth elsewhere.[27,28] To illustrate this point we note that in the proton rest frame, one of the contributions to (B2) can be written approximately as $-[(l_\sigma l_\tau)/(2m_p p)]\ln[(E+p)/(E-p)]$, where we recall that E and p are the electron energy and three-momentum. Replacing $l_\sigma \rightarrow \zeta l_\sigma$, $p \rightarrow \zeta p$, $E \rightarrow \zeta p$, we see that this term is a continuous function of ζ and tends to 0 as $\zeta=0$. However, it is not a continuous function of E and p as $E=p$ is a line of logarithmic singularities. This is the only contribution to Eq. (B2) which involves mass singularities. It is, however, a term of higher order in the lepton momentum in the sense of Sec. III. By examining the continuity properties with respect to ζ we avoid, so to speak, the light cone of "the massless electron."

We note that for the general argument of Sec. III it is not actually necessary to assume the continuity or even the boundedness of the second term of Eq. (18b). The minimal required property is that such contribution should not diverge faster than $\ln\zeta$ as $\zeta \rightarrow 0$.

[26] If real photons are undetected, there may be additional contributions from the nonclassical contributions of the inner bremsstrahlung. As it stands, Eq. (21) refers to a hypothetical experiment in which only soft photons are undetected.

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