Nonrelativistic Sum Rules and the Binding Energy of the Deuteron

K. CHADAN AND A. MONTES LOZANO
Laboratoire de Physique Théorique et Hautes Energies, Orsay, France
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Nonrelativistic sum rules obtained by one of the authors in potential scattering with potentials of finite radius are used to calculate in a very simple way the deuteron binding energy from the neutron-proton triplet-S phase shift. In order to use these sum rules, one must know the imaginary zeros in the k plane ($k=\sqrt{E}$) of the regular solution $\varphi_l(k,r)$ of the radial Schrödinger equation. These are studied in detail for an arbitrary potential (whether having a finite radius or not), and their properties are established. It is found that these zeros are closely related to the bound states. In particular, their number is the same as the number of bound states. If we denote the zeros by $i\gamma_j(r)$, $j=1\cdots n$, it is shown that each $\gamma_j(r)$ is an increasing function of r and has a finite limit for $r\to\infty$, which corresponds to a bound state of energy $E_j=-\gamma_j^2(\infty)$. These facts, together with the sum rules, enable us to calculate the bound-state energies in a very simple and direct manner from the phase shift and the interaction radius, which is an arbitrary parameter in the sum rules and may be chosen as large as necessary in order to be sure that one is really outside the range of the interaction. The method is applied to the deuteron, assuming that the n-p interaction in the $^3S_1$ state has a radius of the order of the triplet scattering length ($a_t=5.4$ F), i.e., more than 3 times the triplet effective range ($r_{0t}=1.73$ F). The result is 2.30 MeV, in good agreement with the experimental value 2.224 MeV. This particular example shows that although, in general, the phase shift and the binding energies are (except for the Levinson theorem) independent of each other, it may be possible, for interactions of short range, to calculate the latter from the former in a simple way by making the physically reasonable assumption that the interaction vanishes identically beyond some radius, which may be taken as large as necessary. Some possible applications of the sum rules are briefly indicated at the end.

I. INTRODUCTION

IN this paper we consider the problem of scattering and bound states of a nonrelativistic particle of mass M in a central potential $V(r)$. The potential is assumed to satisfy the customary conditions

$$\int_0^A r|V(r)|dr<\infty\,, \tag{1a}$$

$$\int_B^\infty r^2|V(r)|dr<\infty\,, \tag{1b}$$

where $A<\infty$ and $B\geq 0$. Under these assumptions, it is well known[1,2] that the scattering problem is well defined, and can be developed in terms of the Jost solutions and the Jost functions. Moreover, the inverse-scattering problem in a given angular momentum state—i.e., to find the potential from the knowledge of the phase shift at all energies and of the binding energies in the angular momentum state under consideration—is also well defined and leads to the Gel'fand-Levitan integral equation. The following theorem can then be proved[3]: Given any real continuous function $\delta_l(k)$ of the real positive variable k,[4] and n_l arbitrary real negative numbers,

$$0>E_1>E_2>\cdots>E_{n_l},$$

such that $\delta_l(\infty)=0$ and

$$\delta_l(0)=n_l\pi\,, \tag{2}$$

then there is a family of potentials, the so-called phase-equivalent family, which admits, in the angular momentum state l, the phase shift $\delta_l(k)$ and n_l bound states with energies $E_1\cdots E_{n_l}$.

This family depends on n_l arbitrary positive parameters, $\lambda_1\cdots\lambda_{n_l}$. These parameters are related to the residues of the S matrix, $S_l=\exp 2i\delta_l$, at the poles E_j.[5] In other words, except for the Levinson theorem (2), there is no relationship, in general, between the phase shift and the binding energies. Each one can be chosen arbitrarily. The potential is therefore a definite universal function (functional) of all the above quantities:

* Laboratoire associé au C.N.R.S.

† Postal address: Laboratoire de Physique Théorique et Hautes Energies, Bâtiment 211, Faculté des Sciences, 91-Orsay, France.

[1] R. G. Newton, J. Math. Phys. **1**, 319 (1960). We shall follow the notations of this review article, to which the reader is referred for a good collection of references on classical works on potential scattering and the inverse-scattering problem.

[2] M. L. Goldberger and K. M. Watson, *Collision Theory* (John Wiley & Sons, Inc., New York, 1964); R. G. Newton, *Scattering Theory of Waves and Particles* (McGraw-Hill Book Company, Inc., New York, 1966); V. de Alfaro and T. Regge, *Potential Scattering* (North-Holland Publishing Company, Amsterdam, 1965).

[3] See Ref. 1, Sec. 8 and references therein.

[4] As usual, we take $\hbar=2M=1$. The energy is then given by $E=k^2$, when k is the wave number. For negative values of k, the phase shifts are defined by $\delta_l(-k)=-\delta_l(k)$, i.e., they are all odd functions of k.

[5] In the case of n-p interaction in the $^3S_1$ state, the phase-equivalent family depends on one free parameter λ. One can show, in general [R. G. Newton, Phys. Rev. **101**, 1588 (1956)], that if the phase-equivalent family contains a potential decreasing at infinity like $\exp(-2\chi r)$, $\chi>\chi_{n_l}$, this potential is unique among all the potentials of the family. It has the shortest range. On the contrary, if one of them behaves like $\exp(-2\chi r)$, $\chi\leq\chi_1$, they all do so. For the n-p interaction in the state $^3S_1$, the value of the free parameter λ can be obtained from the cross section of the deuteron photodisintegration at low energies, or from radiative capture of thermal neutrons by protons, because the wave function of the bound state depends explicitly on λ. See K. Chadan, J. Phys. Radium **16**, 843 (1955); Compt. Rend. Acad. Sci. Paris, **242**, 1964 (1956); R. G. Newton, Phys. Rev. **105**, 763; **107**, 1025 (1957); J. S. Levinger and M. L. Rustgi, *ibid.* **106**, 607 (1957). It is to be noticed that this determination of λ in the simple model considered for calculations (Eckart potentials) gives precisely the potential with the shortest range.

$V=V(\delta_l(k);\ E_1\cdots E_{n_l};\ \lambda_1\cdots\lambda_{n_l};\ r)$. The situation is quite different if we know beforehand that the potential must have some definite properties (given asymptotic behavior, etc.). In that case, we may be able to find some relations (compatibility conditions) among the above quantities. The simplest example is the one discussed in Ref. 5.

The case we consider here is the one in which the potential has a finite radius R:

$$V\equiv 0 \quad \text{for} \quad r>R. \tag{3}$$

It is obvious here that $\delta_l(k)$, E_j, and λ_j are no longer independent of each other. In fact, according to (3), we have a continuous infinity of relations among them. This, of course, means not only that the above quantities are not independent of each other, but also that the phase shift itself, as is well known, can not be arbitrarily chosen. Therefore, using (3), we can in principle calculate the binding energies and λ_j if we know the phase shift. Even if the potential has an infinite tail, it may happen that putting $V\equiv 0$ for $r>R$, and R large enough, will lead to good approximate values for the binding energies. Unfortunately, the Gel'fand-Levitan integral equation is of the Fredholm type, so that its general solution is not explicitly known. Despite this, the situation is not hopeless, because, assuming (3), we can obtain other types of functional dependence between the phase shift and the binding energies. A first type of relation is given by the sum rule[6]

$$r+\frac{d\delta}{dk}=-\sum_{j=1}^{n}\frac{2\chi_j}{k^2+\chi_j^2}-\frac{1}{\pi}\int_{-\infty}^{+\infty}dz \times\frac{\ln\left|\cos[zr+\delta(k+z)-\delta(k)+\theta(k+z)-\theta(k)]\right|}{z^2}, \tag{4}$$

valid for all $r\geq R$ and all real k. χ_j is defined by $E_j=-\chi_j^2$, and

$$\theta(x)=-2\sum_j \tan^{-1}\frac{\chi_j}{x}, \tag{5}$$

the sum over j being limited, of course, to bound states of angular momentum l. At zero energy we get

$$r-a_0\delta_{l_0}=-\sum_j\frac{2}{\chi_j}-\frac{2}{\pi}\int_0^{\infty}dk \times\frac{\ln\left|\cos[kr+\delta(k)+\theta(k)]\right|}{k^2}, \tag{6}$$

where a_0 is the S-wave scattering length and δ_{l0} is the Kronecker symbol. If there are no bound states, one must simply put all $\chi_j=0$ in (4).

A second, more general type of sum rules has also been obtained, which applies to all cases,[7] i.e., whether the potential has a finite radius or not. If the potential has a finite radius R, and for S wave, they read[8]

$$r+\frac{d\delta}{dk}=C(k,r)-2\sum_j\frac{\chi_j}{k^2+\chi_j^2}-\frac{1}{\pi}P\int_{-\infty}^{+\infty}dk' \times\frac{\ln\left|k\sin[k'r+\delta(k')]/k'\sin[kr+\delta(k)]\right|}{(k'-k)^2}. \tag{7}$$

Once more, this formula applies for all $r\geq R$ and all real k. P means the principal value of the integral, and $C(k,r)$ is given by

$$C(k,r)=\sum_n\frac{|\sin\theta_n'(k,r)|}{|k_n'(k,r)|}, \tag{8}$$

where θ_n' and $|k_n'|$ are, respectively, the argument and the modulus of the complex zeros, in the k' plane, of the regular solution $\varphi_l(k'+k,r)$ of the reduced radial Schrödinger equation

$$\Psi_l''+\left[E-\frac{l(l+1)}{r^2}\right]\Psi_l=V\Psi_l, \tag{9}$$

$$\lim_{r\to 0}\frac{(2l+1)!!}{r^{l+1}}\varphi_l(k,r)=1. \tag{9'}$$

For other waves, one simply has to replace in (7) the sine functions by appropriate combinations of the spherical Hankel functions $h_l^{(\pm)}(kr)$ and $\exp(\pm i\delta_l)$. One can also take the limit $k\to 0$ in (7) and obtain a formula similar to (6).

The above sum rules may be used for many purposes. We list here the most interesting applications.

A. Physical Definition of the Radius of Interaction

Knowing the phase shift—even if it does not correspond to a potential of finite radius R—and the binding energies, one may use Eq. (4) to define an interaction radius at the energy $E=k^2$. Indeed, it is clear that (4) and (7) should hold for very large values of r. The larger r is, the better the right-hand side will agree with the left-hand side. The agreement is, of course, complete for $r\geq R$ if the potential vanishes beyond R. In the limit of $r\to\infty$, both (4) and (7) are exact whatever the phase shift and the binding energies may be, under the sole restriction (2). This is because cutting off a given potential with infinite tail, satisfying (1b), at a very large distance R, does not affect appreciably $\delta(k)$ and E_j, so that, by making $R\to\infty$, all the above quantities reach smoothly and continuously their limits. Everything in (4) and (7) being continuous (continuous functionals) and the integrals absolutely convergent, one finds that

[6] K. Chadan, Nuovo Cimento **40**, 1194 (1965). To simplify the writing, we shall henceforth drop the subscript l in all the sum rules.

[7] K. Chadan, Nuovo Cimento **41**, 115; **44**, 838 (1966).

[8] To obtain Eq. (7), or its generalization for other waves, we just have to replace, for $r>R$, the Jost solutions $f_l(\pm k,r)$ by the free solutions (spherical Hankel functions) in Eq. (3.1) of the first paper of Ref. 7.

both are exact for $r\to\infty$, i.e., the difference between the left-hand side and the right-hand side goes to zero in that limit. If we now start from a given large value of r, at which we have a good agreement between the two sides of (4), and make it smaller and smaller, the agreement will break down at some value of r. We may then take this value as a definition of the interaction radius at the energy $E=k^2$. Note that our sum rules make use only of well-defined physical quantities. They do not contain anything (the potential or the wave function) referring to the interaction region. This point is very important because the concept of the radius of interaction should not depend on these quantities in the interior region. It is given by the region of space where particles move freely.

Of course, in practice, we do not know the phase shift at all energies, nor do we know it exactly at low energies. However, at low energies (small values of k) the high-energy part of the integrals, because of their rapid convergence, gives a very small contribution of the order of 1%. On the other hand, the uncertainties on the values of δ at low energies, for neutron-proton scattering, give an error of the order of a few percent. One may assume that the radius of interaction is given by the point where the disagreement reaches 5–10%. This procedure has been applied to both singlet and triplet n-p interactions in S wave at low energies. The details of calculation will be reported in a forthcoming paper. The interesting results one finds are as follows: First, in both singlet and triplet cases, the agreement between the two sides of (4) is excellent, and stays within 2 or 3% down to values of r given by $R_s=2.4$ F and $R_t=1.7$ F, respectively. However, making r smaller than these values by 0.1 F, one suddenly gets a large disagreement, of the order of 10%, which increases very rapidly when one makes r smaller and smaller. The second interesting feature is that the above radii are very close to the effective range values $r_{0s}=2.51$ F and $r_{0t}=1.73$ F.[9] Also R_s and R_t are almost independent of energy in the range 0–10 MeV. Therefore, we see that the concept of interaction radius, as defined through (4), seems useful, at least for nucleon-nucleon scattering at low energies, and is well defined because of its sharpness. We are therefore in a position to affirm that, at low energies, the n-p interaction in the ^3S state is quite negligible, as far as (4) and (7) are concerned, at distances of the order of the triplet scattering length $a_t=5.4$ F. This point is very important in what follows, and leads to a very good value for the binding energy of the deuteron.

B. Energy of Bound States

We may use (4) or (7) to calculate the energy of the bound states if everything else is known and if r is chosen large enough. In practice it is preferable to use (7) because the binding energies do not appear in the integrand, and despite the presence of $C(k,r)$, which we shall study in the next section. As will be seen, the complex zeros of $\varphi_l(k,r)$ have many interesting properties which relate them closely to the bound states. In fact, it is this connection between the complex zeros of φ_l and the bound states which gives us the key for the use of (7) to calculate χ_j.

[9] R. Wilson, *The Nucleon-Nucleon Interaction* (Interscience Publishers, Inc., New York, 1963).

II. COMPLEX ZEROS OF $\varphi_l(k,r)$

The regular solution $\varphi_l(k,r)$ of (9) is an even entire function of k.[1,2] We are in fact looking at the zeros of $\varphi_l(k'+k,r)$ in the k' plane, k (real positive) and $r(>0)$ being fixed. Let $k_n(r)$ be the zeros of $\varphi_l(k,r)$. We have $k_n'(k,r)=k_n(r)-k$, and so $C(k,r)$ can be written

$$C(k,r)=\sum_n \frac{|k_n||\sin\theta_n|}{|k_n|^2+k^2-2k|k_n|\cos\theta_n}. \tag{10}$$

Now, combining the Schrödinger equation (9) at $E=k^2$ with its complex conjugate, we find the well-known relation

$$(\varphi_l\varphi_l'^*-\varphi_l'\varphi_l^*)|_r=4i\,\mathrm{Re}k\,\mathrm{Im}k\int_0^r|\varphi_l|^2dr. \tag{11}$$

It follows that, for $r>0$, the zeros of $\varphi_l(k,r)$ are purely real or purely imaginary. φ_l being even in k, the zeros are symmetrical with respect to the origin, so that we have to sum only over the zeros in the upper half-plane, and multiply by a factor of 2. If we denote these zeros by $k_n(r)=i\gamma_n(r)$, we have

$$C(k,r)=2\sum_n\frac{\gamma_n(r)}{\gamma_n^2(r)+k^2},\quad \gamma_n(r)>0. \tag{12}$$

We now have to look at the variations of $\gamma_n(r)$ with respect to r. First of all, the asymptotic behavior of φ_l in the k plane is given by

$$\varphi_l(k,r)\underset{|k|\to\infty}{\longrightarrow}k^{-l-1}\sin(kr-\tfrac{1}{2}l\pi)+O(|k|^{-l-1}\exp(|\mathrm{Im}k|r)), \tag{13}$$

uniformly in r. It follows that $\varphi_l(i\gamma,r)\to\infty$ as $\gamma\to\infty$. Consequently, for each fixed $r(\neq0)$, there are no complex (imaginary) zeros beyond some value $i\Gamma(r)$. Also, because of the analyticity of φ_l in k, their number is finite. We have to show now that $\Gamma(r)$ remains bounded when $r\to\infty$. We have the following upper bounds[1]:

$$|\varphi_l-k^{-l-1}u_l|<C_l\exp(|\mathrm{Im}k|r)\times\left(\frac{r}{1+|k|r}\right)^{l+1}\int_0^r dr'|V(r')|\frac{r'}{1+|k|r'}, \tag{14}$$

where $u_l(kr)$ is the free solution (spherical Bessel function), and C_l is a numerical constant. For S wave, this

reads[10] ($\gamma>0$)

$$\left|\varphi_0(i\gamma,r)-\frac{\sinh\gamma r}{\gamma}\right|<C\frac{\exp(\gamma r)}{\gamma}D(\gamma), \tag{15}$$

$$D(\gamma)=\int_0^\infty dr'\,|V(r')|\frac{r'}{1+\gamma r'}. \tag{16}$$

It follows that $\varphi_0(i\gamma,r)$ is strictly positive whenever $2CD(\gamma)+\exp(-2\gamma r)<1$. Now, according to the definition of $D(\gamma)$, and (1a) and (1b), this inequality is satisfied whenever r and γ are large. Therefore, $\Gamma(r)$ is bounded when $r\to\infty$. The argument goes the same way for other waves if we use the asymptotic behavior of $u_l(z)$ for large values of z. As we shall see, the maximum value of $\Gamma(r)$ is reached for $r\to\infty$, and corresponds to the largest binding energy (the ground state) $E_{n_l}=-\Gamma^2(\infty)$. If the potential is absolutely integrable at the origin, we get, from (15) and (16), the upper bound

$$\Gamma(\infty)<2C_l\int_0^\infty |V(r')|\,dr', \tag{17}$$

which is very similar to the Schwinger bound.[11]

Let us now study the motion of the zeros $i\gamma_j(r)$ with r. The wave function φ_l being an entire function of k for all finite values of r, and at least once continuously differentiable with respect to r for all values of k, the equation $\varphi_l(i\gamma,r)=0$ will lead in general to well-defined and differentiable solutions $\gamma_j(r)$ as long as $\partial\varphi_l/\partial\gamma|_{r;\gamma_j(r)}\neq 0$. To simplify the writing, let us denote by $\gamma(r)$ one of these solutions. We have

$$\frac{d\gamma}{dr}=-\left.\frac{(\partial\varphi_l/\partial r)}{(\partial\varphi_l/\partial\gamma)}\right|_{r,\gamma(r)}. \tag{18}$$

Now, combining (9) for two infinitesimally close energies, we get the familiar formula

$$\frac{\partial\varphi_l}{\partial r}\frac{\partial\varphi_l}{\partial k}-\varphi_l\frac{\partial^2\varphi_l}{\partial r\partial k}=2k\int_0^r \varphi_l^2(k,r')dr'. \tag{19}$$

At this point we notice that, for k either real or purely imaginary, the right side of (19) is different from zero if both k and r are $\neq 0$.[10] Therefore, for $r\neq 0$, φ_l and $\partial\varphi_l/\partial k$ cannot both vanish at the same time, except at $k=0$. If we substitute now $\partial\varphi_l/\partial r$ from (19) into (18), we get, remembering that $\varphi_l=0$,

$$\frac{d\gamma}{dr}=\frac{2\gamma(r)}{(\partial\varphi_l/\partial\gamma)^2|_{r,\gamma(r)}}\int_0^r \varphi_l^2(i\gamma(r),r')dr'. \tag{20}$$

This equation, when integrated, gives

$$\gamma(r)=\gamma(r_1)\exp\int_{r_1}^r \Phi(r')dr', \tag{21}$$

$$\Phi(r)=2\int_0^r \varphi_l^2(i\gamma(r),r')dr'\Big/\left(\frac{\partial\varphi_l}{\partial\gamma}\right)^2|_{r,\gamma(r)}. \tag{22}$$

Now $\Phi(r)$ is real positive because $\varphi_l(i\gamma,r)$ and $\partial\varphi_l/\partial\gamma$ are both real for γ real. It follows that:

(i) $\gamma(r)$ is defined on every interval (r_1,r_2) where $\Phi(r)$ is integrable. The only singular points are those where $\partial\varphi_l/\partial\gamma$ vanishes in such a way as to make Φ nonintegrable. This is because $\gamma(r)$ stays bounded, as we saw before, and so does the numerator of Φ. Therefore, the only singular points are those at which $\gamma(r)=0$ $(k=i\gamma)$.

(ii) Starting from a singular point r_s and increasing r, the function $\gamma(r)$ becomes continuous and differentiable, keeps a constant sign, and increases monotonically when r increases. It does not vanish any more, and so we never meet a singular point again.

(iii) The trajectory $\gamma(r)$ reaches its maximum value (finite) at $r=\infty$. At this point we have $\varphi_l(i\gamma(\infty),\infty)=0$, i.e., $\varphi_l(i\gamma(\infty),r)$ corresponds to a bound state of energy $E=-\gamma^2(\infty)$. This means that to each bound state there corresponds a zero trajectory $\gamma(r)$, the squared maximum of which gives the binding energy. The number of complex zeros of φ_l is therefore the same as that of the bound states, at least when r is large enough. In particular, φ_l has no complex zeros if there are no bound states.

(iv) For small values of r ($r<r_s$), the imaginary zeros disappear: They go through the origin at $r=r_s$ (r_s different for each trajectory), become real for $r<r_s$, and stay so. In fact, at $r=r_s$, we have two zeros $\pm i\gamma(r)$ which meet at the origin. If $i\gamma(r)$ for $r<r_s$ goes to the lower half-plane, $-i\gamma(r)$ goes then to the upper half-plane, so that we would have a new imaginary zero with $d\gamma/dr<0$. This is impossible because of (20). Where these (real) zeros go when $r\to 0$ is of no importance. In general, they terminate at infinity on the real axis because $\varphi_l(k,r=0)$ vanishes independently of k.

The above considerations may be illustrated very easily with the square-well potential. We consider an attractive potential of radius R and depth $-V_0=-k_0^2$, and take for simplicity the S wave. The wave function is

$$\begin{aligned}\varphi_0(k,r)&=(\sin Kr)/K, && r\leq R\\ &=A\exp(ikr)+B\exp(-ikr), && r\geq R\end{aligned}$$

with $K=(k^2+k_0^2)^{1/2}$, and

$$A=\exp(-ikR)\frac{ik\sin KR+K\cos KR}{2ikK},$$

$$B=\exp(ikR)\frac{ik\sin KR-K\cos KR}{2ikK}.$$

[10] In general, φ_l is real for real or purely imaginary values of k. This is so because (9) is real for E real ($\gtrless 0$), and the boundary condition defining φ_l is also real and in fact independent of k. See Ref. 1.

[11] J. Schwinger, Proc. Nat. Acad. Sci. (U. S.) **47**, 122 (1961).

On the imaginary axis $k=i\gamma$, we have $K=(k_0^2-\gamma^2)^{1/2}$. It is then easily seen that the total number of imaginary zeros is N or $N+1$, according to whether $\tan k_0R$ is >0 or <0, N being given by the greatest integer such that $N\pi\leq k_0R$. The complex trajectories $\gamma_n(r)$ begin at the (singular) points r_n^0, $\gamma_n(r_n^0)=0$, $n=1, \cdots, N$, $r_n^0=n\pi/k_0$, and the point $r_{N+1}^0=R-(1/k_0)\tan k_0R$ whenever the last trajectory exists. Notice that, in this case, $r_{N+1}^0>R$, whereas $r_n^0\leq R$ for $n=1, \cdots, N$. Then one finds that, for $r\to\infty$, all the trajectories go upward in a monotonic way until their limiting values, given by

$$\cot K_n(\infty)R=-\gamma_n(\infty)/K_n(\infty),$$
$$K_n(\infty)=[k_0^2-\gamma_n^2(\infty)]^{1/2}.$$

The above equation is exactly that of bound states and corresponds to $B=0$. We finally note that, in the case where an $(N+1)$th trajectory exists ($r_{N+1}^0>R$), the scattering length a_0 must be greater than R. Indeed, the wave function at zero energy is given by

$$\varphi_0(0,r)=C(r-a_0), \quad r>R.$$

Therefore, the $(N+1)$th complex trajectory begins at $r_{N+1}^0=a_0$, and so one must have $a_0>R$. For potentials of finite radius R, it is obvious that there is at most one trajectory beginning at $r_0>R$, all the others having $r_n^0<R$.

The total bound-state term in (7) can now be written

$$S(k,r)=C(k,r)-\sum_n \frac{2\chi_n}{k^2+\chi_n^2}$$
$$=2\sum_n\left[\frac{\gamma_n(r)}{k^2+\gamma_n^2(r)}-\frac{\gamma_n(\infty)}{k^2+\gamma_n^2(\infty)}\right]. \quad (23)$$

If the phase shift is known, one can calculate $S(k,r)$, using (7) and study its dependence on r ($r>R$) and k. In practice, it would be sufficient to calculate $S(k,r)$ from (7), for a given value of r ($r>R$), at a finite number of values of k in order to determine the parameters $\gamma_j(r)$ and $\gamma_j(\infty)$. The trajectories of the zeros would then be determined outside the interaction radius by varying r. This in turn would enable us to make a best fit for $\gamma_j(\infty)$ by taking r large and using the fact that $S(k,\infty)=0$.

III. BINDING ENERGY OF THE DEUTERON

The method developed in the previous sections has been applied to calculate the deuteron binding energy from the $^3S_1$ n-p phase shift. We assume that the interaction has a radius R less than or equal to the triplet scattering length $a_t=5.4$ F. This is a very natural and plausible assumption in that both r_{0t} and R_t are close to 1.7 F. The phase shift we use is obtained as follows. For E in the range 0–24 MeV ($k=0$–106 MeV/c) we use an effective-range expansion with four terms. The first two coefficients (the scattering length and the effective range) are those of Noyes.[12] The two other coefficients are chosen in such a way as to make the phase shift in the above energy range join as smoothly as possible the phase shift in the energy region 24–400 MeV ($k=106$–433 MeV/c), taken from Arndt and MacGregor.[13] In this last energy range, we use in fact an interpolation formula that reproduces in a very smooth way the experimental results. Finally, in order to estimate the importance of the high-energy contribution, we use an asymptotic tail, up to $k=2000$ MeV/c, inspired from the p-p phase-shift analysis of Hama and Hoshizaki.[14] It is found that this asymptotic tail gives a very small contribution, of the order of at most 2%, to the integrals. The major sources of errors are in fact the computational errors (3%) and the experimental uncertainties on the phase shift (6%). The net result is $E_0=2.30(1\pm0.1)$ MeV, in good agreement with the experimental value 2.224 MeV. The calculations have been performed for several values of r ($r\geq a_t$). The result, except for very small fluctuations, is independent of r. This confirms once more that at distances greater than a few fermis, the interaction is really quite negligible.

In principle, one should obtain the best result for $k=0$ in (7). However, this cannot be done very easily. We have used several values of k in the range 0–20 MeV/c. The binding energy seems to depend slightly on k, although a constant value is not excluded in the above energy range. In fact, E_0 oscillates around the value given above, with a slight tendency to increase when k increases. The extrapolation at $k=0$ gives $E_0=2.28(1\pm0.1)$ MeV.

The above example has to be considered as a test for the practical usefulness and the reliability of the sum rules in the case of interactions with a short range. Among many applications of interest we mention the two following: (i) In nuclear physics, the sum rules may serve as a subsidiary condition for eliminating some sets of experimental phase shifts in neutron-nucleus scattering if the target and the neutron can form a bound state (isotope.)[15] The case of proton-nucleus scattering needs some further work because of the Coulomb interaction. (ii) In elementary-particle physics at low energies, one may use the sum rules for the same purpose of removing some of the ambiguities of the experimental phase shifts. Here we do not very often have a bound state.[15] However, we may use (4) or (7) to determine, as was explained in Sec. I, the interaction radii corresponding to different phase shifts, hoping that some of these may lead to interactions with an unreasonably large range.

[12] P. Noyes, Phys. Rev. **130**, 2025 (1963).
[13] R. A. Arndt and M. H. MacGregor, Phys. Rev. **141**, 873 (1966).
[14] Y. Hama and N. Hoshizaki, Progr. Theoret. Phys. (Kyoto) **34**, 584 (1965).
[15] Many-channel sum rules are needed here in most of the cases. Work in this direction is under progress and some results have already been obtained. We hope to develop the generalized sum rules in a forthcoming paper.