Unified Formulation of Effective Nonlinear Pion-Nucleon Lagrangians*

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(Received 18 July 1967)

It is shown that a nonlinear Lagrangian model with chiral $SU(2)\otimes SU(2)$ symmetry for the π-N system proposed earlier, when complemented with an exact partially conserved axial-vector current term, provides a systematic and unified treatment of various phenomenological Lagrangians that are found in the recent literature. A detailed comparison is made for three special cases, showing agreement with current algebra and experimental data for π-N and π-π scattering lengths.

1. INTRODUCTION

ONE of the successful applications of current commutation relations and the hypothesis of partially conserved axial-vector current[1,2] (PCAC) deals with the calculation of the pion-nucleon scattering lengths and pion-pion scattering lengths at threshold energy.[3,4] The same results can be reproduced in the perturbation expansion from a phenomenological Lagrangian chosen in such a way that the vector and axial-vector currents that follow from it satisfy the current commutation relations and the PCAC assumption. Some specific models have been studied by several authors[5-8] and the results are all in agreement with each other up to the second-order expansion in the coupling constant. Such a Lagrangian approach provides a simple calculational scheme for observable quantities in pion physics even if Gell-Mann's current-algebra method and the PCAC assumption are regarded to be more fundamental and primary.

The main purpose of this paper is to show that a general, partially chiral-invariant nonlinear theory of π-N interaction proposed some time ago[9-11] provides a systematic and unified approach to the problem of constructing effective Lagrangians for baryon-meson systems. Three specific models (of which one is extensively studied in a paper by Brown[7]) are introduced as particular cases of the general model to show explicitly the relation of the theory to the recent phenomenological models.[3] In the renormalizable σ model[2] a 0^+ meson is introduced in the Lagrangian to be later eliminated after assuming a high mass for σ.[5] In the aforementioned general model the Lagrangian is nonlinear to start with and involves only baryons and 0^- mesons. The comparison is facilitated by using a canonical transformation previously proposed.[9] Not surprisingly, one of the models leads exactly to the phenomenological Lagrangian considered by Schwinger, and they all agree with Weinberg's effective Lagrangian up to the second order in the pion-nucleon pseudovector coupling constant. The slight discrepancies in the S-wave pion-pion scattering are discussed in Secs. 4 and 5.

2. GENERAL NONLINEAR MODEL WITH PCAC TERM

The charge-independent Lagrangian of the pion-nucleon system with a Yukawa coupling reads

$$\mathcal{L}=-\bar{\psi}(\gamma_\mu\partial_\mu+m)\psi-\tfrac{1}{4}\,\mathrm{tr}(\partial_\mu\Phi\partial_\mu\Phi)-\tfrac{1}{4}\mu^2\,\mathrm{tr}(\Phi\Phi)+\mathcal{L}_{\mathrm{int}}\,,\quad(2.1)$$

where $\Phi=\boldsymbol{\tau}\cdot\boldsymbol{\varphi}$ and $\boldsymbol{\varphi}$ is the Hermitian pion field. The interaction Lagrangian may be nonderivative (renormalizable model)

$$\mathcal{L}_{\mathrm{int}}=ig\bar{\psi}\gamma_5\Phi\psi\,,\quad(2.2)$$

or, of a derivative coupling type

$$\mathcal{L}_{\mathrm{int}}=(f/\mu)\bar{\psi}\gamma_\mu\gamma_5\partial_\mu\Phi\psi\,.\quad(2.3)$$

The isospin symmetry of the Lagrangian (2.1) is expressed by

$$\psi\rightarrow\psi'=e^{\frac{1}{2}i\tau\cdot\omega}\psi\,,$$
$$\Phi\rightarrow\Phi'=e^{\frac{1}{2}i\tau\cdot\omega}\Phi\,e^{-\frac{1}{2}i\tau\cdot\omega}\,.$$

Now if we consider the following chiral transformation for nucleon fields:

$$\psi\rightarrow\psi'=e^{\frac{1}{2}i\gamma_5\tau\cdot\mathbf{a}}\psi\,,\quad(2.4a)$$

the Lagrangian (2.1) is no longer invariant under such chiral transformation due to the presence of not only the pion-mass term, but also the nucleon-mass term $m\bar{\psi}\psi$. Actually the perfect chiral $SU(2)$ symmetry of the Lagrangian of the pion-nucleon system is not

* Work partially supported by the U. S. Atomic Energy Commission.
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expected, because of the nonconservation of the axial-vector current which is constructed from the chiral $SU(2)$ gauge transformation. Furthermore the PCAC assumption tells us that the divergence of the axial-vector current is proportional to the meson mass, and hence, in the limit of the pion mass μ going to zero, the axial-vector current becomes conserved, leading to a perfect chiral $SU(2)$ symmetry in this limit.[9,10] In order to keep chiral symmetry valid in this limit in the presence of a nonvanishing nucleon mass term, the nonlinear model was proposed as a solution.[9,10] In this model, the nucleon mass term $m\bar{\psi}\psi$ is simply replaced by $m\bar{\psi}\mathbf{U}\psi$, $\mathbf{U}$ being an 8×8 matrix function[12] of $i\gamma_5\Phi = i\gamma_5\boldsymbol{\tau}\cdot\boldsymbol{\varphi}$. It is easily seen that

$$\bar{\psi}(\gamma_\mu\partial_\mu+m\mathbf{U})\psi$$

is invariant under the isospin $SU(2)$ transformation defined as

$$\psi\rightarrow\psi'=e^{\frac{1}{2}i\tau\cdot\omega},$$
$$\mathbf{U}\rightarrow\mathbf{U}'=e^{\frac{1}{2}i\tau\cdot\omega}\mathbf{U}\,e^{-\frac{1}{2}i\tau\cdot\omega},$$

and the chiral transformation

$$\psi\rightarrow\psi'=e^{\frac{1}{2}i\gamma_5\tau\cdot\mathbf{a}}\psi,\tag{2.4a}$$
$$\mathbf{U}\rightarrow\mathbf{U}'=e^{-\frac{1}{2}i\gamma_5\tau\cdot\mathbf{a}}\mathbf{U}\,e^{-\frac{1}{2}i\gamma_5\tau\cdot\mathbf{a}},\tag{2.4b}$$

which combine into a 6-parameter $SU(2)\times SU(2)$ chiral group (see Appendix B).

In conjunction with the replacement of the nucleon-mass term $m\bar{\psi}\psi$ by $m\bar{\psi}\mathbf{U}\psi$ we can also write down the kinetic-energy term of the mesons in the following form[13]:

$$-\frac{1}{16f^2}\mathrm{tr}(\partial_\mu\mathbf{U}\,\partial_\mu\mathbf{U}^\dagger).$$

Some specific models that will reduce to the ordinary pion kinetic energy $-\frac{1}{2}\partial_\mu\boldsymbol{\varphi}\cdot\partial_\mu\boldsymbol{\varphi}$ as f goes to zero are

$$\mathbf{U}=e^{2if\gamma_5\tau\cdot\varphi},$$
$$\mathbf{U}=\frac{1+if\gamma_5\boldsymbol{\tau}\cdot\boldsymbol{\varphi}}{1-if\gamma_5\boldsymbol{\tau}\cdot\boldsymbol{\varphi}},$$

and

$$\mathbf{U}=(1-4f^2\varphi^2)^{1/2}+2if\gamma_5\boldsymbol{\tau}\cdot\boldsymbol{\varphi}.$$

Hence, the part of the Lagrangian, symmetrical with respect to chiral $SU(2)\otimes SU(2)$ transformation, is

$$\mathcal{L}'=-\bar{\psi}(\gamma_\mu\partial_\mu+m\mathbf{U})\psi-(1/16f^2)\,\mathrm{tr}(\partial_\mu\mathbf{U}\,\partial_\mu\mathbf{U}^\dagger).\tag{2.5}$$

The total Lagrangian will contain an extra term which will break the symmetry of chiral transformation, but must still be invariant under the $SU(2)$ group because of the conservation of vector currents. The chiral symmetry-breaking term has to be constructed such that it fulfills the PCAC assumption. The explicit form of the extra term in the Lagrangian is determined by the specific models one uses. If one writes $\mathbf{U}$ in the general form (see Appendix A)

$$\mathbf{U}(if\gamma_5\Phi)=\sigma(f^2\varphi^2)+2if\gamma_5\Phi\rho(f^2\varphi^2),$$

where

$$\sigma^2(f^2\varphi^2)+4f^2\varphi^2\rho^2(f^2\varphi^2)=1,$$

it will be shown in Appendix C that the chiral symmetry-breaking term can be expressed as

$$\mathcal{L}_{\mathrm{S.B.}}=\frac{\mu^2}{2f^2}\int_0^{f^2\varphi^2}\frac{\alpha^{1/2}\sigma'(\alpha)}{[1-\sigma^2(\alpha)]^{1/2}}d\alpha,\quad \sigma'(\alpha)=\frac{d\sigma(\alpha)}{d\alpha}.\tag{2.6}$$

The total Lagrangian is therefore

$$\mathcal{L}=\mathcal{L}'+\mathcal{L}_{\mathrm{S.B.}}=-\bar{\psi}(\gamma_\mu\partial_\mu+m\mathbf{U})\psi-\frac{1}{16f^2}\mathrm{tr}(\partial_\mu\mathbf{U}\,\partial_\mu\mathbf{U}^\dagger)+\frac{\mu^2}{2f^2}\int_0^{f^2\varphi^2}\frac{\alpha^{1/2}\sigma'(\alpha)}{[1-\sigma^2(\alpha)]^{1/2}}d\alpha.\tag{2.7}$$

The lower limit in the integration for $\mathcal{L}_{\mathrm{S.B.}}$ is chosen such that when f goes to zero, i.e., in the absence of interaction, the Lagrangian will reduce to the sum of free-fermion plus free-pion Lagrangians. The lower-limit zero here is consistent with the three specific models we considered previously.

3. CANONICAL TRANSFORMATION

The Lagrangian which provides a simple way to calculate the pion scattering lengths can be obtained by the following canonical transformation[11]:

$$\xi=W\psi W^\dagger=\mathbf{U}^{1/2}\psi,\tag{3.1a}$$
$$\boldsymbol{\pi}=W\boldsymbol{\varphi}W^\dagger=\boldsymbol{\varphi},\tag{3.1b}$$

where W is a unitary operator, the detailed structure of which depends upon the specific model. For example, the W associated with the exponential model

$$\mathbf{U}=e^{2if\gamma_5\tau\cdot\varphi}$$

is given by

$$W=\exp[if\int d\mathbf{x}\,\bar{\psi}(x)\gamma_5\boldsymbol{\tau}\cdot\boldsymbol{\varphi}(x)\psi(x)].$$

From (3.1a) we have

$$\bar{\psi}=\psi^\dagger\gamma_4=(\mathbf{U}^{-1/2}\xi)^\dagger\gamma_4=\xi^\dagger(\mathbf{U}^{-1/2})^\dagger\gamma_4=\bar{\xi}\gamma_4(\mathbf{U}^{-1/2})^\dagger\gamma_4.$$

It is shown in Appendix D that

$$(\mathbf{U}^{1/2})^\dagger=\mathbf{U}^{-1/2},\quad \gamma_\mu\mathbf{U}^{\pm1/2}=\mathbf{U}^{\mp1/2}\gamma_\mu,$$

so that we have

$$\bar{\psi}=\bar{\xi}\mathbf{U}^{1/2},$$

and the Lagrangian in Eq. (2.7), after the canonical

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[12] $\mathbf{U}$ is a matrix regarded as a direct product of 2-dimensional and 4-dimensional matrices.

[13] It is understood that the trace is only performed in the $SU(2)$ part of the direct product matrix.

transformation, becomes

$$W\mathcal{L}W^\dagger = -\bar{\xi}(\gamma_\mu\partial_\mu + m)\xi - \bar{\xi}\gamma_\mu(\mathbf{U}^{1/2}\partial_\mu\mathbf{U}^{-1/2})\xi - \frac{1}{16f^2}\,\mathrm{tr}(\partial_\mu\mathbf{U}\,\partial_\mu\mathbf{U}^\dagger) + \frac{\mu^2}{2f^2}\int_0^{f^2\pi^2}\frac{\alpha^{1/2}\sigma'(\alpha)}{[1-\sigma^2(\alpha)]^{1/2}}d\alpha. \tag{3.2}$$

The Lagrangian in Eq. (3.2) contains the classical form of the free-nucleon Lagrangian $-\bar{\xi}(\gamma_\mu\partial_\mu+m)\xi$, plus the pion-nucleon interaction term $-\bar{\xi}\gamma_\mu(\mathbf{U}^{1/2}\partial_\mu\mathbf{U}^{-1/2})\xi$ and the pion Lagrangian. The free-pion Lagrangian and the pion-pion interaction Lagrangian are absorbed in the last two terms on the right-hand side of Eq. (3.2). The familiar form of the free-pion Lagrangian

$$-\tfrac{1}{2}\partial_\mu\boldsymbol{\pi}\cdot\partial_\mu\boldsymbol{\pi} - \tfrac{1}{2}\mu^2\boldsymbol{\pi}\cdot\boldsymbol{\pi}$$

will appear in the lowest order in the expansion as a power series in f.

Let us turn to consider the transformation properties of the physical nucleon and pion field ξ and $\boldsymbol{\pi}$, respectively, that appear in the transformed Lagrangian of Eq. (3.2).

Denote

$$V = e^{\frac{1}{2}i\gamma_5\boldsymbol{\tau}\cdot\mathbf{a}},$$

then Eqs. (2.4a) and (2.4b) read

$$\psi \to \psi' = V\psi, \tag{2.4a}$$

$$\mathbf{U} \to \mathbf{U}' = V^{-1}\mathbf{U}V^{-1}, \tag{2.4b}$$

where V is unitary ($VV^\dagger = V^\dagger V = 1$). Let Λ be the transformation matrix for the ξ field; then

$$\xi \to \xi' = \Lambda\xi.$$

By the canonical transformation (3.1a), (3.1b) and the chiral transformation (2.4a), (2.4b), we have

$$\xi' = \Lambda\xi = \Lambda\mathbf{U}^{1/2}\psi = \Lambda\mathbf{U}^{1/2}V^{-1}V\psi = \Lambda\mathbf{U}^{1/2}V^{-1}\psi' = \mathbf{U}'^{1/2}\psi'.$$

Hence

$$\mathbf{U}'^{1/2} = \Lambda\mathbf{U}^{1/2}V^{-1}, \tag{3.3a}$$

and

$$\Lambda = \mathbf{U}'^{1/2}V\mathbf{U}^{-1/2}. \tag{3.3b}$$

It is easily checked that Λ is also unitary:

$$\Lambda\Lambda^\dagger = (\mathbf{U}'^{1/2}V\mathbf{U}^{-1/2})(\mathbf{U}'^{1/2}V\mathbf{U}^{-1/2})^\dagger = 1.$$

Furthermore, we have (2.4b)

$$\mathbf{U}' = \mathbf{U}'^{1/2}\mathbf{U}'^{1/2} = V^{-1}\mathbf{U}V^{-1} = V^{-1}\mathbf{U}^{1/2}\mathbf{U}^{1/2}V^{-1} = V^{-1}\mathbf{U}^{1/2}\Lambda^{-1}\Lambda\mathbf{U}^{1/2}V^{-1},$$

hence

$$\mathbf{U}'^{1/2} = V^{-1}\mathbf{U}^{1/2}\Lambda^{-1}, \tag{3.4a}$$

and

$$\Lambda = \mathbf{U}'^{1/2}V^{-1}\mathbf{U}^{1/2}. \tag{3.4b}$$

By (3.3b), (3.4b), and making use of

$$\gamma_\mu V = V^{-1}\gamma_\mu, \quad \gamma_\mu\mathbf{U}^{\pm1/2} = \mathbf{U}^{\mp1/2}\gamma_\mu,$$

it is easily shown that Λ commutes with γ_μ, i.e.,

$$[\Lambda,\gamma_\mu] = 0,$$

and therefore it commutes with the 16 Γ matrices. This enables us, according to Schur's lemma, to express Λ in the general form

$$\Lambda = e^{i\boldsymbol{\tau}\cdot\mathbf{b}},$$

where $\mathbf{b}$ depends upon the pion field $\boldsymbol{\pi}$, $\mathbf{a}$, and the coupling constant f. For infinitesimal transformation, we will show in Appendix F that

$$\Lambda(\mathbf{b}) = e^{i\boldsymbol{\tau}\cdot\mathbf{b}} \simeq 1 + i\boldsymbol{\tau}\cdot\mathbf{b} = 1 - \tfrac{1}{2}if\boldsymbol{\tau}\cdot(\boldsymbol{\pi}\times\mathbf{a}).$$

Also in Appendix E, we see that (to the lowest order in f)

$$\mathbf{a} = -2f\delta\boldsymbol{\pi}.$$

Hence, we have

$$\Lambda = 1 + if^2\boldsymbol{\tau}\cdot(\boldsymbol{\pi}\times\delta\boldsymbol{\pi}),$$

which is just the infinitesimal transformation of the nucleon field discussed by Schwinger.[6]

4. EXPLICIT FORMS FOR SPECIAL CASES

Let us now discuss in more detail the three specific models of $\mathbf{U}$ mentioned at the beginning of this paper. The interaction Lagrangian will be calculated; the axial-vector currents and symmetry-breaking term will also be derived in each case.

A. Exponential Model

$$\mathbf{U} = e^{2if\gamma_5\boldsymbol{\tau}\cdot\boldsymbol{\pi}} = \cos(2f\sqrt{\pi^2}) + 2if\gamma_5\boldsymbol{\tau}\cdot\boldsymbol{\pi}\frac{\sin(2f\sqrt{\pi^2})}{2f\sqrt{\pi^2}},$$

$$\rho(f^2\pi^2) = \frac{\sin(2f\sqrt{\pi^2})}{2f\sqrt{\pi^2}} = \frac{\sin(2\sqrt{\alpha})}{2\sqrt{\alpha}},$$

$$\sigma(f^2\pi^2) = \cos(2f\sqrt{\pi^2}) = \cos(2\sqrt{\alpha}),$$

where

$$\alpha = f^2\pi^2.$$

From Eq. (2.6) and Appendix D,

$$\mathcal{L}_{\mathrm{S.B.}} = -\tfrac{1}{2}\mu^2\boldsymbol{\pi}\cdot\boldsymbol{\pi},$$

$$\mathbf{J}_{5\mu} = -\bar{\xi}\gamma_\mu\gamma_5\frac{\boldsymbol{\tau}}{2}\xi + \frac{\partial_\mu\boldsymbol{\pi}}{2f}\left(\frac{\sin(2f\sqrt{\pi^2})\cos(2f\sqrt{\pi^2})}{2f\sqrt{\pi^2}}\right) + \frac{f^2\boldsymbol{\pi}(\partial_\mu\pi^2)}{2f^2\pi^2}\left(1 - \frac{\sin(2f\sqrt{\pi^2})\cos(2f\sqrt{\pi^2})}{2f\sqrt{\pi^2}}\right). \tag{4.1a}$$

Since

$$\mathbf{U}^{1/2}=e^{if\gamma_5\tau\cdot\pi},$$

$$\mathbf{U}^{1/2}\partial_\mu\mathbf{U}^{-1/2}=i\boldsymbol{\tau}\cdot(\boldsymbol{\pi}\times\partial_\mu\boldsymbol{\pi})\frac{\sin^2 f\sqrt{\pi^2}}{\pi^2}$$

$$-i\gamma_5\left\{\frac{\boldsymbol{\tau}\cdot\partial_\mu\boldsymbol{\pi}}{\sqrt{\pi^2}}\sin(f\sqrt{\pi^2})\cos(f\sqrt{\pi^2})\right.$$

$$\left.+\frac{(\boldsymbol{\tau}\cdot\boldsymbol{\pi})(\boldsymbol{\pi}\cdot\partial_\mu\boldsymbol{\pi})}{\pi^2}f\left(1-\frac{\sin(f\sqrt{\pi^2})\cos(f\sqrt{\pi^2})}{f\sqrt{\pi^2}}\right)\right\},$$

and

$$-\frac{1}{16f^2}\operatorname{tr}(\partial_\mu\mathbf{U}\,\partial_\mu\mathbf{U}^\dagger)=-\tfrac{1}{2}\partial_\mu\boldsymbol{\pi}\cdot\partial_\mu\boldsymbol{\pi}\left(\frac{\sin^2(2f\sqrt{\pi^2})}{4f^2\pi^2}\right)$$

$$-\frac{(\boldsymbol{\pi}\cdot\partial_\mu\boldsymbol{\pi})(\boldsymbol{\pi}\cdot\partial_\mu\boldsymbol{\pi})}{2\pi^2}\left(1-\frac{\sin^2(2f\sqrt{\pi^2})}{4f^2\pi^2}\right).$$

The interaction Lagrangians for this model are

$$\mathcal{L}_{\text{int}}(\xi\text{-}\pi)=-\bar{\xi}\gamma_\mu(\mathbf{U}^{1/2}\partial_\mu\mathbf{U}^{-1/2})\xi$$

$$=-\bar{\xi}\gamma_\mu\left\{i\boldsymbol{\tau}\cdot(\boldsymbol{\pi}\times\partial_\mu\boldsymbol{\pi})\frac{\sin^2(f\sqrt{\pi^2})}{\pi^2}\right.$$

$$-i\gamma_5\left[\frac{\boldsymbol{\tau}\cdot\partial_\mu\boldsymbol{\pi}}{\sqrt{\pi^2}}\sin(f\sqrt{\pi^2})\cos(f\sqrt{\pi^2})\right.$$

$$\left.\left.+\frac{(\boldsymbol{\tau}\cdot\boldsymbol{\pi})(\boldsymbol{\pi}\cdot\partial_\mu\boldsymbol{\pi})}{\pi^2}f\left(1-\frac{\sin(f\sqrt{\pi^2})\cos(f\sqrt{\pi^2})}{f\sqrt{\pi^2}}\right)\right]\right\}\xi, \tag{4.1b}$$

and

$$\mathcal{L}_{\text{int}}(\pi\text{-}\pi)=-\tfrac{1}{2}\partial_\mu\boldsymbol{\pi}\cdot\partial_\mu\boldsymbol{\pi}\left(\frac{\sin^2(2f\sqrt{\pi^2})}{4f^2\pi^2}-1\right)$$

$$-\frac{(\boldsymbol{\pi}\cdot\partial_\mu\boldsymbol{\pi})(\boldsymbol{\pi}\cdot\partial_\mu\boldsymbol{\pi})}{2\pi^2}\left(1-\frac{\sin^2(2f\sqrt{\pi^2})}{4f^2\pi^2}\right). \tag{4.1c}$$

The expression of (4.1a)–(4.1c) in the power series of f are

$$\mathbf{J}_{5\mu}=-\bar{\xi}\gamma_\mu\gamma_5\frac{\boldsymbol{\tau}}{2}\xi+\frac{1}{2f}\partial_\mu\boldsymbol{\pi}+O(f), \tag{4.2a}$$

and

$$\mathcal{L}_{\text{int}}(\xi\text{-}\pi)=\bar{\xi}\gamma_\mu[if\gamma_5\boldsymbol{\tau}\cdot\partial_\mu\boldsymbol{\pi}-if^2\boldsymbol{\tau}\cdot(\boldsymbol{\pi}\times\partial_\mu\boldsymbol{\pi})]\xi+O(f^3). \tag{4.2b}$$

We also find[14]

$$\mathcal{L}_{\text{int}}(\pi\text{-}\pi)=f^2[(\boldsymbol{\pi}\cdot\boldsymbol{\pi})(\partial_\mu\boldsymbol{\pi}\cdot\partial_\mu\boldsymbol{\pi})+\tfrac{1}{3}\mu^2(\boldsymbol{\pi}\cdot\boldsymbol{\pi})^2]+O(f^4). \tag{4.2c}$$

B. Second Model

$$\mathbf{U}=\frac{1+if\gamma_5\boldsymbol{\tau}\cdot\boldsymbol{\pi}}{1-if\gamma_5\boldsymbol{\tau}\cdot\boldsymbol{\pi}}=\frac{1-f^2\pi^2}{1+f^2\pi^2}+2if\gamma_5\frac{\boldsymbol{\tau}\cdot\boldsymbol{\pi}}{1+f^2\pi^2},$$

$$\sigma=\frac{1-f^2\pi^2}{1+f^2\pi^2},\quad \rho=\frac{1}{1+f^2\pi^2}.$$

The symmetry-breaking term for this model is then

$$\mathcal{L}_{\text{S.B.}}=\frac{\mu^2}{2f}\int_0^{f^2\pi^2}\frac{\alpha^{1/2}\sigma'(\alpha)}{[1-\sigma^2(\sigma)]^{1/2}}d\alpha$$

$$=-\frac{\mu^2}{2f^2}\int_0^{f^2\pi^2}\frac{d\alpha}{1+\alpha}=-\frac{\mu^2}{2f^2}\ln(1+f^2\pi^2).$$

The axial-vector current is (see Appendix D)

$$\mathbf{J}_{5\mu}=-\bar{\xi}\gamma_\mu\gamma_5\frac{\boldsymbol{\tau}}{2}\xi+\frac{\partial_\mu\boldsymbol{\pi}}{2f}\frac{1-f^2\pi^2}{(1+f^2\pi^2)^2}+\tfrac{1}{2}f\boldsymbol{\pi}\frac{(\partial_\mu\pi^2)}{(1+f^2\pi^2)^2}. \tag{4.3a}$$

Since

$$\mathbf{U}^{\pm1/2}=\frac{1\pm if\gamma_5\boldsymbol{\tau}\cdot\boldsymbol{\pi}}{(1+f^2\pi^2)^{1/2}},$$

it is easily checked that

$$\mathbf{U}^{1/2}\partial_\mu\mathbf{U}^{-1/2}=(1+f^2\pi^2)^{-1}\times[-if\gamma_5\boldsymbol{\tau}\cdot\partial_\mu\boldsymbol{\pi}+if^2\boldsymbol{\tau}\cdot(\boldsymbol{\pi}\times\partial_\mu\boldsymbol{\pi})],$$

and also

$$-\frac{1}{16f^2}\operatorname{tr}(\partial_\mu\mathbf{U}\,\partial_\mu\mathbf{U}^\dagger)=-\frac{1}{2}\frac{\partial_\mu\boldsymbol{\pi}\cdot\partial_\mu\boldsymbol{\pi}}{(1+f^2\pi^2)^2}.$$

Hence the Lagrangian in Eq. (2.7) is

$$\mathcal{L}=-\bar{\xi}(\gamma_\mu\partial_\mu+m)\xi+\bar{\xi}\gamma_\mu\{[if\gamma_5\boldsymbol{\tau}\cdot\partial_\mu\boldsymbol{\pi}-if^2\boldsymbol{\tau}\cdot(\boldsymbol{\pi}\times\partial_\mu\boldsymbol{\pi})](1+f^2\pi^2)^{-1}\}\xi$$

$$-\tfrac{1}{2}\partial_\mu\boldsymbol{\pi}\cdot\partial_\mu\boldsymbol{\pi}(1+f^2\pi^2)^{-2}-\frac{\mu^2}{2f^2}\ln(1+f^2\pi^2).$$

One can recognize immediately that the Lagrangian is the one proposed by Schwinger with the identification $f=f_0/m_\pi$.

The two parts of the interaction Lagrangian are

$$\mathcal{L}_{\text{int}}(\xi\text{-}\pi)=\bar{\xi}\gamma_\mu\{if\gamma_5\boldsymbol{\tau}\cdot\partial_\mu\boldsymbol{\pi}-if^2\boldsymbol{\tau}\cdot(\boldsymbol{\pi}\times\partial_\mu\boldsymbol{\pi})\}\times(1+f^2\pi^2)^{-1}\xi, \tag{4.3b}$$

$$\mathcal{L}_{\text{int}}(\pi\text{-}\pi)=-\tfrac{1}{2}\partial_\mu\boldsymbol{\pi}\cdot\partial_\mu\boldsymbol{\pi}[(1+f^2\pi^2)^{-2}-1]+\tfrac{1}{2}\mu^2[\boldsymbol{\pi}\cdot\boldsymbol{\pi}-(1/f^2)\ln(1+f^2\pi^2)]. \tag{4.3c}$$

[14] By adding the divergence $\frac{1}{6}\partial_\mu(\boldsymbol{\pi}\cdot\boldsymbol{\pi}\partial_\mu(\boldsymbol{\pi}\cdot\boldsymbol{\pi}))$ to the expansion (4.2c), and using the equation of motion, one finds

$$\mathcal{L}_{\text{int}}(\pi\text{-}\pi)=\tfrac{1}{6}\partial_\mu(\boldsymbol{\pi}\cdot\boldsymbol{\pi}\partial_\mu(\boldsymbol{\pi}\cdot\boldsymbol{\pi}))+\tfrac{2}{3}f^2[(\boldsymbol{\pi}\cdot\boldsymbol{\pi})(\partial_\mu\boldsymbol{\pi}\cdot\partial_\mu\boldsymbol{\pi})-(\boldsymbol{\pi}\cdot\partial_\mu\boldsymbol{\pi})(\boldsymbol{\pi}\cdot\partial\boldsymbol{\pi})]+O(f^4)=f^2((\boldsymbol{\pi}\cdot\boldsymbol{\pi})(\partial_\mu\boldsymbol{\pi}\cdot\partial_\mu\boldsymbol{\pi})+\tfrac{1}{3}\mu^2(\boldsymbol{\pi}\cdot\boldsymbol{\pi})^2)+O(f^4).$$

The expansion of (4.3a)–(4.3c) becomes

$$\mathbf{J}_{5\mu}=-\bar{\xi}\gamma_\mu\gamma_5\frac{\boldsymbol{\tau}}{2}\xi+\frac{1}{2f}\partial_\mu\boldsymbol{\pi}+O(f)\,, \tag{4.4a}$$

$$\mathcal{L}_{\rm int}(\xi\text{-}\pi)=\bar{\xi}\gamma_\mu[if\gamma_5\boldsymbol{\tau}\cdot\partial_\mu\boldsymbol{\pi}-if^2\boldsymbol{\tau}\cdot(\boldsymbol{\pi}\times\partial_\mu\boldsymbol{\pi})]\xi+O(f^3)\,, \tag{4.4b}$$

$$\mathcal{L}_{\rm int}(\pi\text{-}\pi)=f^2[(\boldsymbol{\pi}\cdot\boldsymbol{\pi})(\partial_\mu\boldsymbol{\pi}\cdot\partial_\mu\boldsymbol{\pi})+\tfrac{1}{4}\mu^2(\boldsymbol{\pi}\cdot\boldsymbol{\pi})^2]+O(f^4)\,. \tag{4.4c}$$

C. Third Model

$$\mathbf{U}=(1-4f^2\pi^2)^{1/2}+2if\gamma_5\boldsymbol{\tau}\cdot\boldsymbol{\pi}.$$

This model has been studied by Brown extensively.[7] Instead of expressing the Lagrangian in terms of meson fields $\boldsymbol{\varphi}$ and the corresponding canonical conjugate field $\boldsymbol{\pi}_\mu$ as in his paper, we still write the Lagrangian in the general form given in Eq. (2.7). The results are in agreement with Brown's in the second-order expansion. In this model we have

$$\sigma(\alpha)=(1-4\alpha)^{1/2},\quad \rho(\alpha)=1$$

hence,

$$\mathcal{L}_{\rm S.B.}=\frac{\mu^2}{4f^2}[(1-4f^2\pi^2)^{1/2}-1].$$

The axial-vector current is calculated to be

$$\mathbf{J}_{5\mu}=-\bar{\xi}\gamma_\mu\gamma_5\frac{\boldsymbol{\tau}}{2}\xi+\frac{\partial_\mu\boldsymbol{\pi}}{f}(1-4f^2\pi^2)^{1/2}+f\boldsymbol{\pi}\frac{(\partial_\mu\pi^2)}{(1-4f^2\pi^2)^{1/2}}. \tag{4.5a}$$

Since

$$\mathbf{U}^{\pm1/2}=\frac{1}{\sqrt{2}}\left\{[1+(1-4f^2\pi^2)^{1/2}]^{1/2}\pm\frac{2if\gamma_5\boldsymbol{\tau}\cdot\boldsymbol{\pi}}{[1+(1-4f^2\pi^2)^{1/2}]^{1/2}}\right\},$$

we have

$$\mathbf{U}^{1/2}\partial_\mu\mathbf{U}^{-1/2}=-if\gamma_5\boldsymbol{\tau}\cdot\partial_\mu\boldsymbol{\pi}+if^2\boldsymbol{\tau}\cdot(\boldsymbol{\pi}\times\partial_\mu\boldsymbol{\pi})\frac{1}{2f^2\pi^2}[1-(1-4f^2\pi^2)^{1/2}]-4if^3\gamma_5\frac{(\boldsymbol{\tau}\cdot\boldsymbol{\pi})(\boldsymbol{\pi}\cdot\partial_\mu\boldsymbol{\pi})}{[1+(1-4f^2\pi^2)^{1/2}](1-4f^2\pi^2)^{1/2}},$$

and also

$$-(1/16f^2)\,\mathrm{tr}(\partial_\mu\mathbf{U}\partial_\mu\mathbf{U}^\dagger)=-\tfrac{1}{2}\partial_\mu\boldsymbol{\pi}\cdot\partial_\mu\boldsymbol{\pi}-2f^2(\boldsymbol{\pi}\cdot\partial_\mu\boldsymbol{\pi})(\boldsymbol{\pi}\cdot\partial_\mu\boldsymbol{\pi})(1-4f^2\pi^2).$$

Therefore the interaction Lagrangians are

$$\mathcal{L}_{\rm int}(\xi\text{-}\pi)=\bar{\xi}\gamma_\mu\left\{if\gamma_5\boldsymbol{\tau}\cdot\partial_\mu\boldsymbol{\pi}-if^2\boldsymbol{\tau}\cdot(\boldsymbol{\pi}\times\partial_\mu\boldsymbol{\pi})\times\frac{1}{2f^2\pi^2}[1-(1-4f^2\pi^2)^{1/2}]-4if^2\gamma_5\frac{(\boldsymbol{\tau}\cdot\boldsymbol{\pi})(\boldsymbol{\pi}\cdot\partial_\mu\boldsymbol{\pi})}{[\frac{1}{2}+(1-4f^2\pi^2)^{1/2}](1-4f^2\pi^2)^{1/2}}\right\}\xi, \tag{4.5b}$$

$$\mathcal{L}_{\rm int}(\pi\text{-}\pi)=-2f^2(\boldsymbol{\pi}\cdot\partial_\mu\boldsymbol{\pi})(\boldsymbol{\pi}\cdot\partial_\mu\boldsymbol{\pi})(1-4f^2\pi^2)+\tfrac{1}{2}\mu^2\left[\pi\cdot\pi+\frac{(1-4f^2\pi^2)^{1/2}-1}{2f^2}\right]. \tag{4.5c}$$

The expansions of (4.5a)–(4.5c) are given in the following:

$$\mathbf{J}_{5\mu}=-\bar{\xi}\gamma_\mu\gamma_5\frac{\boldsymbol{\tau}}{2}\xi+\frac{1}{2f}\partial_\mu\boldsymbol{\pi}+O(f)\,, \tag{4.6a}$$

$$\mathcal{L}_{\rm int}(\xi\text{-}\pi)=\bar{\xi}\gamma_\mu\{if\gamma_5\boldsymbol{\tau}\cdot\partial_\mu\boldsymbol{\pi}-if^2\boldsymbol{\tau}\cdot(\boldsymbol{\pi}\times\partial_\mu\boldsymbol{\pi})\}\xi+O(f^3)\,, \tag{4.6b}$$

$$\mathcal{L}_{\rm int}(\pi\text{-}\pi)=f^2((\boldsymbol{\pi}\cdot\boldsymbol{\pi})(\partial_\mu\boldsymbol{\pi}\cdot\partial_\mu\boldsymbol{\pi})+\tfrac{1}{2}\mu^2(\boldsymbol{\pi}\cdot\boldsymbol{\pi}))+O(f^4)\,. \tag{4.6c}$$

By taking the divergence of the axial-vector current $\mathbf{J}_{5\mu}$ in (4.2a) and making use of equation of motion, the PCAC assumption will follow immediately as expected. An important point to note is that we add the PCAC term before the canonical transformation. This accounts for the absence of π-N cross terms in the expression of the axial-vector current that appears in Schwinger's model.[6]

5. TRANSITION TO EFFECTIVE LAGRANGIANS

The effective Lagrangian, from which the pion scattering lengths are calculated, can be obtained by replacing the bare masses and coupling constants appearing in the Lagrangian models we have, by the renormalized masses and coupling constants, and by treating the fields as renormalized operators if we regard our general Lagrangian as the limit of a renormalizable theory like the σ model when the σ meson is eliminated. Because the chiral $SU(2)\otimes SU(2)$ symmetry is broken, the axial-vector coupling constant will differ from the vector coupling constant. Further, the coupling constant associated with the axial-vector current will be renormalized while the vector-current coupling constant will not, since the isospin $SU(2)$ symmetry is assumed to be exact. Conventionally, we denote by g_A the axial-vector renormalization constant. The bare coupling constant f in the interaction Lagrangians will be then changed into g_Af for the axial-vector current part. We also leave the vector-current part unchanged. Then, the effective pion-nucleon interaction Lagrangian becomes

$$\mathcal{L}_{\text{int}}^{(\text{eff})}(\xi\text{-}\pi)=\bar{\xi}\gamma_\mu[ig_A f\boldsymbol{\tau}\cdot\partial_\mu\boldsymbol{\pi} -if^2\boldsymbol{\tau}\cdot(\boldsymbol{\pi}\times\partial_\mu\boldsymbol{\pi})]\xi+O(f^3) \quad (5.1)$$

for all three models. The effective pion-pion interaction Lagrangians are

$$\mathcal{L}_{\text{int}}^{(\text{eff})}(\pi\text{-}\pi)=f^2[(\boldsymbol{\pi}\cdot\boldsymbol{\pi})\partial_\mu\boldsymbol{\pi}\cdot\partial_\mu\boldsymbol{\pi} +\tfrac{1}{3}m_\pi^2(\boldsymbol{\pi}\cdot\boldsymbol{\pi})^2]+O(f^4) \quad (5.2a)$$

for the exponential model,

$$\mathcal{L}_{\text{int}}^{(\text{eff})}(\pi\text{-}\pi)=f^2[(\boldsymbol{\pi}\cdot\boldsymbol{\pi})\partial_\mu\boldsymbol{\pi}\cdot\partial_\mu\boldsymbol{\pi} +\tfrac{1}{4}m_\pi^2(\boldsymbol{\pi}\cdot\boldsymbol{\pi})^2]+O(f^4) \quad (5.2b)$$

for the model $\mathbf{U}=(1+if\gamma_5\boldsymbol{\tau}\cdot\boldsymbol{\pi})/(1-if\gamma_5\boldsymbol{\tau}\cdot\boldsymbol{\pi})$, and

$$\mathcal{L}_{\text{int}}^{(\text{eff})}(\pi\text{-}\pi)=f^2[(\boldsymbol{\pi}\cdot\boldsymbol{\pi})\partial_\mu\boldsymbol{\pi}\cdot\partial_\mu\boldsymbol{\pi} +\tfrac{1}{2}m_\pi^2(\boldsymbol{\pi}\cdot\boldsymbol{\pi})^2]+O(f^4) \quad (5.2c)$$

for the model $\mathbf{U}=(1-4f^2\pi^2)^{1/2}+2if\gamma_5\boldsymbol{\tau}\cdot\boldsymbol{\pi}$.

If we compare the interaction with the one obtained by Weinberg from the σ model, we can notice immediately that pion-nucleon interaction Lagrangians are all equivalent to each other up to the second-order expansion by the identification of f with $-(G/2m_N)(g_V/g_A)$.

The nonderivative pion-pion interaction Lagrangians of the first (the exponential model) and the second model differ from the third model which is equivalent to Weinberg's by factors of $\frac{2}{3}$ and $\frac{1}{2}$, respectively. On the other hand all three models have the same derivative pion-pion interaction term in agreement with Weinberg's model.

Let us consider the general pion-pion interaction for the three models in the following form:

$$\mathcal{L}_{\text{int}}^{(\text{eff})}(\pi\text{-}\pi)=\lambda'(\boldsymbol{\pi}\cdot\boldsymbol{\pi})(\partial_\mu\boldsymbol{\pi}\cdot\partial_\mu\boldsymbol{\pi})+\lambda\mu^2(\boldsymbol{\pi}\cdot\boldsymbol{\pi})^2 \quad (\mu=m_\pi).$$

It can be shown that to first order in λ and λ', the transition matrix element T is

$$T(ab:cd)=16\mu^2[\delta_{ab}\delta_{cd}(\tfrac{1}{2}(\lambda-\lambda')+(t/4\mu^2)\lambda')+\delta_{ad}\delta_{bc} \times(\tfrac{1}{2}(\lambda-\lambda')+(u/4\mu^2)\lambda') +\delta_{ac}\delta_{bd}(\tfrac{1}{2}(\lambda-\lambda')+(s/4\mu^2)\lambda')]. \quad (5.3).$$

T is related to the S matrix by

$$S_{fi}-\delta_{fi}=i(2\pi)^4\delta(p_f-p_i)T_{fi},$$

and

$$s=-(k_a+k_c)^2,$$
$$t=-(k_a-k_b)^2,$$
$$u=-(k_a-k_d)^2,$$

where k_a is the momentum of the pion with isotopic index a. The scattering lengths are then given by

$$a_0=(\mu/4\pi)(5\lambda+\lambda'), \quad (5.4a)$$

$$a_2=(\mu/4\pi)(2\lambda-2\lambda'). \quad (5.4b)$$

The ratio a_2/a_0 for the three models is found to have the values $-\frac{1}{2}, -\frac{2}{3}$, and $-2/7$ for the exponential model, second, and third models, respectively. The second number agrees with Schwinger's while the third is the same as Weinberg's.

The numerical value of a_0 is

$$a_0=\frac{8}{3}\frac{f^2}{4}\mu=0.15\mu^{-1} \text{ for the exponential model,}$$

$$a_0=\frac{9}{4}\frac{f^2}{4}\mu=0.13\mu^{-1} \text{ for the second model,}$$

and

$$a_0=\frac{7}{2}\frac{f^2}{4}\mu=0.20\mu^{-1} \text{ for the third model.}$$

The slight differences in the pion-pion scattering lengths for the three cases do not invalidate the usefulness of our unified Lagrangian model because the values quoted are still compatible with the experimental results obtained from K_{e4} decay.[3]

ACKNOWLEDGMENT

The authors would like to thank Professor L. S. Brown for many stimulating discussions.

APPENDIX A

$\mathbf{U}(if\gamma_5\Phi)$ is a $2\otimes4$ direct product matrix with argument $\gamma_5\Phi=\gamma_5\boldsymbol{\tau}\cdot\boldsymbol{\varphi}$, where the 2-dimensional matrix of the direct product shuffles the two different nucleon fields among each other, while the 4-dimensional part operates on the spinor components of each nucleon field alone. The introduction of i in the argument of the functional matrix $\mathbf{U}(if\gamma_5\Phi)$ makes f a real number, namely the coupling constant. The matrix $\mathbf{U}(if\gamma_5\Phi)$ satisfies the following conditions:

(a) $\mathbf{U}=\mathbf{U}(if\gamma_5\Phi)$, $\mathbf{U}(0)=1$,

(b) $\mathbf{U}\mathbf{U}^\dagger=\mathbf{U}^\dagger\mathbf{U}=1$, unitarity condition

(c) $\mathbf{U}^\dagger=\mathbf{U}(-if\gamma_5\Phi)$, reality condition.

From these three conditions, $\mathbf{U}$ can be expressed generally as

$$\mathbf{U}(if\gamma_5\Phi)=\sigma(f^2\varphi^2)+2if\gamma_5\rho(f^2\varphi^2)\Phi,$$

with

$$\sigma(0)=1,$$
$$\sigma^2(f^2\varphi^2)+4f^2\varphi^2\rho(f^2\varphi^2)=1. \quad (A.1)$$

The determinant of $\mathbf{U}$ is also unity, as a consequence of the properties of the direct product and the diagonality of the matrix γ_5 in the Weyl representation,

$$\det\mathbf{U}=(\sigma^2+4f^2\varphi^2\rho^2)^4=1.$$

APPENDIX B

Let $\psi_L=\frac{1}{2}(1+\gamma_5)\psi$ and $\psi_R=\frac{1}{2}(1-\gamma_5)\psi$ be associated with the irreducible representations of homogeneous Lorentz group $(\frac{1}{2},0)$ and $(0,\frac{1}{2})$, respectively.[15] The transformations under the two commuting $SU(2)$ groups G_L and G_R for the nucleon and the pion fields are defined as

G_L:

$$\psi_L\rightarrow\psi_L'=e^{\frac{1}{2}i\tau\cdot\mu}\psi_L\,,$$
$$\psi_R\rightarrow\psi_R'=\psi_R\,,$$

which can be combined into

$$\psi\rightarrow\psi'=e^{\frac{1}{2}i(1+\gamma_5)(\tau/2)\mu}\psi\,,$$

and for the pion field we have

$$\mathbf{U}\rightarrow\mathbf{U}'=e^{\frac{1}{2}i(1-\gamma_5)(\tau/2)\cdot\mu}\mathbf{U}\,e^{-\frac{1}{2}i(1+\gamma_5)(\tau/2)\cdot\mu}.$$

G_R:

$$\psi_L\rightarrow\psi_L'=\psi_L\,,$$
$$\psi_R\rightarrow\psi_R'=e^{\frac{1}{2}i\tau\cdot\nu}\psi_R\,,$$

which can be written, in the compact form

$$\psi\rightarrow\psi'=e^{\frac{1}{2}i(1-\gamma_5)(\tau/2)\cdot\nu}\psi\,,$$

and

$$\mathbf{U}\rightarrow\mathbf{U}'=e^{\frac{1}{2}i(1+\gamma_5)(\tau/2)\cdot\nu}\mathbf{U}\,e^{-\frac{1}{2}i(1-\gamma_5)(\tau/2)\cdot\nu}.$$

The combined $SU(2)\otimes SU(2)$ transformation G^4 is then

$G^{(4)}$:

$$\psi\rightarrow e^{\frac{1}{2}i(1+\gamma_5)(\tau/2)\cdot\mu+\frac{1}{2}i(1-\gamma_5)(\tau/2)\cdot\nu}\psi\,,$$
$$\mathbf{U}\rightarrow e^{\frac{1}{2}i(1-\gamma_5)(\tau/2)\cdot\mu+\frac{1}{2}i(1+\gamma_5)(\tau/2)\cdot\nu}\mathbf{U}\times e^{-\frac{1}{2}i(1+\gamma_5)(\tau/2)\cdot\mu-\frac{1}{2}i(1-\gamma_5)(\tau/2)\cdot\nu}.$$

We are interested in particular cases

$$\boldsymbol{\mu}=\boldsymbol{\nu}=\boldsymbol{\omega}\quad\text{and}\quad\boldsymbol{\mu}=-\boldsymbol{\nu}=\mathbf{a}.$$

In the first case we have

(a)

$$\psi\rightarrow e^{\frac{1}{2}i\tau\cdot\omega}\psi\,,$$
$$\mathbf{U}\rightarrow e^{\frac{1}{2}i\tau\cdot\omega}\mathbf{U}\,e^{-\frac{1}{2}i\tau\cdot\omega},$$

which is the ordinary isospin transformation. In the second case $\boldsymbol{\mu}=-\boldsymbol{\nu}=\mathbf{a}$,

(b)

$$\psi\rightarrow e^{i\gamma_5(\tau/2)\cdot\mathbf{a}}\psi\,,$$
$$\mathbf{U}\rightarrow e^{-i\gamma_5(\tau/2)\cdot\mathbf{a}}\mathbf{U}\,e^{-i\gamma_5(\tau/2)\cdot\mathbf{a}},$$

which is the chiral transformation that is of special interest to us in this paper.

[15] They are in fact, the $(\frac{1}{2},0)+(\frac{1}{2},0)$ and $(0,\frac{1}{2})+(0,\frac{1}{2})$ representation because we consider the 8-component spinors of the nucleon fields.

APPENDIX C

Consider the infinitesimal chiral transformation defined in Appendix B:

$$\psi\rightarrow\psi'=\psi+\delta\psi=(1+\tfrac{1}{2}i\lambda_5\boldsymbol{\tau}\cdot\mathbf{a})\psi\,,\tag{C.1}$$
$$\mathbf{U}\rightarrow\mathbf{U}'=\mathbf{U}+\delta\mathbf{U}=\mathbf{U}-\tfrac{1}{2}i\{\gamma_5\boldsymbol{\tau}\cdot\mathbf{a},\mathbf{U}\}\,,\tag{C.2}$$

where

$$\delta\psi=\tfrac{1}{2}i\gamma_5\boldsymbol{\tau}\cdot\mathbf{a}\psi\,,$$
$$\delta\mathbf{U}=-\tfrac{1}{2}i\{\gamma_5\boldsymbol{\tau}\cdot\mathbf{a},\mathbf{U}\}\,.$$

The construction of the axial-vector current from the variation of the Lagrangian under the infinitesimal transformation will be carried out by regarding the components of $\mathbf{a}$ to be arbitrary functions of the space-time coordinates. Therefore the variation of the Lagrangian

$$\mathcal{L}=-\bar{\psi}(\gamma_\mu\partial_\mu+m\mathbf{U})\psi-(1/16f^2)\,\mathrm{tr}(\partial_\mu\mathbf{U}\partial_\mu\mathbf{U}^\dagger)+\mathcal{L}_{\text{S.B.}}$$

gives

$$\delta\mathcal{L}=-\bar{\psi}\gamma_\mu\gamma_5\frac{\boldsymbol{\tau}}{2}\psi\cdot\partial_\mu\mathbf{a}-\frac{1}{16f^2}\times\mathrm{tr}[(\partial_\mu\mathbf{U})(\partial_\mu\delta\mathbf{U}^\dagger)+(\partial_\mu\delta\mathbf{U})(\partial_\mu\mathbf{U}^\dagger)]+\delta\mathcal{L}_{\text{S.B.}}.$$

Using (C.2), it is easily checked that

$$\mathrm{tr}[(\partial_\mu\mathbf{U})(\partial_\mu\delta\mathbf{U}^\dagger)+(\partial_\mu\delta\mathbf{U})(\partial_\mu\mathbf{U}^\dagger)]=2i\,\mathrm{tr}\{[(\partial_\mu\mathbf{U})\mathbf{U}^\dagger+\mathbf{U}^\dagger(\partial_\mu\mathbf{U})]\gamma_5\tau/2\}\cdot\partial_\mu\mathbf{a}.$$

Hence we have

$$\delta\mathcal{L}=-\left(\bar{\psi}\gamma_\mu\gamma_5\frac{\boldsymbol{\tau}}{2}\psi+\frac{i}{8f^2}\,\mathrm{tr}\left\{[(\partial_\mu\mathbf{U})\mathbf{U}^\dagger+\mathbf{U}^\dagger(\partial_\mu\mathbf{U})]\gamma_5\frac{\boldsymbol{\tau}}{2}\right\}\right)\cdot\partial_\mu\mathbf{a}+\delta\mathcal{L}_{\text{S.B.}}=\mathbf{J}_{5\mu}\cdot\partial_\mu\mathbf{a}+\delta\mathcal{L}_{\text{S.B.}}\,,\tag{C.3}$$

where

$$\mathbf{J}_{5\mu}=-\bar{\psi}\gamma_\mu\gamma_5\frac{\boldsymbol{\tau}}{2}\psi-\frac{i}{8f^2}\,\mathrm{tr}\left\{[(\partial_\mu\mathbf{U})\mathbf{U}^\dagger+\mathbf{U}^\dagger(\partial_\mu\mathbf{U})]\gamma_5\frac{\boldsymbol{\tau}}{2}\right\}$$

is the axial-vector current.

Let us write (C.3) in the form

$$\delta\mathcal{L}=\mathbf{J}_{5\mu}\cdot\partial_\mu\mathbf{a}+\delta\mathcal{L}_{\text{S.B.}}=\partial_\mu(\mathbf{J}_{5\mu}\cdot\mathbf{a})-(\partial_\mu\mathbf{J}_{5\mu})\cdot\mathbf{a}+\delta\mathcal{L}_{\text{S.B.}}.\tag{C.4}$$

If the chiral symmetry-breaking term does not appear in the Lagrangian, then $\delta\mathcal{L}_{\text{S.B.}}=0$ and $\delta\mathcal{L}=0$ require that $\partial_\mu\mathbf{J}_{5\mu}=0$, because $\mathbf{a}$ is completely arbitrary, and we have a conserved axial-vector current. Now we introduce $\mathcal{L}_{\text{S.B.}}$ so as to agree with the PCAC assumption,

$$\partial_\mu\mathbf{J}_{5\mu}=\frac{\mu^2}{2f}\boldsymbol{\varphi}.$$

Then (C.4) becomes

$$\delta\mathcal{L}=\partial_\mu(\mathbf{J}_{5\mu}\cdot\mathbf{a})-\frac{\mu^2}{2f}\boldsymbol{\varphi}\cdot\mathbf{a}+\delta\mathcal{L}_{\text{S.B.}}\,,\tag{C.4'}$$

and the equations of motion require

$$\delta\mathcal{L}_{\mathrm{S.B.}}=\frac{\mu^2}{2f}\boldsymbol{\varphi}\cdot\mathbf{a}. \qquad (C.5)$$

Because of the conservation of the vector current, the solution has to be invariant under isospin $SU(2)$ transformations, which allows us to assume $\mathcal{L}_{\mathrm{S.B.}}$ to be a function of $\boldsymbol{\varphi}\cdot\boldsymbol{\varphi}$ only.[16] Let $\alpha=f^2\varphi^2$; then (C.5) reduces to

$$\frac{d\mathcal{L}_{\mathrm{S.B.}}}{d\alpha}2f^2\boldsymbol{\varphi}\cdot\delta\boldsymbol{\varphi}=\frac{\mu^2}{2f}\boldsymbol{\varphi}\cdot\mathbf{a}$$

or

$$\frac{d\mathcal{L}_{\mathrm{S.B.}}}{d\alpha}=\frac{\mu^2}{4f^3}\frac{\boldsymbol{\varphi}\cdot\mathbf{a}}{\boldsymbol{\varphi}\cdot\delta\boldsymbol{\varphi}}. \qquad (C.6)$$

The solution can be obtained easily if we know the relation between $\mathbf{a}$ and $\delta\boldsymbol{\varphi}$.

Let us now write

$$\mathsf{U}=\sigma(\alpha)+2if\gamma_5\rho(\alpha)\Phi.$$

(C.2) then can be expressed as

$$\delta\mathsf{U}=-\tfrac{1}{2}i\{\gamma_5\boldsymbol{\tau}\cdot\mathbf{a},\mathsf{U}\}=-i\gamma_5\sigma\boldsymbol{\tau}\cdot\mathbf{a}+2f\rho\boldsymbol{\varphi}\cdot\mathbf{a}. \qquad (C.7a)$$

On the other hand, we have

$$\delta\mathsf{U}=2f^2\sigma'\boldsymbol{\varphi}\cdot\delta\boldsymbol{\varphi}+2if\gamma_5 \times\{\rho\boldsymbol{\tau}\cdot\delta\boldsymbol{\varphi}+2f^2\rho'(\boldsymbol{\tau}\cdot\boldsymbol{\varphi})(\boldsymbol{\varphi}\cdot\delta\boldsymbol{\varphi})\}. \qquad (C.7b)$$

By equating (C.6) and (C.7) we obtain the following relations:

$$\rho\mathbf{a}=f\sigma'\delta\boldsymbol{\varphi}, \qquad (C.8a)$$

$$\sigma\mathbf{a}=-2f\rho\delta\boldsymbol{\varphi}-4f^3\rho'\boldsymbol{\varphi}(\boldsymbol{\varphi}\cdot\delta\boldsymbol{\varphi}). \qquad (C.8b)$$

Combining (C.6) and (C.8a), we have

$$\frac{d\mathcal{L}_{\mathrm{S.B.}}}{d\alpha}=\frac{\mu^2}{4f^3}\frac{f\sigma'}{\varphi}=\frac{\mu^2}{4f^2}\frac{2(\alpha)^{1/2}\sigma'(\alpha)}{[1-\sigma^2(\alpha)]^{1/2}},$$

$$\mathcal{L}_{\mathrm{S.B.}}=\frac{\mu^2}{2f^2}\int_0^{f^2\varphi^2}\frac{(\alpha)^{1/2}\sigma'(\alpha)}{[1-\sigma^2(\alpha)]^{1/2}}d\alpha. \qquad (C.9)$$

The lower limit of the integration zero is chosen such that the Lagrangian will reduce to the free Lagrangian of nucleons and pions as the coupling constant f goes to zero. The symmetry-breaking term $\mathcal{L}_{\mathrm{S.B.}}$ given by (C.9) becomes the pion-mass term $-\frac{1}{2}\mu^2\boldsymbol{\varphi}\cdot\boldsymbol{\varphi}$ as $f\to 0$ for all three models.

APPENDIX D

This appendix is devoted to the construction of the vector current $\mathbf{J}_\mu$, and the axial-vector current $\mathbf{J}_{5\mu}$.

A. Construction of J_μ

Consider the infinitesimal transformation of $SU(2)$ given in Appendix B:

$$\psi\to\psi'=\psi+\delta\psi=(1+\tfrac{1}{2}i\boldsymbol{\tau}\cdot\boldsymbol{\omega})\psi,$$

$$\mathsf{U}\to\mathsf{U}'=\mathsf{U}+\delta\mathsf{U}=\mathsf{U}+\tfrac{1}{2}i[\tau\cdot\omega,\mathsf{U}],$$

where

$$\delta\psi=\tfrac{1}{2}i\boldsymbol{\tau}\cdot\boldsymbol{\omega}\psi,\quad \delta\mathsf{U}=\tfrac{1}{2}i[\boldsymbol{\tau}\cdot\boldsymbol{\omega},\mathsf{U}].$$

Using the method in Appendix C, we can obtain easily the vector current

$$\mathbf{J}_\mu=-\bar{\psi}\gamma_\mu\frac{\boldsymbol{\tau}}{2}\psi-\frac{i}{8f^2}\mathrm{tr}\left[(\mathsf{U}^\dagger\partial_\mu\mathsf{U}+\mathsf{U}\partial_\mu\mathsf{U}^\dagger)\frac{\boldsymbol{\tau}}{2}\right].$$

If we express $\mathsf{U}=\sigma+2if\psi_5\rho\Phi$, and make use of the trace properties of $\boldsymbol{\tau}$ matrix

$$\mathrm{tr}(\boldsymbol{\tau}\cdot\mathbf{A})=0,$$

$$\mathrm{tr}\left[(\boldsymbol{\tau}\cdot\mathbf{A})\frac{\boldsymbol{\tau}}{2}\right]=\mathbf{A},$$

we find

$$\mathrm{tr}\left(\mathsf{U}^\dagger\partial_\mu\mathsf{U}\frac{\boldsymbol{\tau}}{2}\right)=\mathrm{tr}\left\{(\sigma-2if\rho\boldsymbol{\tau}\cdot\boldsymbol{\varphi})\left[\sigma'f^2(\partial_\mu\varphi^2)+2if\rho\boldsymbol{\tau}\cdot\partial_\mu\boldsymbol{\varphi}+2if^3\rho'\boldsymbol{\tau}\cdot\boldsymbol{\varphi}(\partial_\mu\varphi^2)\right]\frac{\boldsymbol{\tau}}{2}\right\}$$

$$=2if\sigma\rho\partial_\mu\boldsymbol{\varphi}+2if^3\boldsymbol{\varphi}(\partial_\mu\varphi^2)(\sigma\rho'-\rho\sigma')+4if^2\rho^2\boldsymbol{\varphi}\times\partial_\mu\boldsymbol{\varphi}.$$

$$\mathrm{tr}\left(\mathsf{U}\partial_\mu\mathsf{U}^\dagger\frac{\boldsymbol{\tau}}{2}\right)=\mathrm{tr}\left\{(\sigma+2if\rho\boldsymbol{\tau}\cdot\boldsymbol{\varphi})\left[\sigma'f^2(\partial_\mu\varphi^2)-2if\rho\boldsymbol{\tau}\cdot\partial_\mu\boldsymbol{\varphi}-2if^3\rho'\boldsymbol{\tau}\cdot\boldsymbol{\varphi}(\partial_\mu\varphi^2)\right]\frac{\boldsymbol{\tau}}{2}\right\}$$

$$=-2if\sigma\rho\partial_\mu\boldsymbol{\varphi}-2if^3\boldsymbol{\varphi}(\partial_\mu\varphi^2)(\sigma\rho'-\rho\sigma')+4if^2\rho^2\boldsymbol{\varphi}\times\partial_\mu\boldsymbol{\varphi}.$$

$$\mathbf{J}_\mu=-\bar{\psi}\gamma_\mu\frac{\boldsymbol{\tau}}{2}\psi-\frac{i}{8f^2}\mathrm{tr}\left[(\mathsf{U}^\dagger\partial_\mu\mathsf{U}+\mathsf{U}\partial_\mu\mathsf{U}^\dagger)\frac{\boldsymbol{\tau}}{2}\right]$$

$$=-\bar{\psi}\gamma_\mu\frac{\boldsymbol{\tau}}{2}\psi+\rho^2\boldsymbol{\varphi}\times\partial_\mu\boldsymbol{\varphi},$$

[16] Terms like $\partial_\mu(\boldsymbol{\varphi}\cdot\boldsymbol{\varphi})$ and $\boldsymbol{\varphi}\cdot\partial_\mu\boldsymbol{\varphi}$ are also invariant under isospin transformations. But we do not want $\mathcal{L}_{\mathrm{S.B.}}$ to depend upon them because we require that $\delta\mathcal{L}_{\mathrm{S.B.}}$ does not contribute to $\mathbf{J}_{5\mu}$ under the gauge transformation (C.1) and (C.2).

and after the canonical transformation $\mathbf{J}_\mu$ becomes

$$\mathbf{J}_\mu = -\bar{\xi}\gamma_\mu\frac{\boldsymbol{\tau}}{2}\xi + \rho^2\boldsymbol{\pi}\times\partial_\mu\boldsymbol{\pi}.$$

B. Construction of $J_{5\mu}$

In Appendix C, we have already derived

$$\mathbf{J}_{5\mu} = -\bar{\psi}\gamma_\mu\gamma_5\frac{\boldsymbol{\tau}}{2}\psi - \frac{i}{8f^2}\,\mathrm{tr}\left\{[(\partial_\mu\mathbf{U})\mathbf{U}^\dagger + \mathbf{U}^\dagger(\partial_\mu\mathbf{U})]\gamma_5\frac{\boldsymbol{\tau}}{2}\right\}.$$

It is easily verified that

$$\mathrm{tr}\left[(\partial_\mu\mathbf{U})\mathbf{U}^\dagger\frac{\boldsymbol{\tau}}{2}\right] = 2if\gamma_5\sigma\rho\partial_\mu\boldsymbol{\varphi} + 2if^3\boldsymbol{\varphi}(\partial_\mu\varphi^2)(\sigma\rho' - \rho\sigma') - 4if^2\rho^2\boldsymbol{\varphi} + \partial_\mu\boldsymbol{\varphi},$$

and

$$\mathrm{tr}\left[\mathbf{U}^\dagger(\partial_\mu\mathbf{U})\frac{\boldsymbol{\tau}}{2}\right] = 2if\gamma_5\sigma\rho\partial_\mu\boldsymbol{\varphi} + 2if^3\boldsymbol{\varphi}(\partial_\mu\varphi^2)(\sigma\rho' - \rho\sigma') - 4if^2\rho^2\boldsymbol{\varphi}\times\partial_\mu\boldsymbol{\varphi}.$$

Hence,

$$\mathbf{J}_{5\mu} = -\bar{\psi}\gamma_\mu\gamma_5\frac{\boldsymbol{\tau}}{2}\psi + \frac{\rho\sigma}{2f}\partial_\mu\boldsymbol{\varphi} + \tfrac{1}{2}f\boldsymbol{\varphi}(\partial_\mu\varphi^2)(\sigma\rho' - \rho\sigma'),$$

and after canonical transformation

$$\mathbf{J}_{5\mu} = -\bar{\xi}\gamma_\mu\gamma_5\frac{\boldsymbol{\tau}}{2}\xi + \frac{\rho\sigma}{2f}\partial_\mu\boldsymbol{\pi} + \tfrac{1}{2}f(\partial_\mu\pi^2)(\sigma\rho' - \rho\sigma').$$

If the specific models are considered, we have

(a)

$$\mathbf{U} = e^{2if\gamma_5\boldsymbol{\tau}\cdot\boldsymbol{\pi}},$$

$$\mathbf{J}_\mu = -\bar{\xi}\gamma_\mu\frac{\boldsymbol{\tau}}{2}\xi + \frac{\sin^2(2f\sqrt{\pi^2})}{4f^2\pi^2}\boldsymbol{\pi}\times\partial_\mu\boldsymbol{\pi},$$

$$\mathbf{J}_{5\mu} = -\bar{\xi}\gamma_\mu\gamma_5\frac{\boldsymbol{\tau}}{2}\xi + \frac{\partial_\mu\boldsymbol{\pi}}{2f}\left(\frac{\sin(2f\sqrt{\pi^2})\cos(2f\sqrt{\pi^2})}{2f\sqrt{\pi^2}}\right) + \frac{f\boldsymbol{\pi}(\partial_\mu\pi^2)}{2f^2\pi^2}\left(1 - \frac{\sin(2f\sqrt{\pi^2})\cos(2f\sqrt{\pi^2})}{2f\sqrt{\pi^2}}\right).$$

(b)

$$\mathbf{U} = \frac{1 + if\gamma_5\boldsymbol{\tau}\cdot\boldsymbol{\pi}}{1 - if\gamma_5\boldsymbol{\tau}\cdot\boldsymbol{\pi}},$$

$$\mathbf{J}_\mu = -\bar{\xi}\gamma_\mu\frac{\boldsymbol{\tau}}{2}\xi + (1 + f^2\pi^2)^{-2}\boldsymbol{\pi}\times\partial_\mu\boldsymbol{\pi},$$

$$\mathbf{J}_{5\mu} = -\bar{\xi}\gamma_\mu\gamma_5\frac{\boldsymbol{\tau}}{2}\xi + \frac{\partial_\mu\boldsymbol{\pi}}{2f}\frac{1 - f^2\pi^2}{(1 + f^2\pi^2)^2} + \frac{f}{2}\boldsymbol{\pi}\partial_\mu(\pi^2)(1 + f^2\pi^2)^{-2}.$$

(c)

$$\mathbf{U} = (1 - 4f^2\pi^2)^{1/2} + 2if\gamma_5\boldsymbol{\tau}\cdot\boldsymbol{\pi},$$

$$\mathbf{J}_\mu = -\bar{\xi}\gamma_\mu\frac{\boldsymbol{\tau}}{2}\xi + \boldsymbol{\pi}\times\partial_\mu\boldsymbol{\pi},$$

$$\mathbf{J}_{\mu 5} = -\bar{\xi}\gamma_\mu\gamma_5\frac{\boldsymbol{\tau}}{2}\xi + \frac{\partial_\mu\boldsymbol{\pi}}{2f}(1 - 4f^2\pi^2)^{1/2} + f\boldsymbol{\pi}(\partial_\mu\pi^2)\frac{1}{(1 - 4f^2\pi^2)^{1/2}}.$$

To the lowest order in f, all three models have the same expansion for $\mathbf{J}_\mu$ and $\mathbf{J}_{5\mu}$:

$$\mathbf{J}_\mu = -\bar{\xi}\gamma_\mu\frac{\boldsymbol{\tau}}{2}\xi + \boldsymbol{\pi}\times\partial_\mu\boldsymbol{\pi} + O(f^2),$$

$$\mathbf{J}_{5\mu} = -\bar{\xi}\gamma_\mu\gamma_5\frac{\boldsymbol{\tau}}{2}\xi + \frac{1}{2f}\partial_\mu\boldsymbol{\pi} + O(f).$$

APPENDIX E

Since $\mathbf{U}$ is a function of the direct product of γ_5 and $\boldsymbol{\tau}$, the solution for $\mathbf{U}^{1/2}$ can be generally expressed as a linear function of the direct product $\{1,\gamma_5\}\times\{1,\boldsymbol{\tau}\}$, or explicitly

$$\mathbf{U}^{1/2} = a + b\gamma_5 + \boldsymbol{\tau}\cdot\mathbf{c} + \gamma_5\boldsymbol{\tau}\cdot\mathbf{d},$$

$$\mathbf{U} = \mathbf{U}^{1/2}\mathbf{U}^{1/2} = a^2 + b^2 + \mathbf{c}\cdot\mathbf{c} + \mathbf{d}\cdot\mathbf{d} + 2ab\gamma_5 + 2a\boldsymbol{\tau}\cdot\mathbf{c} + 2a\gamma_5\boldsymbol{\tau}\cdot\mathbf{d} + 2b\gamma_5\boldsymbol{\tau}\cdot\mathbf{c} + 2\gamma_5\mathbf{c}\cdot\mathbf{d}.$$

Comparing the latter and $\mathbf{U} = \sigma + 2if\gamma_5\boldsymbol{\tau}\cdot\boldsymbol{\varphi}$, we have the following relations:

(a) $b = \mathbf{c} = 0$,

(b) $a\mathbf{d} = if\rho\boldsymbol{\varphi}$,

(c) $a^2 + \mathbf{d}^2 = \sigma$,

from which we can obtain the solutions

$$a = \tfrac{1}{2}\sqrt{2}[\sigma + (\sigma^2 + 4f^2\rho^2\varphi^2)^{1/2}]^{1/2} = \tfrac{1}{2}\sqrt{2}(1+\sigma)^{1/2},$$

$$\mathbf{d} = (i/a)f\rho\boldsymbol{\varphi} = 2if\rho\boldsymbol{\varphi}/[2(1+\sigma)]^{1/2}.$$

Therefore,

$$\mathbf{U}^{1/2} = \tfrac{1}{2}\sqrt{2}\left\{(1+\sigma)^{1/2} + \frac{2if\rho\gamma_5\boldsymbol{\tau}\cdot\boldsymbol{\varphi}}{[2(1+\sigma)]^{1/2}}\right\}.$$

Similarly,

$$\mathbf{U}^{-1/2} = \tfrac{1}{2}\sqrt{2}\left\{(1+\sigma)^{1/2} - \frac{2if\rho\gamma_5\boldsymbol{\tau}\cdot\boldsymbol{\varphi}}{(1+\sigma)^{1/2}}\right\}.$$

Since $\{\gamma_\mu,\gamma_\nu\} = 2\delta_{\mu\nu}$, $\{\gamma_5,\gamma_\mu\} = 0$ in the Weyl representation, it follows that

$$\mathbf{U}^{\pm 1}\gamma_\mu = \mathbf{U}^{\mp 1}\gamma_\mu,$$

$$\mathbf{U}^{\pm 1/2}\gamma_\mu = \mathbf{U}^{\mp 1/2}\gamma_\mu.$$

APPENDIX F

From (3.3b), we have

$$\Lambda = \mathbf{U}'^{1/2} V \mathbf{U}^{-1/2},$$

where

$$\mathbf{U}' - V\mathbf{U}V^{\dagger},$$
$$V = e^{i\gamma_5(\tau/2)\cdot\mathbf{a}}.$$

Let us evaluate Λ for the exponential model

$$\mathbf{U} = e^{2if\gamma_5\tau\cdot\pi}.$$

We have

$$\mathbf{U}' = e^{-\frac{1}{2}i\gamma_5\tau\cdot\mathbf{a}}\, e^{2if\gamma_5\tau\cdot\pi}\, e^{-\frac{1}{2}i\gamma_5\tau\cdot\mathbf{a}} = [\cos(\tfrac{1}{2}\sqrt{\mathbf{a}^2}) - \sin(\tfrac{1}{2}\sqrt{\mathbf{a}^2})]\cos(2f\sqrt{\pi^2})$$
$$+2\left[\frac{\sin(\frac{1}{2}\sqrt{\mathbf{a}^2})\cos(\frac{1}{2}\sqrt{\mathbf{a}^2})}{\frac{1}{2}\sqrt{\mathbf{a}^2}}\right]\left(\frac{\sin(2f\sqrt{\pi^2})}{2f\sqrt{\pi^2}}\right) f\mathbf{a}\cdot\boldsymbol{\pi} + i\gamma_5\left\{2f\boldsymbol{\tau}\cdot\boldsymbol{\pi}\left(\frac{\sin(2f\sqrt{\pi^2})}{2f\sqrt{\pi^2}}\right)\right.$$
$$\left. -\left[\frac{\sin(\frac{1}{2}\sqrt{\mathbf{a}^2})\cos(\frac{1}{2}\sqrt{\mathbf{a}^2})}{\frac{1}{2}\sqrt{\mathbf{a}^2}}\right]\cos(2f\sqrt{\pi^2})\boldsymbol{\tau}\cdot\mathbf{a} - \left[\frac{\sin(\frac{1}{2}\sqrt{a^2})}{\frac{1}{2}\sqrt{a^2}}\right]^2\left(\frac{\sin(2f\sqrt{\pi^2})}{2f\sqrt{\pi^1}}\right) f\boldsymbol{\pi}\cdot\mathbf{a}\boldsymbol{\tau}\cdot\mathbf{a}\right\}. \tag{F.1}$$

Hence,

$$\mathbf{U}'^{1/2} = \tfrac{1}{2}\sqrt{2}\left[(1+\beta)^{1/2} + \frac{i\gamma_5\delta}{(1+\beta)^{1/2}}\right], \tag{F.2}$$

where

$$\beta = [\cos^2(\tfrac{1}{2}\sqrt{a^2}) - \sin(\tfrac{1}{2}\sqrt{a^2})]\cos(2f\sqrt{\pi^2}) + 2f\mathbf{a}\cdot\boldsymbol{\pi}\left[\frac{\sin(\frac{1}{2}\sqrt{a^2})\cos(\frac{1}{2}\sqrt{a^2})}{\frac{1}{2}\sqrt{a^2}}\right]\left[\frac{\sin(2f\sqrt{\pi^2})}{2f\sqrt{\pi^2}}\right],$$

$$\delta = 2f\boldsymbol{\tau}\cdot\boldsymbol{\pi}\left[\frac{\sin(2f\sqrt{\pi^2})}{2f\sqrt{\pi^2}}\right] - \left\{\left[\frac{\sin(\frac{1}{2}\sqrt{a^2})\cos(\frac{1}{2}\sqrt{a^2})}{\frac{1}{2}\sqrt{a^2}}\right]\cos(2f\sqrt{\pi^2}) + \left[\frac{\sin(\frac{1}{2}\sqrt{a^2})}{\frac{1}{2}\sqrt{a^2}}\right]^2\left[\frac{\sin(2f\sqrt{\pi^2})}{2f\sqrt{\pi^2}}\right] f\boldsymbol{\pi}\cdot\mathbf{a}\right\}\boldsymbol{\tau}\cdot\mathbf{a},$$

and

$$\Lambda = \tfrac{1}{2}\sqrt{2}\left\{(1+\beta)^{1/2} + \frac{i\gamma_5\delta}{(1+\beta)^{1/2}}\right\} e^{\frac{1}{2}i\gamma_5\tau\cdot a}\, e^{-if\gamma_5\tau\cdot\pi}. \tag{F.3}$$

The expansion of Λ as a power series of $\mathbf{a}$ and f will be

$$\Lambda \simeq 1 + \tfrac{1}{2}if\tau\cdot(\mathbf{a}\times\boldsymbol{\pi}) + O(f^3, f^2a, fa^2, a^3). \tag{F.4}$$

Furthermore we have derived in Appendix C that

$$\mathbf{a} = \frac{f\sigma'}{\rho}\delta\boldsymbol{\pi}.$$

To the lowest order in f, we have

$$\mathbf{a} \simeq -2f\delta\boldsymbol{\pi}.$$

Hence (F.4) becomes

$$\Lambda \simeq 1 + if^2(\boldsymbol{\pi}\times\delta\boldsymbol{\pi}). \tag{F.5}$$

The infinitesimal transformation for the ξ field and $\boldsymbol{\pi}$ field are

$$\xi \to \xi' = \Lambda\xi = [1 + if^2\tau\cdot(\boldsymbol{\pi}\times\delta\boldsymbol{\pi})]\xi, \tag{F.6a}$$
$$\boldsymbol{\pi} \to \boldsymbol{\pi}' = \boldsymbol{\pi} + \delta\boldsymbol{\pi}. \tag{F.6b}$$

The other two models have the same expansion as the exponential model up to order f^3. It is therefore easy to convince oneself that (F.6a), (F.6b) will hold for the three models.