Low-Energy Pion-Nucleon Scattering from Current Commutation Relations and Partial Conservation of the Axial-Vector Current*

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Starting with current commutation relations and the hypothesis of partial conservation of the axial-vector current, relations are obtained for the pion-nucleon scattering amplitude considered to the second order in the energy variable ν. The structure-dependent contributions to the second-order terms in the matrix element involving the axial-vector currents are approximated by the direct and exchanged poles and resonances, particularly the N^* resonances with spins $\frac{1}{2}$ and $\frac{3}{2}$. The contribution of the ρ meson to this matrix element is estimated to be small. Expressions are obtained for the P-wave scattering lengths and the S-wave and P-wave phase shifts at low energies. Numerical estimates are made for these and are compared with the available data; it is found that reasonable agreement with experiment can be obtained except for the P-wave scattering length $a_{1-}^{(-)}$ and the partial-wave amplitude $f_{1-}^{(-)}$. Some questions relating to the off-mass-shell extrapolation are discussed. We have also examined the validity of the assumption made, in the usual derivation of the relations for the S-wave scattering lengths, that the terms of higher order in ν may be neglected. We suggest that at least for the isospin-symmetric scattering length this is not a good approximation, so that this scattering length cannot be obtained adequately by using only the current commutation relations and the hypothesis of a partially conserved axial-vector current. The scalar term arising from the commutator $[\mathcal{A}^0(x),\phi(y)]$ is estimated; we suggest that the contribution of this term is important and cannot be ignored.

I. INTRODUCTION

STARTING with current commutation relations and the hypothesis of partial conservation of the axial-vector current (PCAC), results have been obtained for pion-nucleon scattering which are in good agreement with experiment. These are the Adler-Weisberger sum rule[1] and the relations for the S-wave πN scattering lengths.[2–7] Some results have also been obtained for the πN P-wave scattering lengths.[8–11] These require additional dynamical assumptions about the matrix element of the product of two axial-vector currents, and the low-energy theorems obtained for the P-wave scattering lengths are therefore approximate.

In this paper we examine in detail the relations obtained for the low-energy πN scattering amplitude considered to second order in the energy variable ν, starting with current commutation relations and the PCAC hypothesis and specific assumptions about the matrix element of the product of two axial-vector currents between single-nucleon states.

Our basic assumptions are the following: (i) the existence of equal-time commutation relations (C.R.'s) between certain components of the axial-vector current densities, of the form

$$[\mathcal{A}_\alpha{}^0(x),\mathcal{A}_\beta{}^\nu(y)]\delta(x_0-y_0) = if_{\alpha\beta\gamma}\mathcal{V}_\gamma{}^\nu(x)\delta(x-y) + \text{a possible singular term;} \quad (1.1)$$

and (ii) the hypothesis of partial conservation of the axial-vector current (PCAC):

$$\partial_\mu\mathcal{A}_\alpha{}^\mu(x)=C_\alpha\varphi_\alpha(x). \quad (1.2)$$

In (1.1) and (1.2), the subscripts $\alpha, \beta, \gamma, \cdots$ are SU_3 octet indices, $\mathcal{V}^\nu(x)$ is the vector current density, and $if_{\alpha\beta\gamma}$ are the completely antisymmetric structure constants of SU_3.

The constant C_α in (1.2) can be related, for $\alpha=\pi$, to the pion decay lifetime or to the coupling of the pion to the baryons. Taking matrix elements of both sides of (1.2), for $\alpha=\pi$, between single-nucleon states gives the following expression for C_π:

$$C_\pi=mg_A\mu^2/[G_{NN\pi}K_{NN\pi}(0)]. \quad (1.3)$$

Here, m and μ are the nucleon and pion masses, g_A is the $N\rightarrow N$ axial-vector renormalization constant, $G_{NN\pi}$

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[1] S. L. Adler, Phys. Rev. Letters **14**, 1051 (1965); Phys. Rev. **140**, B736 (1965); W. I. Weisberger, Phys. Rev. Letters **14**, 1047 (1965).
[2] Y. Tomozawa, Nuovo Cimento **46**, 803 (1967).
[3] K. Raman and E. C. G. Sudarshan, Phys. Letters **21**, 450 (1966).
[4] K. Raman and E. C. G. Sudarshan, Phys. Rev. **154**, 1499 (1967); see also K. Raman, in *High Energy and Particle Physics*, edited by W. E. Brittin, A. D. Barut, and M. Guenin (Gordon and Breach, Science Publishers, Inc., New York, 1967), Vol. 9B, p. 393.
[5] A. P. Balachandran, M. Gundzik, and F. Nicodemi, Nuovo Cimento **44**, 1257 (1966).
[6] B. Hamprecht (unpublished).
[7] S. Weinberg, Phys. Rev. Letters **17**, 616 (1966).
[8] K. Raman, Phys. Rev. Letters **17**, 983 (1966); **17**, 1248(E) (1966); **18**, 432(E) (1967). In the paper quoted here, the error was made of omitting the second-order terms (in ν) arising from the nonBorn part of $M_{AA}{}^{\mu\nu}$ in (2.1); this was corrected in the Erratum and in Ref. 9 below.
[9] K. Raman, Phys. Rev. **159**, 1501 (1967).
[10] Relations similar to those in Refs. 8 and 9 have been discussed by N. H. Fuchs, Phys. Rev. **150**, 1241 (1960); C. Bouchiat, G. Flamand, and Ph. Meyer (unpublished); and L. Maiani and G. Preparata (to be published); H. Goldberg and F. Gross, Phys. Rev. **162**, 1350 (1967); and I. S. Gerstein, Phys. Rev. **161**, 1631 (1967). In the first two papers here, the same error was made as in Ref. 8, as was noticed by the respective authors.
[11] A. P. Balachandran, M. Gundzik, and F. Nicodemi (to be published) have given expressions for the P-wave scattering lengths and S-wave effective ranges which involve essentially only the contribution of the nucleon Born term and are independent of the commutator term. Our basic approximations, procedure, and results are quite different from those of these authors.

is the πNN pseudoscalar coupling constant, and $K_{NN\pi}(q^2)$ is the form factor at the πNN vertex, normalized to unity at $q^2=\mu^2$. We shall usually write merely $K(0)$ for $K_{NN\pi}(0)$.

Because of the nonzero mass of the pion, one requires not only (1.1) and (1.2), but also assumptions regarding the relations between the πN scattering amplitude with physical (massive) pions and the amplitude with pions of zero external mass $(q^2)^{1/2}$ (where q^2 is the square of the four-momentum of an external pion). These assumptions will be discussed in the course of this paper.

In Sec. II we briefly derive and discuss the basic equations. In Sec. III we examine the validity of the assumption made, in the usual derivation of the results for the S-wave scattering lengths, of neglecting the higher-order terms in ν; we suggest that at least for $a^{(+)}$, this is not a good approximation. We discuss the scalar term occurring in the relation for $a^{(+)}$ and argue that it may not be small, in contrast to what is suggested by most earlier work. In Sec. IV we discuss our approximation and method of extrapolation. In Sec. V we briefly discuss the various contributions to the scattering lengths and the partial-wave amplitudes. In Sec. VI we discuss the numerical results, and in Sec. VII we summarize our conclusions.

In Appendix A we give some definitions and relations used in the paper. In Appendix B we discuss an alternative method of extrapolating the amplitudes from $q^2=0$ to $q^2=\mu^2$ and examine whether there is any significant difference between the different methods of extrapolation. In Appendix C we discuss the relation obtained from current C.R.'s for the $\sigma\pi\pi$ coupling and its application to the σ-exchange contribution to the relations we obtain for the πN amplitude. Equations in the Appendix are numbered (A1), etc., and are referred to in the text as such.

While this work was being completed, it came to our attention that work similar to ours has been done recently by Schnitzer,[12] who has approximated the second-order terms arising from the matrix element of the product of the axial-vector currents by the $(\frac{3}{2},\frac{3}{2})$ N_1^* contribution, and has obtained results for the P-wave scattering lengths and S-wave effective ranges. Our scheme of approximation is more elaborate than Schnitzer's, and our work and results are more detailed. Where our work overlaps that of Schnitzer, they differ in detail, partly because of the differences in the method of extrapolation and in the evaluation of the N_1^* contribution.

II. GENERAL DERIVATION

Our starting point is the equation[13]

$$q_{1\mu}q_{2\nu}M_{AA}{}^{\mu\nu}=(\mu_\alpha{}^2-q_1{}^2)^{-1}(\mu_\beta{}^2-q_2{}^2)^{-1}C_\alpha C_\beta M_{PP}\\ -if_{\alpha\beta\beta'}q_{2\nu}F_{\beta'}{}^{\nu}(\gamma\sigma)\\ +d_{\beta\alpha\alpha'}b_\alpha{}^{-1}G_{\alpha'}(\gamma\sigma)+q_{1k}q_{2l}S_{\alpha\beta}{}^{kl}. \quad (2.1)$$

[12] H. Schnitzer, Phys. Rev. **158**, 1471 (1967).

[13] For a more detailed discussion of (2.1), see Refs. 4 and 9.

Here, the time-ordered amplitude $M_{AA}{}^{\mu\nu}$ is defined as follows:

$$i(2\pi)^4\delta(p_i+q_1-p_f-q_2)T_{AA}{}^{\mu\nu}\\ =i^2\left[\frac{m_im_f}{E_iE_f}\right]^{1/2}\int d^4xd^4y\,\exp[i(q_2\cdot y-q_1\cdot x)]\\ \times\langle B(p_f)|T(\mathcal{A}_\alpha{}^\mu(x)\mathcal{A}_\beta{}^\nu(y))|B(p_i)\rangle; \quad (2.2)$$

$$T_{AA}{}^{\mu\nu}=\bar{u}(p_f)M_{AA}{}^{\mu\nu}u(p_i). \quad (2.3)$$

In (2.2), $\mathcal{A}_\alpha{}^\mu(x)$ and $\mathcal{A}_\beta{}^\nu(y)$ are axial-vector current densities, and $B(p_i)$ and $B(p_f)$ are baryon states with momenta p_i and p_f. The baryons are kept on the mass shell throughout.

The transition amplitude $T_{PP}=\bar{u}(p_f)M_{PP}u(p_i)$ for the meson-baryon scattering process

$$\underset{q_1}{\varphi_\alpha}+\underset{p_i}{B_\sigma}\rightarrow\underset{q_2}{\varphi_\beta}+\underset{p_f}{B_\gamma}, \quad (2.4)$$

with momenta as indicated above, is defined as follows:

$$i(2\pi)^4\delta(p_f+q_2-p_i-q_1)T_{PP}\\ =i^2\left[\frac{m_im_f}{E_iE_f}\right]^{1/2}\int d^4xd^4y\\ \times\exp[i(q_2\cdot y-q_1\cdot x)](\mu_\alpha{}^2-q_1{}^2)(\mu_\beta{}^2-q_2{}^2)\\ \times\langle B(p_f)|T(\varphi_\alpha(x)\varphi_\beta(y))|B(p_i)\rangle. \quad (2.5)$$

The function $F_{\beta'}{}^\nu$ in (2.1) is related to the matrix element of the vector current $\mathcal{V}_{\beta'}{}^\nu$ by

$$\langle B(p_f)|\mathcal{V}_{\beta'}{}^\nu(0)|B(p_i)\rangle\\ =i^2[m_im_f/E_iE_f]^{1/2}\bar{u}(p_f)F_{\beta'}{}^\nu u(p_i), \quad (2.6)$$

and may be decomposed into the Dirac and Pauli form factors $F_1(t)$ and $F_2(t)$ in the usual manner. The term involving G_α in (2.1) is obtained by evaluating $[\mathcal{A}_\beta{}^0(x),\varphi_\alpha(y)]\delta(x_0-y_0)$ in a quark model,[14] and using (C3), (C5), and the definition

$$\langle B(p_f)|S_{\alpha'}(0)|B(p_i)\rangle\\ =i^2[m_im_f/E_iE_f]^{1/2}\bar{u}(p_f)G_{\alpha'}u(p_i). \quad (2.7)$$

The last term in (2.1), which arises from the singular term[15] in the C.R. (1.1), does not contribute any terms of order q_0 (see Refs. 4 and 9), but could contribute, through terms of the second order in q_0, to the low-energy relations we shall be interested in. However, as this term is unknown, we shall omit it in all subsequent equations.[16]

[14] M. Gell-Mann, Phys. Letters **8**, 214 (1964); Physics **1**, 63 (1964).

[15] T. Gotô and T. Imamura, Progr. Theoret. Phys. (Kyoto) **14**, 396 (1955); J. Schwinger, Phys. Rev. Letters **3**, 296 (1959); K. Johnson, Nucl. Phys. **25**, 431 (1961); S. Okubo, Nuovo Cimento **44**, 1015 (1966); J. Schwinger, Phys. Rev. **130**, 406 (1963); L. S. Brown, *ibid.* **150**, 1338 (1966).

[16] It has been suggested that quite generally, the contribution of the singular term in a C.R. such as (1.1) to relations for amplitudes is exactly cancelled by other contributions (from the "seagull" terms). See, for instance, the paper by Adler and Dothan in Ref. 36 below.

We now consider πN scattering and restrict the indices α, β, $\cdots$ to the isospin subgroup. We make the spin decomposition of the amplitudes M_{PP} and $q_{1\mu}q_{2\nu}\bar{M}_{AA}{}^{\mu\nu}$ in the form

$$\begin{aligned} M_{PP}&=A+(\gamma\cdot Q)B;\\ q_{1\mu}q_{2\nu}\bar{M}_{AA}{}^{\mu\nu}&=D+(\gamma\cdot Q)R; \end{aligned} \tag{2.8}$$

and the usual isospin decomposition into the isospin-symmetric and isospin-antisymmetric amplitudes $A^{(\pm)}$, $B^{(\pm)}$, $D^{(\pm)}$, and $R^{(\pm)}$. Here, $\bar{M}_{AA}{}^{\mu\nu}$ is the covariant amplitude corresponding to $M_{AA}{}^{\mu\nu}$.[4,9]

From (2.1) we obtain the following equations:

$$A^{(+)}(s,t)=(\mu^2-q_1^2)(\mu^2-q_2^2)\times C_\pi^{-2}\{D^{(+)}(s,t)-C_\pi H(t)\}; \tag{2.9a}$$

$$A^{(-)}(s,t)=(\mu^2-q_1^2)(\mu^2-q_2^2)\times C_\pi^{-2}\{D^{(-)}(s,t)-\tfrac{1}{4}(s-u)F_2{}^V(t)\}; \tag{2.9b}$$

$$B^{(+)}(s,t)=(\mu^2-q_1^2)(\mu^2-q_2^2)C_\pi^{-2}R^{(+)}(s,t); \tag{2.9c}$$

$$B^{(-)}(s,t)=(\mu^2-q_1^2)(\mu^2-q_2^2)C_\pi^{-2}\{R^{(-)}(s,t)+\tfrac{1}{2}[F_1{}^V(t)+2mF_2{}^V(t)]\}. \tag{2.9d}$$

Here, $H(t)$ is defined by

$$d_{\beta\alpha\alpha'}b_\alpha{}^{-1}G_{\alpha'}(t)\rightarrow\delta_{\beta\alpha}H(t),$$

when the indices α, β, $\cdots$ are restricted to be isospin indices for the pion.

We may note that equating the residues at the nucleon pole (in s, say) in (2.9c) or (2.9d) gives

$$[G_{\pi NN}(q^2)]^2=(\mu_\pi{}^2-q^2)^2C_\pi{}^{-2}m^2g_A{}^2, \tag{2.10}$$

which is just the relation obtained by taking the matrix element of the PCAC relation (1.2) between one-nucleon states. Similarly, comparing the residues at an N^* pole gives the relation obtained by taking the matrix element of (1.2) between a nucleon and an N^*.[17]

At the poles in t, among which we may consider those from a ρ meson and a possible σ meson, one may obtain a relation between the $\rho\pi\pi$ coupling and the couplings of a ρ meson to two axial-vector currents, at $p_\rho{}^2=m_\rho{}^2$, $q_1^2=q_2^2=0$, where p_ρ is the 4-momentum of the ρ meson and q_1, q_2 are the 4-momenta of the pions (and a similar relation for the couplings of the σ meson). The more interesting type of relation for the ρ couplings is, however, the one that does not involve the coupling of the ρ to the axial-vector currents. This has been discussed by various authors.[18]

From Eqs. (2.9) one may obtain relations for the f_1 and f_2 amplitudes [see (A3)]. Corresponding relations for the partial-wave amplitudes may be projected out by using the following equations (see Chew, Goldberger, Low, and Nambu[19]):

$$f_{0+}=f_1(0)-2k^2[f_1'(0)-\tfrac{1}{3}f_2'(0)]+O(k^4); \tag{2.11a}$$

$$f_{1-}=f_2(0)-2k^2[f_2'(0)-\tfrac{1}{3}f_1'(0)]+O(k^4); \tag{2.11b}$$

$$f_{1+}=\tfrac{2}{3}k^2f_1'(0)+O(k^4); \tag{2.11c}$$

$$f_{2-}=O(k^4);\ \text{etc.} \tag{2.11d}$$

In (2.11), the argument zero in $f_1(0)$, etc., on the right-hand side, denotes $t=0$, and $f_i'(0)=[\partial f_i(t)/\partial t]_{t=0}$, etc.

Equations (2.9) and the equations for the partial-wave amplitudes obtained from them are our basic equations. In obtaining the simplest low-energy theorems, one essentially compares the coefficients of different powers of ν on the two sides of the Eqs. (2.9), setting $q_1^2=q_2^2=0$, $t=0$. However, since an extrapolation in q^2 is necessary and some contributions to the amplitudes vary rapidly with q^2, it appears to be better to treat some parts of the amplitude separately. (This will be discussed in Sec. IV.) By taking the t dependence of Eqs. (2.9) into account through the projection into partial-wave amplitudes, a wider class of low-energy relations is obtained. A study of these is the object of this paper.

We conclude this section with some remarks about the zero-energy theorems obtained from (2.9).

The zero-energy theorems for $B^{(-)}$ and $A^{(+)}$, obtained from (2.9) at $\nu=0$, $q_1^2=q_2^2=0$, $t=0$, may be extrapolated to the physical threshold and re-expressed as relations involving the P-wave scattering lengths, as shown in Refs. 8 and 9.

An alternative way of expressing the zero-energy theorems is to assume dispersion relations for $B^{(\pm)}$ and $A^{(\pm)}$, substitute them in the zero-energy theorems, and use the relation analogous to (2.1) for the absorptive parts of the amplitudes. For $B^{(-)}$, for instance, this gives a relation involving

$$\int_{\mu+\mu^2/2m}^{\infty} d\nu'\,\mathrm{Im}B^{(-)}(\nu')/\nu'$$

and a term arising from the matrix element of the axial-vector currents. Such relations were discussed by the authors of Ref. 10.

III. S-WAVE SCATTERING LENGTHS; THE SCALAR TERM

In deriving the results for the S-wave scattering lengths,[2–7] one may start with the following relations for the nonpole parts $T_p(\nu,t)$ of the amplitudes in the limit

[17] Such equations may be used for obtaining information about the couplings of baryon resonances; see R. H. Graham, S. Pakvasa, and K. Raman, Phys. Rev. **163**, 1774 (1967).

[18] K. Kawarabayashi and M. Suzuki, Phys. Rev. Letters **16**, 255 (1966); *ibid.* **16**, 348(E) (1966); Riazuddin and Fayyazuddin, Phys. Rev. **147**, 1071 (1966); and more recently, S. G. Brown and G. B. West, Phys. Rev. Letters **19**, 812 (1967); D. A. Geffen, *ibid.* **19**, 770 (1967).

[19] G. F. Chew, M. L. Goldberger, F. E. Low, and Y. Nambu, Phys. Rev. **106**, 1337 (1957).

$q^2=0$, $\nu=0$, $t=0$:

$$\lim T_p^{(-)}=0\,,\quad \lim\frac{\partial}{\partial\nu}T_p^{(-)}=\frac{G^2K^2(0)}{2m^2}(g_A^{-2}-1)\,,\tag{3.1a}$$

$$\lim T_p^{(+)}=\frac{G^2K^2(0)}{m}-\frac{GK(0)}{mg_A}h\,,\quad \lim\frac{\partial}{\partial\nu}T_p^{(+)}=0\,,\tag{3.1b}$$

where $h=\mu^2H(0)$. In order to obtain the S-wave scattering lengths using only the predictions (3.1) of PCAC and the C.R.'s, the extrapolation in ν to $\nu=\mu$ was performed assuming that

$$T_p^{(+)}(\mu,0)\approx T_p^{(+)}(0,0)\,;\tag{3.2a}$$

$$T_p^{(-)}(\mu,0)\approx T_p^{(-)}(0,0)+\mu\left[\frac{\partial}{\partial\nu}T_p^{(-)}(\nu,0)\right]_{\nu=0}.\tag{3.2b}$$

These will be a good approximation if the higher-order terms in a Taylor expansion of $T_p^{(\pm)}$ in ν about $\nu=0$ are negligibly small; we shall now examine whether this is in fact true.

For this we first evaluate these higher-order terms (in ν) for $q^2=\mu^2$ using dispersion relations and the observed cross sections, and then estimate the effect of extrapolating to $q^2=0$. From the dispersion relations in ν for $t=0$, namely,

$$\mathrm{Re}T_p^{(+)}(\nu,0)=T_p^{(+)}(\mu,0)+\frac{2(\nu^2-\mu^2)}{\pi}\mathcal{P}\int_\mu^\infty d\nu'\frac{\nu'}{k_L'}\frac{\sigma^{(+)}(\nu')}{(\nu'^2-\nu^2)}\,,\tag{3.3a}$$

$$\mathrm{Re}T_p^{(-)}(\nu,0)=\frac{2\nu}{\pi}\mathcal{P}\int_\mu^\infty d\nu'\frac{k_L'\sigma^{(-)}(\nu')}{(\nu'^2-\nu^2)}\,,\tag{3.3b}$$

which give

$$T_p^{(\pm)(n)}\equiv\frac{1}{n!}\frac{\partial^n}{\partial\nu^n}T_p^{(\pm)}(\nu,0)\bigg|_{\nu=0}=\frac{2}{\pi}\int_\mu^\infty d\nu'\frac{k_L'}{(\nu')^n}\sigma^{(\pm)}(\nu')\,,\tag{3.3c}$$

$$[T_p^{(+)}(\mu,0)-T_p^{(+)}(0,0)]=\frac{2}{\pi}\mathcal{P}\int_\mu^\infty d\nu'\frac{\sigma^{(+)}(\nu')}{k_L'\nu'}\,,\tag{3.3d}$$

k_L being the laboratory momentum of the incident pion, we obtain (for $q^2=\mu^2$) the following expressions from the Taylor expansions about $\nu=0$[20]:

$$\frac{1}{4\pi}T_p^{(+)}(\mu,0)=\frac{1}{4\pi}T_p^{(+)}(0,0)+0.0884\mu^2+0.0152\mu^4+0.0022\mu^6+\cdots\tag{3.4a}$$

$$=\frac{1}{4\pi}T_p^{(+)}(0,0)+0.11\,,\tag{3.4b}$$

$$\frac{1}{4\pi}T_p^{(-)}(\mu,0)=-0.0406\mu-0.0064\mu^3+\cdots.\tag{3.5}$$

For comparison with the first term in (3.5), we note that for $q^2=0$, Eq. (3.1a) gives

$$\frac{1}{4\pi}T_p^{(-)(1)}\approx-0.046K^2(0)\,.\tag{3.6}$$

We have evaluated the dispersion integrals in (3.3c) and (3.3d) using the experimental values of the cross sections up to 5 BeV as tabulated by Höhler, Ebel, and Giesecke,[21] and using above 5 BeV the following fits[21,22]:

$$\sigma^{(+)}=1.125+3.54k^{-0.7}\,,\quad \sigma^{(-)}=0.773k^{-0.7}\,,\tag{3.7}$$

in the natural units $\hbar=c=\mu_\pi=1$. The uncertainty in our estimates arising from the uncertainty in the cross sections of Höhler *et al.* and in the parameters in (3.7) is expected to be within 10–15%.

Equation (3.5) shows that the term of order μ^3 in $T_p^{(-)}(\mu,0)$ is appreciably large (about 16% of the term of order μ); this suggests that unless there is a cancellation among the higher-order terms in $T_p^{(-)}$, these could give rise to a significant correction to Eq. (3.2b). Whether there is such a cancellation may be seen from the experimental estimates for $T^{(-)}(\mu,0)$. The work of Samaranayake and Woolcock[23] gives $(1/4\pi)T^{(-)}(\mu,0)\approx0.093$, $(1/4\pi)T_p^{(-)}(\mu,0)\approx-0.0323$, while Hamilton and Woolcock[24] and Roper, Wright, and Feld[25] give 0.086 and -0.0405, respectively, for these quantities. Comparing with (3.5), it is seen that if the results of HW and RWF are correct, the higher-order terms in $T_p^{(-)}$ largely cancel one another [so that (3.2b) would be a good approximation], whereas if the estimates of SW are the better ones, the higher-order terms would give a correction of about 20% to (3.2b), for $q^2=\mu^2$. An accurate determination of $T^{(-)}(\mu,0)$ would enable a more conclusive statement to be made.

For $T_p^{(+)}(\mu,0)$, Eqs. (3.4) show the magnitudes of the higher-order terms. Adding the nucleon pole term gives $T^{(+)}(\mu,0)$. Experimentally, there is an uncertainty in the value of $T^{(+)}(\mu,0)$; however, all the estimates are small. The estimates of SW, for instance, give $(1/4\pi)T^{(+)}(\mu,0)$

[20] An integral similar to (3.3d), but with the kinematics for $q^2=0$, has been evaluated by K. Kawarabayashi and W. Wada [Phys. Rev. **146**, 1209 (1966)], who obtained a value of about 0.13. The quantity $T_p^{(-)(1)}/4\pi$ has been evaluated by Adler, who obtained a value of -0.0414 (see Ref. 1).

[21] G. Höhler, G. Ebel, and J. Giesecke, Z. Physik **180**, 430 (1964).

[22] G. von Dardel, D. Dekkers, R. Mermod, M. Vivargent, G. Weber, and K. Winter, Phys. Rev. Letters **8**, 173 (1962).

[23] V. K. Samaranayake and W. S. Woolcock (to be published); we shall refer to this as SW. These authors give $a_1=0.177\pm0.005$; $a_3=-0.102\pm0.004$, which give $a^{(+)}\approx-0.009$, $a^{(-)}\approx0.093$. We are grateful to Dr. Samaranayake for communicating to us the results of this work. See also V. K. Samaranayake and W. S. Woolcock, Phys. Rev. Letters **15**, 936 (1965).

[24] J. Hamilton and W. S. Woolcock, Rev. Mod. Phys. **35**, 737 (1963); this will be referred to as HW.

[25] L. D. Roper, R. M. Wright, and B. T. Feld, Phys. Rev. **138**, B190 (1965); we shall refer to this as RWF.

TABLE I. Contribution of the low-energy resonances to $\partial^2 T_p^{(+)}/\partial\nu^2$ at $\nu=0$.

Values of q^2 and ν	Contribution of the N^* resonances to $\partial^2 T_p^{(+)}/\partial\nu^2$ N_1^*	N_2^*	N_3^*	N_4^*	N_5^*	N_6^*	Sum of N^* contributions to $\partial^2 T_p^{(+)}/\partial\nu^2$	Value of $\partial^2 T_p^{(+)}/\partial\nu^2$ from dispersion relation
$q^2=0$, $\nu=0$	2.29 $K^2(0)$	0.099 $K^2(0)$	-0.066 $K^2(0)$	0.0088 $K^2(0)$	0.0233 $K^2(0)$	0.02 $K^2(0)$	2.37 $K^2(0)$	$\cdots$
$q^2=\mu^2$, $\nu=0$	1.89	0.095	-0.068	0.0085	0.0227	0.0194	1.97	2.22

≈ -0.01.[26] Comparing with the sum of the higher-order terms as given in (3.4b), one sees that these make an important contribution to $T^{(+)}(\mu,0)$, and that it would not be a good approximation to neglect them, as in (3.2a).[27]

The results given above are for $q^2=\mu^2$ and have to be extrapolated to $q^2=0$. For $T_p^{(-)}$, comparing (3.1a) and (3.6) with the first term in (3.5) indicates that the change in going to $q^2=0$ may be expected to be within 15%. For $T_p^{(+)}$, we examine this in more detail, using a model in which the higher-order terms are approximated by the contributions of the N^* resonances. In particular, the second-order terms are approximated by the N^* resonances with spins $\frac{1}{2}$ and $\frac{3}{2}$; there appear to be six such resonances, here denoted as N_i^*, $i=1, 2, \cdots 6$.[28] Among these, the contribution of the $(\frac{3}{2},\frac{3}{2})N_1^*$ is found to dominate.

In Table I are shown the contributions of the N_i^*, $i=1, \cdots 6$, to $\partial^2 T_p^{(+)}/\partial\nu^2$ for $\nu=0$, $q^2=0$ and for $\nu=0$, $q^2=\mu^2$. In obtaining the values for $q^2=0$, $\nu=0$, we have assumed that the factor giving the variation in the πNN^* couplings between $q^2=0$ and $q^2=\mu^2$ is roughly the same as the corresponding factor $K(0)$ for the πNN vertex, which is assumed to be close to unity. The main part of the variation in q^2 of $\partial^2 T_p^{(+)}/\partial\nu^2$ would then arise from the variation with q^2 of the kinematical factors in $T^{(+)}$.

It is seen from Table I that the sum of the N^* contributions gives $\partial^2 T_p^{(+)}/\partial\nu^2$ at $\nu=0$, $q^2=\mu^2$ correct to about 13% as compared with the value deduced from a dispersion relation using the experimental cross sections [see (3.4a)]; thus we may expect that the qualitative conclusions from our model will be reliable.

Further, the contribution of the resonances for $\nu=0$, $q^2=0$ is seen to be about 20% larger than that for $\nu=0$, $q^2=\mu^2$, if one ignores the factor $K^2(0)$. If $K^2(0)$ is less than unity, the difference would be less than 20%.

If one assumes that the variation with q^2 of the higher derivatives of $T_p^{(+)}$ (which could presumably be approximated by including the contributions of resonances with higher spin) is of the same order, then the above results suggest that terms of order ν^2 and higher in $T_p^{(+)}$ make an important contribution to $T^{(+)}(\mu,0)$ at $q^2=0$ also, and that the approximation (3.2a) (in which such terms are neglected) is not adequate. Similarly, for $T_p^{(-)}$, the qualitative conclusions stated above about the importance of the higher-order terms, for $q^2=\mu^2$, are expected to hold true for $q^2=0$ also.

In summary, it appears that at least for the isospin-symmetric S-wave πN scattering length, a good approximation cannot be obtained from the C.R.'s and PCAC alone; higher-order corrections, which must be evaluated from a model (or by using dispersion relations and the observed total cross sections), must be taken into account.

The approximation (3.2a) together with the assumption that $T_p^{(+)}(\mu,0,\mu^2,\mu^2)\approx T_p^{(+)}(\mu,0,0,0)/K^2(0)$, in the notation $T(\nu,t,q_1^2,q_2^2)$, would suggest that the scalar term in (3.1b) may be ignored. However, it is necessary to reexamine this, since higher-order terms have to be added to (3.2a) to give a good approximation for $T^{(+)}$, and further since slow variation with q^2 seems to be a valid assumption only for the nonresonant part of $T_p^{(+)}$.

Equation (3.1b) (for $q_1^2=q_2^2=0$) and Adler's consistency condition (for $q_1^2=\mu^2$, $q_2^2=0$),[29] respectively, give

$$(1/4\pi)T^{(+)}(0,0,0,0)=-[GK(0)/4\pi mg_A]h \qquad (3.8a)$$

and

$$(1/4\pi)T^{(+)}(0,0,0,\mu^2)=0\,, \qquad (3.8b)$$

while (3.4b) together with SW's estimates for $T^{(+)}(\mu,0,\mu^2,\mu^2)$ gives

$$(1/4\pi)T^{(+)}(0,0,\mu^2,\mu^2)\approx -0.12\,. \qquad (3.8c)$$

(3.8b) and (3.8c) show that $T^{(+)}$ at $\nu=0$ varies appreciably with q_1^2 and q_2^2 [the scale being provided by the experimental value of -0.01 for $(1/4\pi)T^{(+)}(\mu,0)$].

As the contributions of the low-lying N^* resonances[30] (especially the N_1^*) to $T^{(+)}$ vary quite rapidly with q^2

[26] For comparison we note that HW's estimates give $T^{(+)}(\mu,0)/4\pi\approx -0.0029$, while the 0-100-MeV and 0-350-MeV solutions of RWF give -0.0138 and 0.0264, respectively. The statements about the higher-order terms continue to hold when these estimates are used.

[27] Note that although the second term on the right of (3.4b) is small compared to the first term, the latter is almost completely cancelled upon adding the pole term and therefore the magnitude of the second term is important in determining the value of $T^{(+)}(\mu,0)$. That the higher-order terms in $T^{(+)}$ may be important is implicit in the work of Kawarabayashi and Wada (Ref. 20). However, their conclusions about the scalar term are different from ours. We note also that $\sigma^{(+)}$ is always positive, in contrast to $\sigma^{(-)}$, and there can be no cancellation among the higher-order terms in $T_p^{(+)}$.

[28] The resonances N_i^* are the ones listed in Appendix A. For their masses and widths, we have used the values quoted by A. H. Rosenfeld *et al.*, Tables from UCRL 8030 (revised), 1966 (unpublished). The coupling constants are given in Appendix A.

[29] S. L. Adler, Phys. Rev. **137**, B1022 (1965).

(note that it is only the term to the zeroth order in ν in the N_1^* contribution that varies rapidly with q^2), we separate them from (3.8); we also separate the nucleon pole contributions. For the nonpole, nonresonant parts $\bar{T}_p^{(+)}$ of $T^{(+)}$ at $\nu=0$, one then obtains the following (omitting terms of order μ^2/m^2):

$$\frac{1}{4\pi}\bar{T}_p^{(+)}(0,0,0,0)\approx-\frac{GK(0)}{4\pi mg_A}h+\frac{G^2}{4\pi m}K^2(0)\,, \qquad (3.9a)$$

$$\frac{1}{4\pi}\bar{T}_p^{(+)}(0,0,0,\mu^2)\approx 0.075+\frac{G^2}{4\pi m}K(0)\,, \qquad (3.9b)$$

$$\frac{1}{4\pi}\bar{T}_p^{(+)}(0,0,\mu^2,\mu^2)\approx 0.028+\frac{G^2}{4\pi m}\,. \qquad (3.9c)$$

Note that the larger part of the right-hand sides of (3.9), given by the second term,[31] is almost exactly cancelled by the N-pole term in $T^{(+)}$.

The first term on the right of (3.9a) may be taken to correspond to the exchange of simple $I=0$ states in πN scattering. A plausible assumption would be that in (3.9b) and (3.9c) also, the major part of the first term on the right arises from the exchange of these $I=0$ states.[32] To obtain a very rough order-of-magnitude estimate of h, we note that if the first terms on the right of Eq. (3.9) varied approximately linearly with q_1^2 and q_2^2, this would suggest that h is not too small and that it is negative

$$[-GK(0)h/4\pi mg_A\approx 0.12\,,\quad h\approx-0.89K^{-1}(0)]\,,$$

in contrast to some other estimates.[33] Further information is needed for a reliable estimate of the scalar term.

Contrary to the statement sometimes made, the assumption of the exact validity of PCAC, and hence of Adler's consistency condition, does not necessarily imply that this term is small.[34] In Sec. VI we shall make a rough estimate of this term by comparing the predictions for $a^{(+)}$ with the experimental estimates. The result [namely, $h\approx-0.5K(0)$] appears to support the qualitative indication obtained from (3.9) that h is negative and not too small.

IV. APPROXIMATION FOR THE STRUCTURE-DEPENDENT TERMS AND THE EXTRAPOLATION PROCEDURE

In electrodynamics it was shown by Low, by Gell-Mann and Goldberger, and by others that for a process involving photons, one may obtain, for terms of the lowest few orders in the photon energy, exact relations which are independent of the detailed structure of the matrix element and depend only on quantities like the charge and the magnetic moment.[35] Analogous results hold true for matrix elements involving axial-vector currents,[36] when the pions (defined in terms of the divergence of the axial-vector current) have zero external mass q^2. Examples of such (exact) results are the relations (3.1) for the πN scattering amplitude with $q^2=0$.

The extrapolation from pions with $q^2=0$ to physical pions with $q^2=\mu^2$ needs additional assumptions. From the preceding section it appears that even in considering the S-wave scattering lengths, it is inadequate to keep only the terms of the zeroth and first orders in ν (which are the terms that can be obtained exactly from the current algebra and PCAC), at least for the isospin-symmetric amplitude. Detailed assumptions must be made about the dynamics in order to evaluate the higher-order structure-dependent contributions to the amplitude.

In this paper we are interested in determining the πN scattering amplitude to terms of the second order in ν; we shall obtain expressions for the P-wave scattering lengths and the S and P phase shifts at low energies. We now need to know the second-order terms in the amplitude $q_{1\mu}q_{2\nu}M_{AA}{}^{\mu\nu}$ on the left-hand side of (2.1). These depend on the detailed structure of the amplitude $M_{AA}{}^{\mu\nu}$; as the latter can be evaluated only approximately, our results will be approximate.

Our basic assumption will be that the terms to the second order in ν in $q_{1\mu}q_{2\nu}M_{AA}{}^{\mu\nu}$ are given to a good approximation by the contributions of the direct and exchanged poles and resonances to the amplitude $M_{AA}{}^{\mu\nu}$. This will give expressions for the amplitudes $D^{(\pm)}$ and $R^{(\pm)}$ [see Eqs. (2.9)]. In order to evaluate the right-hand sides of Eq. (2.11), we shall need to know the

[30] The sum of the N_1^*, N_2^*, and N_3^* contributions to (3.8a), (3.8b), and (3.8c) are found to be 0, -0.075, and -0.148. (Here the N_2^* contribution has been evaluated using a pseudovector coupling.)

[31] This term, which gives the larger part of $T_p^{(+)}$ at $\nu=0$, probably arises from the short-range forces; it is presumably what necessitates the subtraction in the dispersion relation for $T^{(+)}$.

[32] Note, however, that if σ exchange dominated all these terms, then (3.9) would suggest a substantial variation of $g_{\sigma\pi\pi}$ with q_1^2 and q_2^2. This is perhaps an indication that σ dominance is not an adequate approximation for these terms, and in particular for the scalar term h.

[33] Note that if one assumed that (3.8c) differed from (3.8a) essentially only by a factor $K^2(0)$, as was assumed by Kawarabayashi and Wada (Ref. 20), one would obtain a positive value of h: $h\approx 0.7$. (The quantity M_{eff} in Ref. 20 is equal to $\mu_\pi^2 h$.) However, as noted in Ref. 20, the analogous assumption does not describe correctly the relation of (3.8b) to (3.8a) and (3.8c). We have suggested that this is so because (3.8) includes contributions such as those from the N^* resonances which do not vary slowly with q^2.

[34] The argument sometimes offered that the presence of a term (in the scattering amplitude) varying rapidly with q^2 would contradict the spirit of PCAC seems to be too restrictive an interpretation of PCAC. We believe that what the presence of such terms in the amplitude requires is that the assumption of gentle extrapolation used in conjunction with PCAC should be applied only after separating out the part varying rapidly with q^2, as we suggest in Sec. IV. That such terms can be present is seen when one compares Eq. (3.8b) (derived by assuming PCAC to be exact) with (3.8c) (derived from experiment). We further remark that even in a σ model, the scalar term would be small only if $\mu_\pi^2 m_\sigma^{-2} g_{\sigma NN}$ is small, for which there is no estimate.

[35] F. E. Low, Phys. Rev. **96**, 1428 (1954); Phys. Rev. **110**, 974 (1958); M. Gell-Mann and M. L. Goldberger, *ibid.* **96**, 1433 (1954); E. Kazes, Nuovo Cimento **13**, 1226 (1959).

[36] The extension of the results of Low (Ref. 35) to matrix elements involving axial-vector currents has been discussed by S. L. Adler and Y. Dothan [Phys. Rev. **151**, 1267 (1966)].

parameters describing the contributions of the resonances and the values of the form factors $F_1{}^V(t)$ and $F_2{}^V(t)$ and their derivatives $F_1{}^{V\prime}(t)$ and $F_2{}^{V\prime}(t)$ at $t=0$.

Contributions to $M_{AA}{}^{\mu\nu}$ that are of the second order in ν are obtained from N^* resonances with spins $\frac{1}{2}$ and $\frac{3}{2}$ (in the s and u channels) and the ρ meson and a possible σ meson in the t channel. At $q^2=0$, N^* resonances with spins higher than $\frac{3}{2}$ do not contribute to terms of second order in ν in the amplitude. Among the low-lying N^* resonances, the most significant contributions are expected to be those of the $(\frac{3}{2},\frac{3}{2}+)$ $N_1{}^*$, the $(\frac{1}{2},\frac{1}{2}+)$ Roper resonance $N_2{}^*$, and the $(\frac{1}{2},\frac{3}{2}-)$ $N_3{}^*$ (see Ref. 28 and Appendix A). The contributions of the $N_4{}^*$, $N_5{}^*$, and $N_6{}^*$ are ignored, for simplicity; we expect that these are small and that neglecting them will not introduce much error.[37]

Our procedure for obtaining expressions for the partial-wave amplitudes, starting with Eqs. (2.9) (which hold for $q_1{}^2=q_2{}^2=0$) will be as follows.

We first subtract from each side of each of the Eqs. (2.9) the contribution of the nucleon poles (in the sense of dispersion theory) in s and u. We then separate out the contribution of each N^* resonance to the amplitudes $A^{(\pm)}$ and $B^{(\pm)}$ on the left-hand sides of Eqs. (2.9) and subtract these from each side of these equations, using the Goldberger-Treiman relation relating the πNN^* coupling to the $N^*\to N$ axial-vector renormalization constants.

The contributions to $q_{1\mu}q_{2\nu}M_{AA}{}^{\mu\nu}$ from ρ and σ mesons in the t channel at $q_1{}^2=0$, $q_2{}^2=0$ may be similarly separated out. These vanish at $t=0$,[38] and the contributions from such states to the partial-wave amplitudes arise only through their contributions to $f_1{}'(0)$ and $f_2{}'(0)$ (and the higher derivatives). [See Eq. (2.11).]

The relation for the $\sigma\pi\pi$ coupling that enters here is discussed in Appendix C.

The equations obtained after subtracting the N poles and the N^*, ρ and σ terms involve the nonpole, nonresonant parts of $A^{(\pm)}$, etc., which we denote by $\bar{A}^{(\pm)}$, $\bar{B}^{(\pm)}$, $\bar{D}^{(\pm)}$, and $\bar{R}^{(\pm)}$. We assume that to a good approximation, the amplitudes $\bar{A}^{(\pm)}$ and $\bar{B}^{(\pm)}$ for $q^2=\mu^2$, may be obtained from the amplitudes for $q^2=0$ by dividing the latter by $K^2(0)$:

$$\bar{A}^{(\pm)}(s,t,\mu^2,\mu^2)\approx\bar{A}^{(\pm)}(s,t,0,0)/K^2(0)\,,\text{ etc.,}\qquad(4.1)$$

where we have used the notation $\bar{A}^{(\pm)}(s,t,q_1{}^2,q_2{}^2)$. We now add to $\bar{A}^{(\pm)}$ and $\bar{B}^{(\pm)}$ at $q^2=\mu^2$ the exact pole or resonant contributions of the N, N^*, ρ, and σ, and use the resulting amplitudes $A^{(\pm)}$ and $B^{(\pm)}$ to construct the partial-wave amplitudes.

In the above procedure for obtaining relations for $A^{(\pm)}$ and $B^{(\pm)}$ at $q_1{}^2=q_2{}^2=\mu^2$, we are applying the current C.R.'s and the PCAC hypothesis to the nonpole, nonresonant parts of the amplitudes rather than the total amplitudes.

This is important, firstly, in connection with the extrapolation in q^2. Thus the contributions of the spin-$\frac{3}{2}$ N^* resonances to the πN scattering amplitude, particularly to the isospin-symmetric part of the amplitude, are found to vary rapidly with q^2. (This is true of the terms of zeroth order in ν in these contributions.) For instance, at the threshold $s=(m+\mu)^2$ and at very low energies, even the sign of the $(\frac{3}{2},\frac{3}{2}+)$ N^* contribution is different for $q^2=0$ and $q^2=\mu^2$. Thus when these resonances make significant contributions, simple assumptions such as (4.1) will not be adequate to take into account the variation of the whole amplitude between $q^2=0$ and $q^2=\mu^2$. They may, however, be expected to be better approximations for the nonresonant parts of the amplitudes.

Secondly, the procedure of adding the exact contribution of the resonances to the extrapolated nonresonant amplitude gives a method of obtaining an amplitude with a nonzero imaginary part (if the widths of the resonances are taken into account); this approximation for the imaginary part would be a good approximation if the resonances dominate the imaginary part.

The procedure outlined above is not unique; an alternative method, based on the same assumptions, would be to construct the partial-wave amplitudes for $q^2=0$ and extrapolate them, taking care to ensure the correct threshold behavior. The differences between this extrapolation and the one we have outlined above are discussed in Appendix B.

V. EVALUATION OF THE VARIOUS CONTRIBUTIONS

The contributions to the partial-wave amplitudes $f_{0+}{}^{(\pm)}$ and $f_{1\pm}{}^{(\pm)}$ from the nucleon Born term in $D^{(\pm)}$ and $R^{(\pm)}$ and the terms involving the form factors $F_{1,2}{}^V(t)$ may be evaluated by starting with (2.9) at $q_1{}^2=q_2{}^2=0$, extrapolating as discussed in Sec. IV, finding the contributions to f_1 and f_2 [see (A3)], and substituting in (2.11). We give below the contributions to the P-wave scattering lengths from the nucleon and C.R. terms:

$$a_{1-}{}^{(+)}(N)\approx 2a_{1-}{}^{(+)}(N)\approx -2a_{1+}{}^{(+)}(N)$$
$$\approx 2a_{1+}{}^{(-)}(N)\qquad(5.1\text{a})$$

$$\approx\frac{-G^2}{12\pi m\mu(m+\mu)}\left(1-\frac{\mu}{2m}\right)^{-2},\qquad(5.1\text{b})$$

---

[37] In connection with this we note that when one tries to saturate the Adler-Weisberger sum rule with the contributions of the N^* resonances, one finds that keeping just the $N_1{}^*$, $N_2{}^*$, and $N_3{}^*$ gives a good approximation. The author is indebted to Dr. H. Harari for information regarding his results on this.

[38] This may be shown by writing the coupling of two axial-vector currents to the σ meson in the form (C17) and the corresponding coupling of the ρ meson in the form

$$\begin{aligned}M_{AA\rho}{}^{\mu\nu\alpha}=&f_1(t)g^{\mu\nu}Q^\alpha+f_2(t)(Q^\mu g^{\nu\alpha}+Q^\nu g^{\mu\alpha})\\&+[f_3(t)q_2{}^\mu q_1{}^\mu+f_4(t)q_1{}^\mu q_1{}^\nu]Q^\alpha+f_5(t)g^{\alpha\mu}q_2{}^\nu\\&+f_6(t)g^{\alpha\nu}q_1{}^\mu+f_7(t)q_1{}^\mu(Q^\alpha k^\nu+Q^\nu k^\alpha)\\&+f_8(t)q_2{}^\nu(Q^\alpha k^\mu+Q^\mu k^\alpha)+f_9(t)\,\epsilon^{\mu\nu\rho\beta}Q_\rho\epsilon^{\beta\lambda\sigma\alpha}Q_\lambda k_\sigma,\end{aligned}$$

and taking the limit $q_1{}^2=0$, $q_2{}^2=0$, $t=0$. The couplings $M_{AA\rho}{}^{\mu\nu\alpha}$ and $M_{AA\sigma}{}^{\mu\nu}$ given here are free of kinematical singularities in $q_1{}^2$, $q_2{}^2$, and $p_\rho{}^2$ (or $p_\sigma{}^2$).

$$a_{1-}^{(-)}(\text{C.R.})=\frac{G^2}{16\pi g_A{}^2m^2(m+\mu)}\times\left[\left(1+\frac{\mu}{2m}\right)\{F_1{}^V(0)+2mF_2{}^V(0)\}-(\tfrac{2}{3}m-\mu)F_2{}^V(0)+\tfrac{4}{3}m\mu F_1{}^{V\prime}(0)\right], \quad (5.1\text{c})$$

$$a_{1+}^{(-)}(\text{C.R.})=\frac{-G^2}{6\pi g_A{}^2m^2(m+\mu)}\times\{\tfrac{1}{4}F_2{}^V(0)-\tfrac{1}{2}\mu F_1{}^{V\prime}(0)\}. \quad (5.1\text{d})$$

[In (5.1a) and (5.1b) we have neglected terms of higher order in μ/m.] The contributions of the scalar term $H(t)$ are obtained in a similar manner.

To evaluate the contribution of the $(\frac{3}{2},\frac{3}{2}^+)$ $N_1{}^*$, we assume the following form for the matrix element of the axial-vector current between a nucleon and a $\frac{3}{2}^+$ N^*:

$$\langle N^*(p_f)|\mathfrak{A}^\mu(0)|N(p_i)\rangle=-i\left[\frac{Mm_i}{E_fE_i}\right]^{1/2}\bar{\psi}_\rho(p_f)\left\{\mathcal{G}_A(q^2)g^{\mu\rho}+\mathfrak{F}_A(q^2)r^\mu q^\rho+\mathcal{H}_A(q^2)P'^\mu q^\rho+\mathcal{L}_A(q^2)\left(\frac{q^\mu q^\rho}{q^2}-g^{\mu\rho}\right)\right\}u(p_i). \quad (5.2)$$

Here, m and M are the masses of the nucleon and the N^*, and we have defined

$$\begin{aligned}P&=\tfrac{1}{2}(p_i+p_f), & q&=(p_f-p_i),\\ P'&=P-(P\cdot q)q/q^2, & r^\mu&=\epsilon^{\mu\nu\rho\sigma}\gamma_\nu P_\rho q_\sigma\gamma_5.\end{aligned} \quad (5.3)$$

In choosing the form (5.2), our assumption is that the model defined by the coupling (5.2) (and the propagator given below) gives a good description of the off-shell contribution of the N^* (see Ref. 9). For the propagator of the spin-$\frac{3}{2}^+$ N^* in the s channel, we assume the following form[39]:

$$(-i)\times\frac{1}{3}\left[3g^{\mu\nu}-\frac{2P^\mu P^\nu}{M^2}-\gamma^\mu\gamma^\nu+\frac{\gamma\cdot P}{M^2}(P^\mu\gamma^\nu-\gamma^\mu P^\nu)\right]\times\frac{(\gamma\cdot P+M)}{P^2-M^2+i\gamma(k)k^3}, \quad (5.4)$$

where

$$\gamma(k)=g^2(E+m)(W+M)(24\pi W)^{-1}. \quad (5.5)$$

Here we have chosen a complex propagator in order to take into account the finite width of the resonance. The function $\gamma(k)$ has been determined by the requirement that the contribution of the $\frac{3}{2}^+$ N^* to its own partial wave (i.e., the $P_{3/2}$ partial wave) be unitary and have the correct threshold behavior.[40] It is found that replacing $\gamma(k)$ in (5.4) by its (constant) value at the resonance $W=M$ does not make much difference at low energies. However, including a finite width for the $N_1{}^*$ does have an appreciable effect. For a spin-$\frac{3}{2}$ N^* in the u channel, we take a real propagator, obtained by omitting the imaginary part of the denominator in (5.4).

We may now evaluate the $N_1{}^*$ contributions to the partial-wave amplitudes and scattering lengths in terms of the $\pi NN_1{}^*$ coupling constant g_1 [see (A7)], following the procedure outlined in Sec. IV, and using the relation $\mathcal{G}_A(0)=C_\pi\mu_\pi{}^{-2}g_1K(0)$. The $N_1{}^*$ contributions to the P-wave scattering lengths are found to be the following:

$$a_{1\pm}^{(+)}=\tfrac{2}{3}[a_{1\pm}^{(s)}+a_{1\pm}^{(u)}],\quad a_{1\pm}^{(-)}=\tfrac{1}{3}[-a_{1\pm}^{(s)}+a_{1\pm}^{(u)}], \quad (5.6\text{a})$$

where

$$a_{1-}^{(s)}=\frac{g_1{}^2}{48\pi m(m+\mu)}[(m+\mu)^2-M^2]^{-1}\times\{(M+m)(2m+\mu)^2-2M\mu^2-4m\mu M^{-1}(m+\mu)^2-4m^2(M+m+\mu)+2\mu^2M^{-2}(m+\mu-M)[M^2-(m+\mu)^2]\}, \quad (5.6\text{b})$$

$$a_{1-}^{(u)}=\frac{g_1{}^2}{24\pi(m+\mu)}[M^2-(m-\mu)^2]^{-1}\times\left\{2\left(1+\frac{2m\mu}{3M^2}\right)(M+m)(m+\mu)+\frac{2\mu}{M}(m^2-\mu^2)+\frac{\mu^2}{mM^2}(M+3m+\mu)[M^2-(m-\mu)^2]+\frac{2m}{3}(M+m-3\mu)+\frac{8m^2\mu}{3M^2}(m-\mu)\right\}, \quad (5.6\text{c})$$

$$a_{1+}^{(s)}=\frac{g_1{}^2}{12\pi(m+\mu)}(M-m-\mu)^{-1}, \quad (5.6\text{d})$$

$$a_{1+}^{(u)}=\frac{g_1{}^2m}{18\pi(m+\mu)}[M^2-(m-\mu)^2]^{-1}\left\{\frac{1}{2}(M+m-3\mu)+\frac{\mu}{M}\left(1+\frac{2m}{M}\right)(m+\mu)\right\}. \quad (5.6\text{e})$$

[39] With a covariant propagator, the off-shell (nonpole) part of the $N_1{}^*$ contribution (in the s channel) contributes to partial waves other than the $P_{3/2}$ also. A different model would be one in which only the mass-shell part of the $N_1{}^*$ contribution (which is pure $P_{3/2}$) is kept. The N^* resonance in the u channel, of course, contributes to all partial waves in the s channel: this is more easily evaluated in terms of a covariant propagator. The form (5.4) of the propagator corresponds to the projection operator as given, for instance, by Y. Takahashi and H. Umezawa, Progr. Theoret. Phys. (Kyoto) 9, 14 (1953); see also H. Umezawa, *Quantum Theory* (North-Holland Publishing Company, Amsterdam, 1956).

[40] Note that $\text{Re}f_{l\pm}=(\sin2\delta_{l\pm})/2k\sim k^{2l}$, $\text{Im}f_{l\pm}=(\sin^2\delta_{l\pm})/k\sim k^{4l+1}$ as $k\to0$, if we assume that $\delta_{l\pm}\sim k^{2l+1}$ as $k\to0$.

TABLE II. Contributions to S-wave scattering lengths and real parts of S-wave amplitudes at low energies.[a]

Energy (in MeV)	Amplitude	Nucleon Born term	C.R. term	N_1^*	N_2^*	N_3^*	Total of N, C.R., and N^* terms	Scalar term	Total (including scalar term)	Experimental estimates: Estimates of RWF	Experimental estimates: Corrected estimates
0	$a_{0+}^{(+)}$	-0.0105	0	-0.06	-0.0086	0.0037	-0.067	$+0.058$	-0.009	0.023	-0.009
0	$a_{0+}^{(-)}$	0.0008	0.099	0.0012	0	-0.0009	0.1	0	0.1	0.086	0.093
20	$\mathrm{Re}f_{0+}^{(+)}$	-0.0057	0	-0.08	-0.001	0.0041	-0.083	0.054	-0.029	-0.0016	-0.034
20	$\mathrm{Re}f_{0+}^{(-)}$	-0.0054	0.101	0.002	-6×10^{-5}	-0.001	0.096	0	0.096	0.082	0.088
58	$\mathrm{Re}f_{0+}^{(+)}$	0.0046	0	-0.146	-0.001	0.0056	-0.136	0.046	-0.09	-0.02	-0.052
58	$\mathrm{Re}f_{0+}^{(-)}$	-0.021	0.1	0.035	-4×10^{-4}	-0.0022	0.112	0	0.112	0.08	0.087
98	$\mathrm{Re}f_{0+}^{(+)}$	0.0186	0	-0.103	-0.0008	0.0068	-0.078	0.037	-0.041	-0.034	-0.066
98	$\mathrm{Re}f_{0+}^{(-)}$	-0.04	0.092	0.007	-0.0009	-0.0033	0.055	0	0.055	0.08	0.087

[a] The corrected experimental estimates for $a_{0+}^{(\pm)}$ and $f_{0+}^{(\pm)}$ have been obtained in the manner discussed in Sec. VI.

For the $D_{3/2}\,N_3^*$, we proceed similarly, replacing $\gamma(k)k^3$ in (5.4) by $\bar{\gamma}(k)k^5$, where

$$\bar{\gamma}(k)=g_3^2(W+M)(E+m)^{-1}(24\pi W)^{-1}. \qquad (5.7)$$

The results for the P-wave scattering lengths may be obtained by replacing $M_1\to -M_3$, $g_1\to g_3$ in (5.6).

A procedure similar to the treatment of the nucleon Born term gives the contribution of the $\frac{1}{2}^+ N_2^*$; we use the relation

$$\mathcal{G}_{A2}(0)=(2/\sqrt{3})g_2\mu_\pi^{-2}C_\pi K(0)(M+m)^{-1}, \qquad (5.8)$$

where $\mathcal{G}_{A2}$ is defined by (A6). In evaluating the N^* contributions above, we have assumed $g_i(0)/g_i(\mu^2)\approx K(0)$, $i=1, 2, 3$.

Finally, we may write down the contributions of a ρ-meson pole and a possible σ-meson pole in $M_{AA}^{\mu\nu(\pm)}$ (or equivalently, $D^{(\pm)}$ and $R^{(\pm)}$). An important feature of these is that the contributions to the S-wave scattering lengths (i.e., to $a_{0+}^{(-)}$ from the ρ meson and to $a_{0+}^{(+)}$ from the σ meson) are zero. Further, the contributions to the P-wave scattering lengths and the S- and P-wave amplitudes are proportional to $f_{\rho\pi\pi}'(0)$ and $g_{\sigma\pi\pi}'(0)$ [where $f_{\rho\pi\pi}'(0)\equiv\partial f_{\rho\pi\pi}(t)/\partial t$ at $t=0$, etc.]. These latter quantities are not known; however, a rough estimate of $f_{\rho\pi\pi}'(0)$ suggests that it is small.

VI. NUMERICAL RESULTS

We have evaluated the various contributions to the scattering lengths and to the partial-wave amplitudes $f_{0+}^{(\pm)}$, $f_{1-}^{(\pm)}$ and $f_{1+}^{(\pm)}$ at values of the pion laboratory kinetic energy of 20, 58, and 98 MeV. Our low-energy approximations are not expected to be good approximations at energies much higher than 50 MeV; however, the values of the amplitudes at higher values of the energy would indicate at what energy an appreciable discrepancy sets in.

In making the numerical estimates, we have used the values of the πNN^* coupling constants as obtained from the observed widths, and the values of $F_{1,2}^V(0)$ and $F_{1,2}^{V\prime}(0)$ as obtained from the results for the nucleon form factors given by Chan *et al.*,[41] which give $F_1^{V\prime}(0)\approx0.147$, $2mF_2^{V\prime}(0)\approx0.12$. As the terms in the C.R. contribution involving $F_{1,2}^{V\prime}(0)$ are small compared to those involving $F_{1,2}^V(0)$, the uncertainty in the experimental values of $F_{1,2}^{V\prime}(0)$ does not seriously affect the results.

For the experimental values of the πN phase shifts, we have taken the 0–350-MeV solution of RWF.[25] This solution of RWF was obtained by an over-all fit to the data between 0 and 350 MeV and is therefore not expected to give accurate values of the phase shifts at all energies in this range. As noted in Ref. 4, the S-wave scattering length $a_{0+}^{(+)}$ given by this solution, $a_{0+}^{(+)}=0.023$, differs considerably from that obtained, for instance, by SW,[23] whose estimates for the S-wave scattering lengths we believe to be more reliable. This suggests that the amplitudes $f_{0+}^{(+)}$ at low energies, obtained from the phase shifts of RWF, are not reliable. To obtain more reliable estimates of these amplitudes at low energies, we suggest correcting the phase shifts of RWF by replacing the scattering-length parameter in their effective-range formula for the S-wave phase shifts by more reliable values; we have used the values of SW.[23,42] We believe that at low energies, the values of the S-wave phase shifts so corrected are more reliable.[43] We have given both the uncorrected and the corrected experimental values of $f_{0+}^{(\pm)}$; they differ roughly by a constant, as only the S-wave scattering-length parameter has been altered.

Among the P-wave scattering lengths, for $a_{1+}^{(+)}$ and $a_{1+}^{(-)}$, the estimates of RWF agree well with those of other workers. For $a_{1-}^{(+)}$ and $a_{1-}^{(-)}$, the estimates of

[41] L. H. Chan, K. W. Chen, J. R. Dunning, Jr., N. F. Ramsey, J. K. Walker, and R. Wilson, Phys. Rev. **141**, 1298 (1966).

[42] For comparison, we note that the 0–100-MeV solution of RWF gives $a_{0+}^{(+)}\approx-0.012$, $a_{0+}^{(-)}\approx0.084$.

[43] We note that in the effective-range parametrization used by RWF (Ref. 25), the form assumed for $\tan\delta_{l\pm}$ includes both even and odd powers of the 3-momentum k. On the other hand, in an effective-range approximation derived from a relativistic theory, one would obtain an expression for $\tan\delta_{l\pm}$ with only odd powers of k (or equivalently, of $f_{l\pm}$ with only even powers of k): $\tan\delta_l=k^{2l+1}[b_0+b_1k^2+b_2k^4+\cdots]$. This introduces some uncertainty into an assessment of the results of RWF.

TABLE III. Contributions to P-wave scattering lengths.[a]

	Nucleon Born term	C.R. term	N_1^*	N_2^*	N_3^*	Total of N, C.R., and N^* terms	Scalar term	Total	Experimental estimates SW	HW	RWF
$a_{1-}^{(+)}$	-0.108	0	0.038	0.003	-0.002	-0.069	0.0047	-0.064	-0.055	-0.059	-0.069
$a_{1-}^{(-)}$	-0.055	0.041	0.02	0.004	0.0023	0.0123	0	0.0123	-0.013	-0.021	-0.016
$a_{1+}^{(+)}$	0.055	0	0.074	0.0012	-0.0021	0.128	0.005	0.133	0.136	0.134	0.137
$a_{1+}^{(-)}$	-0.055	0.0006	-0.028	-0.0012	0.0006	-0.083	0	-0.083	-0.081	-0.081	-0.08

[a] Among their estimates, RWF (Ref. 28) do not directly quote a value for the $(\frac{3}{2},\frac{3}{2})$ P-wave scattering length a_{33}. We have taken $a_{33}\approx 0.217$ in obtaining the value of $a_{1+}^{(\pm)}$ quoted in the last column. (We note that different workers agree closely in their estimate for a_{33}.)

different workers differ appreciably; however, it is not clear whether any one set of estimates is preferable. We have taken the P-wave phase shifts as quoted by RWF (in their 0–350-MeV solution).

We first examine how the S-wave scattering lengths and low-energy phase shifts given by the sum of the various contributions discussed in the last section compare with the experimental estimates.

In Table II we have listed the various contributions to the amplitudes $f_{0+}^{(\pm)}$ at 0-, 20-, 58-, and 98-MeV pion laboratory kinetic energy, as well as the experimental estimates for these.

Looking at the calculated values of the different contributions to $f_{0+}^{(\pm)}$, we see from Table II that at threshold, for the isospin-antisymmetric amplitude $f_{0+}^{(-)}$, contributions from the N^* terms are very small, whereas for the isospin-symmetric amplitude at threshold these contributions are important.

The contributions of the resonances, when added to those of the nucleon Born term and the vector commutator, give a total contribution of -0.067 to the scattering length $a_{0+}^{(+)}$, which may be compared with the experimental estimate of -0.009 given by Samaranayake and Woolcock.[23] The discrepancy is considerable; thus the nucleon Born term and the N^* resonances do not give a good approximation for $a_{0+}^{(+)}$, or more generally for the amplitude $f_{0+}^{(+)}$.

However, we have seen in Sec. III that the contributions of the lowest few spin-$\frac{1}{2}$ and spin-$\frac{3}{2}$ resonances give the second-order terms (in ν) in $T_p^{(+)}$ correct to about 13%. (One may expect that the higher-order terms may be similarly approximated by higher-spin resonances.) It therefore seems to be reasonable to assume that the discrepancy between the resonant approximation to $f_{0+}^{(+)}$ and the full partial-wave amplitude $f_{0+}^{(+)}$ at low energies arises mainly from the term of zeroth order in ν in $T^{(+)}$. If this assumption is correct, then in order to correct the discrepancy in $a_{0+}^{(+)}$, one would require a term in the amplitude that would give an appreciable contribution to $a_{0+}^{(+)}$ (and $f_{0+}^{(+)}$ at low energies) but considerably smaller contributions to $a_{1+}^{(+)}$ (and $f_{1+}^{(+)}$). Such a contribution to $T^{(+)}$ may be obtained from the scalar term H [see Eq. (2.9a)].[44] [Note that a σ-exchange term in $M_{AA}^{\mu\nu}$ would not be adequate, as it does not contribute to $a_{0+}^{(+)}$ (see Sec. V).]

The contributions of the scalar term are described by two parameters, $H(0)$ and $H'(0)$. We estimate $H(0)$ by requiring that the scalar term should correct the discrepancy in $a_{0+}^{(+)}$. This gives

$$h\equiv\mu_\pi^2 H(0)\approx -0.5K(0)\,,\quad -Gh/4\pi m g_A K(0)\approx 0.064\,. \tag{6.1}$$

Looking at Eqs. (3.9), a contribution of the order of 0.06 arising from the scalar term appears to be not unreasonable. [Note that the estimate of h given in (6.1) is again opposite in sign to the estimate suggested by Kawarabayashi and Wada.[20,33]]

The parameter $H'(0)$ is expected to be smaller in magnitude than $H(0)$; we have chosen a value that gives a reasonable fit to $a_{1+}^{(+)}$:

$$H'(0)\approx -0.064\,. \tag{6.2}$$

Comparing the (corrected) experimental values of $f_{0+}^{(-)}$ with the theoretical values, we see that there is good agreement at 0 and 20 MeV and a discrepancy of about 25% at 58 MeV. At 98 MeV, the energy is probably too high for simple low-energy approximations to hold; we see that the calculated value becomes considerably smaller than the experimental value.

For $f_{0+}^{(+)}$, the amplitude at threshold was fitted to the experimental value; at 20 MeV there is reasonable agreement, while at 58 MeV, there is an appreciable discrepancy. However, the inclusion of the scalar term has made the discrepancy smaller.

We now turn to the P waves. In Tables III and IV are shown the different contributions to the P-wave scattering lengths $a_{1+}^{(\pm)}$ and amplitudes $f_{1+}^{(\pm)}$.

We first note that the different experimental estimates for $a_{1+}^{(+)}$ and $a_{1+}^{(-)}$ agree better with one another than those for $a_{1-}^{(+)}$ and $a_{1-}^{(-)}$.

For $a_{1+}^{(-)}$ and $f_{1+}^{(-)}$, the calculated values, which are dominated by the N and N_1^* contributions, agree well with the experimental values. The choice of the parame-

[44] The scalar term on the right of (2.9a) would presumably help to reproduce correctly the $I=0$ exchange part of $A^{(+)}$ on the left of (2.9a). If an $I=0$ scalar meson (σ) exists, it may be expected to contribute to the scalar term and to the corresponding part of $A^{(+)}$. The existence of an $I=0$ exchange contribution was also suggested by Schnitzer (Ref. 12) from a study of the S-wave effective ranges. However, he did not examine the nature of this contribution. We note that σ exchange in $M_{AA}^{\mu\nu}$ does not seem to be adequate, as it does not contribute to the term of zeroth order in ν in $T^{(+)}$; our analysis suggests that such a contribution is required. (We further suggest that an important part of it is provided by the scalar term.)

TABLE IV. Contributions to the real parts of the low-energy P-wave amplitudes.

	Nucleon Born term	C.R. term	N_1^*	N_2^*	N_3^*	Total of N, C.R., and N^* terms	Scalar term	Total	Experimental values (RWF)
(i) 20 MeV									
$\mathrm{Re} f_{1-}^{(+)}$	-0.02	0	0.008	0.0007	-0.0005	-0.012	0.001	-0.011	-0.0127
$\mathrm{Re} f_{1-}^{(-)}$	-0.01	0.0086	0.004	0.001	0.0006	0.0042	0	0.0042	-0.0036
$\mathrm{Re} f_{1+}^{(+)}$	0.01	0	0.017	0.0003	-0.0005	0.027	0.011	0.038	0.036
$\mathrm{Re} f_{1+}^{(-)}$	-0.01	0.0004	-0.007	-0.0003	-0.0001	-0.017	0	-0.017	-0.019
(ii) 58 MeV									
$\mathrm{Re} f_{1-}^{(+)}$	-0.061	0	0.02	0.0026	-0.0014	-0.039	0.0031	-0.036	-0.026
$\mathrm{Re} f_{1-}^{(-)}$	-0.024	0.029	0.015	0.0036	0.002	0.026	0	0.026	-0.0033
$\mathrm{Re} f_{1+}^{(+)}$	0.023	0	0.068	0.0008	-0.0014	0.091	0.0334	0.124	0.106
$\mathrm{Re} f_{1+}^{(-)}$	-0.023	0.0023	-0.029	-0.0008	-0.0004	-0.051	0	-0.051	-0.052
(iii) 98 MeV									
$\mathrm{Re} f_{1-}^{(+)}$	-0.091	0	0.039	0.005	-0.0024	-0.049	0.0057	-0.043	-0.035
$\mathrm{Re} f_{1-}^{(-)}$	0.052	0.048	0.023	0.007	0.0034	0.133	0	0.133	-0.003
$\mathrm{Re} f_{1+}^{(+)}$	0.032	0	0.156	0.0013	-0.0025	0.186	0.058	0.244	0.195
$\mathrm{Re} f_{1+}^{(-)}$	-0.032	0.008	-0.069	-0.0013	-0.0007	-0.095	0	-0.095	-0.105

ter $H'(0)$ in the scalar term was made so as to give a reasonable fit to $a_{1+}^{(+)}$. For $f_{1+}^{(+)}$ at 20 MeV, there is good agreement; at the higher values of the energy, there is still agreement to about 20%.

For $a_{1-}^{(+)}$, the experimental estimates of different workers differ appreciably; the theoretical estimate lies about halfway between the estimates of RWF and HW and differs from these by 7–10%. (Note that the contribution of the scalar term decreases the magnitude of $a_{1-}^{(+)}$.) For $f_{1-}^{(+)}$ at 20 MeV, again the agreement is reasonable; at higher values of the energy, the calculated values are larger in magnitude than the experimental estimates.

For $a_{1-}^{(-)}$ and $f_{1-}^{(-)}$, there is a discrepancy between the theoretical and experimental estimates. The reason for this disagreement is not clear. To correct it, one would look for a contribution to $T^{(-)}$ that gives a negative contribution to $a_{1-}^{(-)}$ and $f_{1-}^{(-)}$ but does not alter $f_{0+}^{(-)}$ appreciably.

Such a contribution could arise from the ρ-meson pole in $M_{AA}^{\mu\nu}$. However, a rough estimate of this contribution to $a_{1-}^{(-)}$ and $f_{1-}^{(-)}$, suggests that it is too small.

The discrepancy in $a_{1-}^{(-)}$ and $f_{1-}^{(-)}$ would suggest that there is a significant contribution to $T^{(-)}$ that is not taken into account by the simple contributions considered here.

We have also evaluated the contribution to the imaginary parts of the amplitudes arising from the N^* resonances in the direct (or s) channel, using a complex propagator for the resonances as discussed in Sec. IV. Only the $(\frac{3}{2},\frac{3}{2})$ N_1^* makes a significant contribution to the imaginary part at low energies. However, it is found that this simple picture does not describe the imaginary parts of the amplitudes adequately. As the problem does not involve the current algebra or PCAC, we do not discuss it further here.

We finally briefly compare the results in this paper with the approximate low-energy theorems for the P-wave scattering lengths given by us in earlier work.[8,9] The low-energy theorems given there were for the combinations $(a_{1-}^{(+)}-a_{1+}^{(+)})$ and $(a_{1-}^{(-)}-a_{1+}^{(-)})$ of the P-wave scattering lengths.[45] Expressions were obtained for these in terms of the contributions of the nucleon Born term, the commutator term, the structure-dependent terms as approximated by the low-lying N^* resonances, and the quantities $\Delta A^{(+)}$ and $\Delta B_p^{(-)}$ defined by

$$\Delta A^{(+)}=A^{(+)}(\mu,0,\mu^2,\mu^2)-A^{(+)}(0,0,\mu^2,\mu^2) \qquad (6.3)$$

and a similar expression for $\Delta B_p^{(-)}$.

The approximations and extrapolation used in this paper are somewhat different from those used in Ref. 9. There, one first wrote down relations for $A^{(\pm)}$ and $B^{(\pm)}$ at $\nu=0$, $t=0$, $q_1^2=q_2^2=0$, evaluated the N^* contributions (to the terms of order ν^2) at this point, extrapolated the amplitudes to $q^2=\mu^2$, and used dispersion relations to obtain $\Delta A^{(+)}$ and $\Delta B_p^{(-)}$, and thence $A^{(+)}$ and $B^{(-)}$, at the physical threshold. The main difference is in the last step. As $\Delta A^{(+)}$ and $\Delta B_p^{(-)}$ were evaluated in terms of dispersion integrals, using experimental values of the phase shifts, they took into account all possible contributions to the difference between the πN scattering amplitudes at $\nu=0$, $q^2=\mu^2$ and $\nu=\mu$, $q^2=\mu^2$, and not only those which may be well approximated by the contributions of the N^* resonances, the ρ meson and a possible σ meson.

For comparison, we have given in Table V the various contributions to $(a_{1-}^{(\pm)}-a_{1+}^{(\pm)})$ as given by the results of this paper, and in Table VI the various contributions as evaluated in Ref. 9.

Comparing Tables V and VI, we first note that the contributions of the nucleon Born term and the commutator term are the same in both, as they should be.

[45] Note that the combination $(a_{1-}-a_{1+})$ of the P-wave scattering lengths involves only the f_2 amplitudes and not $\partial f_1/\partial t$ or $\partial f_2/\partial t$; because of this, the sum rules given in Refs. 8 and 9 involved the form factors $F_1^V(t)$ and $F_2^V(t)$ at $t=0$ but not their derivatives. Also, the ρ-meson pole in $M_{AA}^{\mu\nu}$ does not contribute to the second-order terms in these sum rules.

TABLE V. Contributions to $(a_{1-}^{(\pm)}-a_{1+}^{(\pm)})$.[a,b]

	Nucleon term	C.R. term	N_1^* term	N_2^* term	N_3^* term	Total	Experimental estimates SW	HW	RWF
$(a_{1-}^{(+)}-a_{1+}^{(+)})$	-0.163	0	-0.036	0.0018	10^{-4}	-0.197	-0.191	-0.192	-0.206 (-0.17)
$(a_{1-}^{(-)}-a_{1+}^{(-)})$	0	0.04	0.048	0.0052	0.002	0.095	0.068	0.066	0.064 (0.081)

[a] The contribution of the scalar term to $(a_{1-}^{(+)}-a_{1+}^{(+)})$ is found to be negligibly small.
[b] The numbers given in parentheses in the last column are obtained from the 0–100-MeV solution of RWF, while the other numbers in this column are obtained from their 0–350-MeV solution. It is not clear which of these are the better estimates.

(Note that in Table VI, the total contribution of the nucleon Born term is the sum of the exact nucleon pole term and the PCAC term, the latter term being the nonpole part of the nucleon Born term in $q_{1\mu}q_{2\nu}M_{AA}{}^{\mu\nu}$.)

The sum of the N^* contributions in the first row of Table V gives -0.034, while the $\Delta A^{(+)}$ term in the first row of Table VI gives -0.0395; the two are roughly the same, which shows that the change in $A^{(+)}$ in going from $\nu=0$, $q^2=\mu^2$ to $\nu=\mu$, $q^2=\mu^2$ is taken into account to a good approximation (to about 15%) by the N^* contributions. For $(a_{1-}^{(+)}-a_{1+}^{(+)})$, the results of both Table V and Table VI agree well with the experimental estimates of HW and SW and the 0–350-MeV solution of RWF.

For $(a_{1-}^{(-)}-a_{1+}^{(-)})$, the sum of the N^* contributions (at the physical threshold) in Table V is a little larger than the sum of the N^* contributions in Table VI [which are evaluated at $\nu=0$, $q^2=0$ and divided by $K^2(0)$]. One may expect that the N^* contribution in $\Delta B_p{}^{(-)}$ in Table VI when added to the explicit N^* contributions will give about the same number as in Table V, assuming that the N^* contributions are well described by our model. However, the sum of $\Delta B_p{}^{(-)}$ and the N^* terms in the second row of Table VI is 0.041, as compared to 0.055 for the sum of the N^* terms in the second row of Table V. This indicates that there is a significant contribution to the isospin-antisymmetric amplitude $q_{1\mu}q_{2\nu}M_{AA}{}^{\mu\nu(-)}$ in addition to that arising from the N^* terms, and that taking it into account gives a better agreement with experiment for $(a_{1-}^{(-)}-a_{1+}^{(-)})$ (as the result of Table VI for this quantity agrees better with the experimental estimates than the result of Table V).

This may be taken to suggest that the basic procedure and assumptions about extrapolation used in this paper are essentially correct, and that the discrepancy in $a_{1-}^{(-)}$ and $f_{1-}^{(-)}$ arises because there is a contribution to the amplitude $q_{1\mu}q_{2\nu}M_{AA}{}^{\mu\nu(-)}$ which cannot be well approximated by the simplest poles and resonances in the various channels.

VII. CONCLUSIONS

In this paper we have examined the results that may be obtained for the πN P-wave scattering lengths and low-energy S-wave and P-wave phase shifts by starting with the current commutation relations and the PCAC hypothesis. We have analyzed the basic relation (2.1) for the nonpole, nonresonant parts of the amplitudes $M_{AA}{}^{\mu\nu}$ and M_{PP} to terms of the second order in ν, making a suitable dynamical approximation for the structure-dependent part of $M_{AA}{}^{\mu\nu}$. The relations thus obtained for the πN scattering amplitude were extrapolated to $q_1{}^2=q_2{}^2=\mu^2$. Adding the exact pole and resonant contributions, results were obtained for the physical πN partial-wave amplitudes.

We have first examined the validity of the assumptions usually made in obtaining the S-wave scattering lengths using only the current C.R.'s and PCAC. Estimating the higher-order terms (in ν) in $T^{(\pm)}$ from dispersion relations and the experimental cross sections (for $q^2=\mu^2$), and using a model to extrapolate to $q^2=0$, we find that at least for $T^{(+)}$ these cannot be neglected. Consequently it appears that the C.R.'s and PCAC are not sufficient to give the scattering length $a^{(+)}$ to a good approximation; one needs detailed dynamical assumptions in order to evaluate the higher-order corrections which should be taken into account in writing a relation for $a^{(+)}$. For $a^{(-)}$, it is possible that the higher-order terms (though individually not negligible) largely cancel one another. Whether in fact they do so may be decided when an accurate experimental value for $a^{(-)}$ becomes available.

We have further examined the magnitude of the scalar term in the relation for $a^{(+)}$, which arises from equal-time commutator $[\mathcal{Q}_\beta{}^0(x),\varphi_\alpha(y)]$; we have suggested that the reasons given in earlier work for ignoring this term are not valid, and that it makes an important contribution.

TABLE VI. Contributions to the sum rules for $(a_{1-}^{(\pm)}-a_{1+}^{(\pm)})$.[a,b]

	Nucleon pole	PCAC term	C.R. term	S-wave scatt. length term	$\Delta A^{(+)}$ or $\Delta B_p{}^{(-)}$	Structure-dependent terms N_1^*	N_2^*	N_3^*	Total
$(a_1^{(+)}-a_{1-}^{(+)})$	⋯	-0.162	⋯	-0.001	-0.0395	⋯	⋯	⋯	-0.202
$(a_1^{(-)}-a_{1+}^{(-)})$	0.012	-0.0117	0.04	-0.0005	-0.0083	0.04	0.005	0.0047	0.081

[a] Here, the PCAC term is the contribution of the nonpole part of the nucleon Born term in $q_{1\mu}q_{2\nu}M_{AA}{}^{\mu\nu}$.
[b] See Refs. 8 and 9.

We have then evaluated and discussed the magnitudes of the various contributions to the partial-wave amplitudes. Some uncertainty arises in these estimates because of the uncertainty in the parameters describing the N^* contributions. However, the most important part of these contributions is that arising from the $(\frac{3}{2},\frac{3}{2}+)$ N_1^*; the parameters for this resonance seem to be fairly well established.

We have suggested that in extrapolating the amplitudes from $q^2=0$ to $q^2=\mu^2$, it is better to extrapolate the nonresonant, nonpole parts of the amplitudes rather than the whole amplitudes and add the exact pole and resonance contributions after extrapolation (see Sec. IV). We have also suggested that it is preferable to extrapolate the invariant amplitudes rather than the partial-wave amplitudes, as the latter require an additional correction for the threshold behavior and further, they sometimes have purely kinematic factors which are sensitive to a variation in q^2.

The results for the isospin-antisymmetric S-wave scattering length and low-energy phase shifts do not seem to depend significantly on the structure-dependent contributions and agree well with the experimental estimates. For the isospin-symmetric S-wave amplitude and scattering length, the sum of the nucleon and N^* contributions does not agree well with the experimental estimates. Adding the contribution of the scalar term (with parameters chosen to fit $a_{0+}^{(+)}$ and $a_{1+}^{(+)}$) improves the agreement with experiment of the predicted values of $f_{0+}^{(+)}$ at low energies. If it is correct to assume that the scalar term $H(t)$ resolves most of the discrepancy in $a_{0+}^{(+)}$ and $f_{0+}^{(+)}$ at low energies, then this term is appreciably large. Our estimate of $H(0)$ [see (6.1)] has the opposite sign to an estimate suggested by Kawarabayashi and Wada.[20]

In comparing the theoretical values of $f_{0+}^{(+)}$ with the experimental estimates, we have noted that the values of $f_{0+}^{(+)}$ at low energies obtained from the phase shifts in the 0–350-MeV solution of RWF (which seem to be the best available estimates) do not seem to be reliable, as they give a scattering length $a_{0+}^{(+)}$ differing in sign and magnitude from the (presumably more reliable) estimates of SW and HW. We have suggested correcting the estimates of $f_{0+}^{(+)}$ (as well as $f_{0+}^{(-)}$) at low energies obtained from the phase shifts of RWF by assuming that the error in these phase shifts arises mainly from the value of the scattering length in the effective-range parametrization of RWF, and that more reliable estimates for the phase shifts at low energies are obtained by replacing this scattering length parameter by the estimate of SW. A more accurate solution for the phase shifts between 0 and 50 MeV would enable more definite statements to be made.

Among the P-wave scattering lengths and amplitudes, for $a_{1+}^{(\pm)}$ and $f_{1+}^{(\pm)}$ there is good agreement with the experimental estimates, while for $a_{1-}^{(+)}$ and $f_{1-}^{(+)}$ at low energies the agreement is reasonable.

On the other hand, for $a_{1-}^{(-)}$ and $f_{1-}^{(-)}$ there is a discrepancy between the theoretical and experimental estimates. Examining the different contributions to $f_{1-}^{(-)}$ and also comparing our results for $(a_{1-}^{(-)}-a_{1-}^{(+)})$ with the (approximate) sum rules for the P-wave scattering lengths obtained in earlier work[8,9] suggests the existence of an additional contribution to $M_{AA}^{\mu\nu(-)}$ that is not taken into account by the simplest pole and resonance terms (like the ρ-pole and the N^* terms).

We conclude by stressing that more accurate experimental estimates of the partial-wave amplitudes, especially $f_{0+}^{(+)}$ and $f_{1-}^{(\pm)}$, at low energies (up to about 50 MeV) are required before any conclusive statements can be made about the validity of our theoretical estimates for these. An over-all view of the results would suggest that our basic assumptions, namely the use of the current commutation relations and PCAC and a specific extrapolation procedure, are correct, and that the main limitation of our work probably arises from the simple model used for approximating the structure of the matrix element of the product of two axial-vector currents (between nucleons).

A study is being made of photoproduction amplitudes and of inelastic and production processes using similar methods, and will be discussed in subsequent papers.

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APPENDIX A

In this Appendix we collect together some definitions and relations used in this paper.

The amplitudes T, M, A, and B for the process (2.4) are defined in terms of the S matrix, S, by the following:

$$S=I+i(2\pi)^4\delta(p_i+q_1-p_f-q_2)\left[\frac{m_im_f}{4E_iE_f\omega_1\omega_2}\right]^{1/2}T, \quad \text{(A1)}$$

$$T=\bar{u}(p_f)Mu(p_i), \quad M=A+(\gamma\cdot Q)B. \quad \text{(A2)}$$

Here, $Q=\frac{1}{2}(q_1+q_2)$. We use the metric defined by $a\cdot b=a_\mu b_\nu g^{\mu\nu}=a_0b_0-\mathbf{a}\cdot\mathbf{b}$. The γ matrices are such that γ_0 is Hermitian and the γ_i for $i=1$, 2, 3 are anti-Hermitian; we have $\{\gamma^\mu,\gamma^\nu\}=2g^{\mu\nu}$. The Dirac equation for the spinors reads $(\gamma\cdot p-m)u(p)=0$. The variables ν, ν_B are defined by $\nu=q_1\cdot(p_i+p_f)/2m$, $\nu_B=-q_1\cdot q_2/2m$; and s, t, u are the usual variables.

The amplitudes f_1 and f_2 are related to A and B as follows:

$$\begin{aligned} f_1&=[(E+m)/8\pi W][A+(W-m)B],\\ f_2&=[(E-m)/8\pi W][-A+(W+m)B]. \end{aligned} \quad \text{(A3)}$$

The isospin decomposition of the various amplitudes T, M, $T_{AA}{}^{\mu\nu}$, $M_{AA}{}^{\mu\nu}$, A, B, etc., is given by

$$T_{\beta\alpha}=\delta_{\beta\alpha}T^{(+)}+\tfrac{1}{2}[\tau_\beta,\tau_\alpha]T^{(-)}\,, \tag{A4}$$

etc. $\sigma^{(\pm)}$ are defined by the optical theorem $\mathrm{Im}T^{(\pm)}=k_L\sigma^{(\pm)}$. The S-wave scattering lengths are related to $T^{(\pm)}$ by

$$a^{(\pm)}=\lim_{t=0,\,\nu=\mu}\frac{m}{4\pi(m+\mu)}T^{(\pm)}(\nu,t)\,. \tag{A5}$$

The matrix element of the axial-vector current $\mathcal{A}^\mu(x)$ between two spin-$\frac{1}{2}$ particles (either two nucleons or a nucleon and a spin-$\frac{1}{2}$ N^*) is written in the form

$$\langle p_f|\,\mathcal{A}^\mu(0)\,|p_i\rangle =\left[\frac{m_im_f}{E_iE_f}\right]^{1/2}\bar{u}(p_f) \times[\mathcal{G}_A(q^2)i\gamma^\mu\gamma^5+\mathcal{H}_A(q^2)q^\mu\gamma^5]u(p_i)\,, \tag{A6}$$

where $q=p_f-p_i$. For the matrix element between nucleons, we have assumed $g_A=g_A(0)=1.18$.

The πNN^* coupling for a spin-$\frac{3}{2}^+$ N^* [described by a Rarita-Schwinger spinor $\psi_\rho(p)$] is written as follows:

$$\langle N^*(p_f)|\,j_\pi(0)\,|N(p_i)\rangle =\left[\frac{m_im_f}{E_iE_f}\right]^{1/2}g(q^2)\bar{\psi}_\rho(p_f)q^\rho u(p_i)\,. \tag{A7}$$

$j_\pi(x)$ is the pion source operator. For a $\frac{3}{2}^-$ N^*, there is an additional factor γ_5 between the spinors.

We give below the values of the masses M, widths Γ, and coupling constants g of the resonances that we have used in this paper (see Ref. 28). The numbers in parentheses indicate (I,J^P), where I is the isospin of the resonance and J^P gives its angular momentum and parity.

(i) $N_1^*(\frac{3}{2},\frac{3}{2}^+)$:
$M_1=1236$ MeV, $\Gamma_1\approx120$ MeV, $g_1^2/4\pi\approx0.38$;

(ii) $N_1^*(\frac{1}{2},\frac{1}{2}^+)$:
$M_2=1400$ MeV, $\Gamma_2\approx120$ MeV, $g_2^2/4\pi\approx6.73$;

(iii) $N_3^*(\frac{1}{2},\frac{3}{2}^-)$:
$M_3=1518$ MeV, $\Gamma_3\approx40$ MeV, $g_3^2/4\pi\approx0.37$;

(iv) $N_4^*(\frac{1}{2},\frac{1}{2}^-)$:
$M_4=1570$ MeV, $\Gamma_4\approx39$ MeV, $g_4^2/4\pi\approx0.063$;

(v) $N_5^*(\frac{1}{2},\frac{1}{2}^-)$:
$M_5=1700$ MeV, $\Gamma_5\approx216$ MeV, $g_5^2/4\pi\approx0.31$;

(vi) $N_6^*(\frac{3}{2},\frac{1}{2}^-)$:
$M_6=1670$ MeV, $\Gamma_6\approx79$ MeV, $g_6^2/4\pi\approx0.116$.

We have also assumed $G_{\pi NN}{}^2/4\pi\approx14.6$.

The coupling constants g_i here are defined such that the width Γ for the decay mode $N^*\to N+\pi$ for an N^* with mass M and spin parity $\frac{1}{2}^\pm$ or $\frac{3}{2}^\pm$ is given by the following:

$$\Gamma(\tfrac{1}{2}^\pm)=\frac{g^2}{4\pi}\frac{p(E\mp m)}{M}\,,\quad \Gamma(\tfrac{3}{2}^\pm)=\frac{g^2}{4\pi}\frac{p^3(E\pm m)}{3M}\,, \tag{A8}$$

where E and p are the energy and 3-momentum of the decay nucleon in the rest frame of the N^*.

Many of the widths quoted above and some of the masses are subject to uncertainty (for instance, see the values quoted in a more recent compilation[46]). However, this is not expected to cause any qualitative changes in our results.

The relation between $\Gamma(\rho\to2\pi)$ and $f_{\rho\pi\pi}$ is the following:

$$\Gamma(\rho\to2\pi)=\frac{f_{\rho\pi\pi}{}^2}{4\pi}\frac{(m_\rho{}^2-4\mu_\pi{}^2)^{3/2}}{12m_\rho{}^2}\,. \tag{A9}$$

Finally, we have defined the $\sigma\pi\pi$ coupling constant $g_{\sigma\pi\pi}$ in terms of the decay width for $\sigma\to2\pi$ as follows:

$$\Gamma(\sigma\to2\pi)=\frac{g_{\sigma\pi\pi}{}^2}{4\pi}\frac{3(m_\sigma{}^2-4\mu_\pi{}^2)^{1/2}}{4m_\sigma{}^2}\,. \tag{A10}$$

APPENDIX B

In this paper, in order to extrapolate the nonresonant, nonpole parts of the amplitudes from $q^2=0$, $s=(m+\mu)^2$ to $q^2=\mu^2$, $s=(m+\mu)^2$, the procedure adopted was to write down the nonresonant, nonpole parts of the amplitudes $A^{(\pm)}$ and $B^{(\pm)}$, assume that the extrapolation of these to the physical threshold could be effected without introducing much error by dividing by $K^2(0)$, and construct the partial-wave amplitudes from these extrapolated amplitudes.

An alternative procedure would be to construct the partial-wave amplitudes at $q^2=0$, $s=(m+\mu)^2$ from the amplitudes A and B and then extrapolate these partial-wave amplitudes $f_{l\pm}$. In this case, besides dividing by $K^2(0)$, one must also ensure that the amplitudes $f_{l\pm}$ for $q^2=\mu^2$ have the correct threshold behavior. A simple model which would take this into account would be one in which the partial-wave amplitudes for $q^2=0$ are multiplied by the factor

$$r_l=\left[\frac{|\mathbf{k}(\mu^2)|}{|\mathbf{k}(0)|}\right]^{2l}, \tag{B1}$$

in addition to the factor $K^{-2}(0)$: here, the argument of $\mathbf{k}$ in (B1) denotes the value of q^2. (This procedure was used by Adler in Ref. 1 as one of his models.)

We here examine whether there is any significant difference between the two methods of extrapolation mentioned above. In each case we assume that the extrapolation is carried out at fixed values of s and t. We take $t=0$ for simplicity.

[46] A. H. Rosenfeld *et al.*, Rev. Mod. Phys. 39, 1 (1967).

Consider the expressions (2.11) for the partial-wave amplitudes. For the S-wave amplitudes, the factor (B1) is unity. With the alternative method of extrapolating the partial-wave amplitudes, the term $f_1(0)$ on the right of (2.11a) would differ from that obtained by extrapolating the A and B amplitudes by a factor

$$[E(0)+m](E+m)^{-1}=[1-\mu^2(W+m)^{-2}]^{-1}, \quad \text{(B2)}$$

while the terms involving $f_1'(0)$ and $f_2'(0)$ would differ by the factors

$$k^2(0)[E(0)+m][k^2(E+m)]^{-1} \quad \text{(B3)}$$

and

$$k^2(0)[E(0)-m][k^2(E-m)]^{-1}, \quad \text{(B4)}$$

respectively. Here $k(0)$ and $E(0)$ are the 3-momentum and the nucleon energy in the c.m. frame for $q^2=0$, while k and E are these quantities for $q^2=\mu^2$.

For the P-wave amplitudes, the factor (B1) is just $k^2/k^2(0)$. The term $f_2(0)$ on the right of (2.11b) as obtained by extrapolating the partial-wave amplitudes differs from that obtained by extrapolating the A and B amplitudes by the factor

$$(E+m)[E(0)+m]^{-1}=[1-\mu^2(W+m)^{-2}], \quad \text{(B5)}$$

while the terms involving $f_2'(0)$ and $f_1'(0)$ in (2.11b) differ by the factors

$$[E(0)-m](E-m)^{-1}=[1-\mu^2(W-m)^{-2}]^{-1} \quad \text{(B6)}$$

and

$$[E(0)+m](E+m)^{-1}, \quad \text{(B7)}$$

respectively. For the right-hand side of (2.11c), the factor is the same as in (B7).

We now note that the factors (B2), (B5), and (B7) are close to unity, while the factors (B3), (B4), and (B6) are quite different from unity at low energies and could give rise to a considerable difference in the corresponding terms.

To examine the quantitative effect of this, we consider the various contributions to the relations for the nonresonant, nonpole parts of the amplitudes, which are the nonpole parts of the nucleon Born term and of Born terms corresponding to spin-$\frac{1}{2}$ N^* resonances (e.g., the N_2^*), the commutator term, and the scalar term.

The nonpole parts of the Born terms corresponding to the nucleon and spin-$\frac{1}{2}$ N^* resonances do not contribute to $\partial f_1/\partial t$ or $\partial f_2/\partial t$ and are therefore not sensitive to the method of extrapolation. The extent to which the values of the commutator term and the scalar term depend on the method of extrapolation is determined by the magnitudes of the derivatives (with respect to t) of the vector and scalar form factors relative to the magnitudes of these form factors at $t=0$.

We have evaluated the contributions of the commutator term using the alternative method of extrapolating the partial-wave amplitudes. For the contribution to $\mathrm{Re}f_{0+}^{(-)}$, the difference is of the order of 3–4%. For the contribution to $\mathrm{Re}f_{1-}^{(-)}$, it is larger, being of the order of 10% (and is of a sign that would decrease $\mathrm{Re}f_{1-}^{(-)}$). However, this is not large enough to affect any qualitative features of our results.

For the scalar term, with the alternative method of extrapolation discussed here, the value of $H'(0)$ required to fit $a_{1+}^{(+)}$ is about the same, while the value of $H(0)$ required to fit $a_{0+}^{(+)}$ is about 10% smaller in magnitude. The values of $\mathrm{Re}f_{0+}^{(+)}$ at low energies are appreciably different from those obtained by extrapolating the invariant amplitudes; for instance, at 20 MeV, the value of $\mathrm{Re}f_{0+}^{(+)}$ is about 30% smaller in magnitude. The values of $f_{1\pm}^{(+)}$ are about the same.

In the method of extrapolating the partial-wave amplitudes discussed here, the variation with q^2, when significant, arises mainly from the change in some purely kinematic factors (which relate $f_{l\pm}$ to A and B) which are sensitive to a change in q^2. Because of this, and because it is not clear how adequately a factor such as (B1) would take into account the change arising from the difference in the kinematics when one varies q^2 from 0 to μ^2, we believe that extrapolating the invariant amplitudes, which was the method adopted in obtaining the results of this paper, is the more satisfactory procedure.

APPENDIX C

In this Appendix we briefly derive the relations analogous to (4.1) and (4.4) for the $\sigma\pi\pi$ coupling. With a view to applying the results to the contribution of σ exchange to the relations (2.1), we have chosen the signs of the 4-momenta so that they correspond to the process $\pi+\sigma\rightarrow\pi$. By replacing $q_1\rightarrow -q_1$ in the following, we may obtain the equations involving the amplitude for the decay $\sigma\rightarrow 2\pi$.

Consider the amplitude $M_{AA\sigma}{}^{\mu\nu}$ defined by

$$\begin{aligned}(2\pi)^4\delta(q_1+p-q_2)M_{AA\sigma}{}^{\mu\nu}(\alpha\beta\gamma) &= F_{AA\sigma}{}^{\mu\nu}(\alpha\beta\gamma) \\ &= \int d^4x d^4y d^4z \exp[i(-q_1\cdot x+q_2\cdot y-p\cdot z)](m_\sigma^2-p^2) \\ &\quad\times\langle 0|T(\mathcal{A}_\alpha{}^\mu(x)\mathcal{A}_\beta{}^\nu(y)\sigma_\gamma(z))|0\rangle. \quad \text{(C1)}\end{aligned}$$

We begin by regarding all the operators as U_3 nonets; we shall later specialize to the isospin subgroup and write the results for the $\sigma\pi\pi$ coupling.

We shall assume the equal-time C.R.'s

$$[\mathcal{A}_\alpha{}^0(x),S_\beta(y)]\delta(x_0-y_0)=-id_{\alpha\beta\gamma}P_\gamma, \quad \text{(C2)}$$

$$[\mathcal{A}_\alpha{}^0(x),P_\beta(y)]\delta(x_0-y_0)=id_{\alpha\beta\gamma}S_\gamma, \quad \text{(C3)}$$

$$[\mathcal{V}_\alpha{}^0(x),S_\beta(y)]\delta(x_0-y_0)=if_{\alpha\beta\gamma}S_\gamma. \quad \text{(C4)}$$

Here, $S_\beta(y)$ and $P_\beta(y)$ are scalar and pseudoscalar operators defined in the quark model[14] (for instance, as quark-antiquark composite field operators). We shall assume that the field operators $\varphi(x)$ and $\sigma(x)$ for the pion and the σ meson are proportional to $P(x)$ and $S(x)$, respectively:

$$S_\alpha(x)=a_\alpha\sigma_\alpha(x)\,,\quad P_\alpha(x)=b_\alpha\sigma_\alpha(x)\,. \tag{C5}$$

In Eqs. (C2) and (C3), the coefficients $d_{\alpha\beta\gamma}$ are symmetric coefficients in U_3 rather than SU_3; they are defined as follows:

$$\{\lambda_\alpha,\lambda_\beta\}=2d_{\alpha\beta\gamma}\lambda_\gamma=2\bar{d}_{\alpha\beta\gamma}\lambda_\gamma+\tfrac{4}{3}\delta_{\alpha\beta}1\,, \tag{C6}$$

where $\gamma=0, 1, \cdots 8$, and $\lambda_0=1$, and $\bar{d}_{\alpha\beta\gamma}$ are the symmetric coefficients in SU_3 as defined by Gell-Mann.[15]

We note that in contrast to the C.R. (1.1), which seems to be valid at least when integrated over all space, as judged by the results derived from it, there is as yet no adequate check of the validity of the C.R.'s (C2)–(C4).

One may write the generalized Ward-Takahashi identity[3,4] for the amplitude defined in (C1), using the PCAC relation (1.2) and the C.R.'s (1.1) and (C2)–(C4)

$$\begin{aligned} q_{1\mu}q_{2\nu}M_{AA\sigma}{}^{\mu\nu}(\alpha\beta\gamma) &=(\mu_\alpha{}^2-q_1{}^2)^{-1}(\mu_\beta{}^2-q_2{}^2)^{-1}C_\alpha C_\beta T(\varphi_\alpha+\sigma_\gamma\to\varphi_\beta)\\ &\quad+d_{\alpha\beta\gamma}\{(m_\sigma{}^2-p^2)a_\gamma{}^{-1}[b_\alpha C_\alpha(\mu_\alpha{}^2-q_1{}^2)^{-1}\\ &\quad+b_\beta C_\beta(\mu_\beta{}^2-q_1{}^2)^{-1}]-a_\gamma b_\alpha{}^{-1}C_\alpha\}\\ &\quad-a_\gamma{}^{-1}\textstyle\sum_\epsilon d_{\alpha\gamma\epsilon}d_{\epsilon\beta\tau}\langle 0|S_\tau(0)|0\rangle\,. \end{aligned} \tag{C7}$$

If one assumes that the vacuum is invariant under SU_3, then the last term on the right of (C7) becomes

$$-d_{\alpha\beta\gamma}\langle 0|\sigma_0(0)|0\rangle\,,$$

where $\sigma_0(0)$ is the unitary singlet component (if any) of the scalar operator $\sigma_\alpha(x)$. Specializing α, β, γ to isospin indices, we now obtain from (C7) the relation

$$\begin{aligned} q_{1\mu}q_{2\nu}M_{AA\sigma}{}^{\mu\nu}(\alpha\beta) &=(\mu_\pi{}^2-q_1{}^2)^{-1}(\mu_\pi{}^2-q_2{}^2)^{-1}C_\pi{}^2T(\pi_\alpha+\sigma\to\pi_\beta)\\ &\quad-\delta_{\beta\alpha}\{\lambda_\sigma-C_\pi{}^2\lambda_\sigma{}^{-1}(m_\sigma{}^2-p^2)\\ &\quad\times[(\mu_\pi{}^2-q_1{}^2)^{-1}+(\mu_\pi{}^2-q_2{}^2)^{-1}]+\tfrac{2}{3}\langle 0|\sigma_0(0)|0\rangle\}\,. \end{aligned} \tag{C8}$$

(Here, it is assumed that the σ meson has zero isospin.)

The amplitude $T(\varphi_\alpha+\sigma_\gamma\to\varphi_\beta)$ in (C7) is defined as follows:

$$\begin{aligned} &i(2\pi)^4\delta(q_1+q_2-p)T(\varphi_\alpha+\sigma_\gamma\to\varphi_\beta)\\ &=i^3[8q_1{}^0q_2{}^0p^0]^{1/2}\int d^4x d^4y d^4z\\ &\quad\times\exp[i(-q_1\cdot x+q_2\cdot y-p\cdot z)](m_\sigma{}^2-p^2)(\mu_\alpha{}^2-q_1{}^2)\\ &\quad\times(\mu_\beta{}^2-q_2{}^2)\langle 0|T(\varphi_\alpha(x)\varphi_\beta(y)\sigma_\gamma(z))|0\rangle\,. \end{aligned} \tag{C9}$$

λ_σ is defined by

$$\lambda_\sigma=C_\pi a_\sigma/b_\pi\,. \tag{C10}$$

We note also the following relations:

$$\langle\sigma(p)|S(0)|0\rangle=[2p_0]^{-1/2}a_\sigma\,, \tag{C11}$$

$$\langle\pi(q)|P(0)|0\rangle=[2q_0]^{-1/2}b_\pi\,. \tag{C12}$$

From (C8), noting the symmetry in α and β of the amplitudes involved, we obtain the following:

$$\begin{aligned} q_{1\mu}q_{2\nu}M_{AA\sigma}{}^{\mu\nu} &=(\mu_\pi{}^2-q_1{}^2)^{-1}(\mu_\pi{}^2-q_2{}^2)^{-1}C_\pi{}^2T(\pi+\sigma\to\sigma)\\ &\quad-\bar{\lambda}_\sigma(p^2,q_1{}^2,q_2{}^2)\,, \end{aligned} \tag{C13}$$

where we have defined

$$\begin{aligned} \bar{\lambda}_\sigma(p^2,q_1{}^2,q_2{}^2) &=\lambda_\sigma-C_\pi{}^2\lambda_\sigma{}^{-1}(m_\sigma{}^2-p^2)[(\mu_\pi{}^2-q_1{}^2)^{-1}+(\mu_\pi{}^2-q_2{}^2)^{-1}]\\ &\quad+\tfrac{2}{3}\langle 0|\sigma_0(0)|0\rangle\,. \end{aligned} \tag{C14}$$

When (C13) is applied to the σ-exchange contribution to πN scattering, the last term in $\bar{\lambda}_\sigma$ will not contribute to the relation for the connected part of the amplitudes and will therefore be omitted. We also note that when the σ meson is on the mass shell the second term on the right of (C14) will not contribute if $m_\sigma\neq\mu_\pi$. In general, however, one is interested in the σ-exchange contribution to the amplitudes $M_{AA}{}^{\mu\nu}$ and M_{PP} [in (2.1)] for values of t other than $t=m_\sigma{}^2$, so that the second term on the right of (C14) would contribute. In particular, for $p_\sigma{}^2=0$, $q_1{}^2=q_2{}^2=0$, (C13) gives

$$q_{1\mu}q_{2\nu}M_{AA\sigma}{}^{\mu\nu}=\mu_\pi{}^{-4}C_\pi{}^2T(\pi+\sigma\to\pi)-\bar{\lambda}_\sigma(0,0,0)\,, \tag{C15}$$

with

$$\bar{\lambda}_\sigma(0,0,0)=\lambda_\sigma-\frac{2C_\pi{}^2}{\lambda_\sigma}\frac{m_\sigma{}^2}{\mu_\pi{}^2}+\tfrac{2}{3}\langle 0|\sigma_0(0)|0\rangle\,. \tag{C16}$$

If we write the coupling of a σ meson to two axial-vector currents in the form[38]

$$\begin{aligned} M_{AA\sigma}{}^{\mu\nu} &=g^{\mu\nu}g_1(t)+q_2{}^\mu q_1{}^\nu g_2(t)+q_1{}^\mu q_2{}^\nu g_3(t)\\ &\quad+q_1{}^\mu q_1{}^\nu g_4(t)+q_2{}^\mu q_2{}^\nu g_5(t)\,, \end{aligned} \tag{C17}$$

then for $q_1{}^2=q_2{}^2=0$, (C13) gives the following:

$$-\tfrac{1}{2}t[g_1(t)-\tfrac{1}{2}tg_2(t)]=\mu^{-4}C_\pi{}^2g_{\sigma\pi\pi}(t,0,0)-\bar{\lambda}_\sigma(t,0,0)\,. \tag{C18}$$

For $t=0$, this gives the relation[38]

$$g_{\sigma\pi\pi}(0,0,0)=\mu_\pi{}^4C_\pi{}^{-2}\bar{\lambda}_\sigma(0,0,0)\,. \tag{C19}$$

We note that in the above derivation we considered the matrix element (C1) with all the particles contracted, because we are interested in the relations obtained when some or all of the momenta q_1, q_2, p are off the mass shell. (For a discussion of this, see Ref. 4.)