Representation of Operators in Quantum Optics*

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The invariance properties of the double-integral and diagonal representations of operators in terms of the coherent states are examined. It is shown that a unique diagonal representation always exists for bounded operators, hence for every density operator, and for unbounded operators which are polynomials in the boson creation and annihilation operators. The associated weight function $P(\alpha)$ is a generalized function in the space Z'. The physical significance of this result is discussed, with particular emphasis on the diagonal representation of the density operator of arbitrary radiation fields. A general formula for the weight function $P(\alpha)$ is derived and is used to calculate the particular form of the weight function for several radiation fields of interest.

I. INVARIANCE PROPERTIES OF REPRESENTATIONS IN TERMS OF THE COHERENT STATES

THE normalized eigenstates of the boson annihilation operator $\hat{a}$, the coherent or quasiclassical states $|\alpha\rangle$, satisfy the completeness relation.[1–4]

$$\frac{1}{\pi}\int |\alpha\rangle\langle\alpha|d^2\alpha=\hat{1}, \tag{1}$$

where $d^2\alpha$ is the real element of area in the complex α plane, and the expansion of $|\alpha\rangle$ in terms of the complete orthonormal set $|n\rangle$ $(n=0,1,2,\cdots)$ is given by

$$|\alpha\rangle=e^{-\frac{1}{2}|\alpha|^2}\sum_{n=0}^{\infty}\frac{\alpha^n}{(n!)^{1/2}}|n\rangle. \tag{2}$$

By virtue of (1), it is always possible to express an arbitrary operator $\hat{A}$ in the form of the double integral

$$\hat{A}=\frac{1}{\pi^2}\int |\alpha\rangle\langle\alpha|\hat{A}|\beta\rangle\langle\beta|d^2\alpha d^2\beta, \tag{3}$$

where the integration is performed over the two complex variables α,β. Since the members of any overcomplete family of states, and in particular the coherent states, are not linearly independent, this representation is, in general, not unique. For example, the vacuum state has the coherent-state representation

$$|0\rangle=\frac{1}{\pi}\int |\alpha\rangle\langle\alpha|0\rangle d^2\alpha=\frac{1}{\pi}\int |\alpha\rangle e^{-\frac{1}{2}|\alpha|^2}d^2\alpha. \tag{4}$$

It then follows from the fundamental properties of the annihilation operator $\hat{a}$ that the sum

$$\sum_{m=1}^{\infty}c_m\hat{a}^m|0\rangle=\frac{1}{\pi}\sum_{m=1}^{\infty}c_m\int \alpha^m|\alpha\rangle e^{-\frac{1}{2}|\alpha|^2}d^2\alpha=0 \tag{5}$$

represents a null vector; the expansion coefficients c_m are arbitrary constants. Multiplying Eq. (5) from the left by an arbitrary bra vector $\langle g|$, we obtain the integral identity

$$\frac{1}{\pi}\sum_{m=1}^{\infty}c_m\int \alpha^m\langle g|\alpha\rangle e^{-\frac{1}{2}|\alpha|^2}d^2\alpha=0. \tag{6}$$

By virtue of Fubini's theorem,[2] we may interchange the order of summation and integration in Eq. (6) provided that the infinite series

$$e^{-\frac{1}{2}|\alpha|^2}\sum_{m=1}^{\infty}|c_m||\alpha|^m|\langle g|\alpha\rangle|$$

is integrable. Now assuming $|g\rangle$ to be normalized, we have by Schwarz's inequality

$$|\langle g|\alpha\rangle|^2\leq\langle\alpha|\alpha\rangle\langle g|g\rangle=1. \tag{7}$$

Hence, if the integral

$$\int\sum_{m=1}^{\infty}|c_m||\alpha|^m e^{-\frac{1}{2}|\alpha|^2}d^2\alpha<\infty \tag{8}$$

exists, the validity of the interchange in Eq. (6) of the operations of summation and integration is assured for all vectors $\langle g|$. The null vector in Eq. (5) then has the form

$$\sum_{m=1}^{\infty}c_m\int \alpha^m e^{-\frac{1}{2}|\alpha|^2}|\alpha\rangle d^2\alpha=\int\sum_{m=1}^{\infty}c_m\alpha^m e^{-\frac{1}{2}|\alpha|^2}|\alpha\rangle d^2\alpha$$
$$=\int g(\alpha)e^{-\frac{1}{2}|\alpha|^2}|\alpha\rangle d^2\alpha=0, \tag{9}$$

where

$$g(\alpha)=\sum_{m=1}^{\infty}c_m\alpha^m.$$

As a result of the last equation, the expansion of an arbitrary state vector $|f\rangle$ is invariant under the

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[1] J. R. Klauder, Ann. Phys. (N. Y.) **11**, 123 (1960).
[2] F. Riesz and B. Sz-Nagy, *Functional Analysis* (Ungar Publishing Company, New York, 1955), p. 83.
[3] R. J. Glauber, Phys. Rev. **131**, 2766 (1963).
[4] R. J. Glauber, Phys. Letters **10**, 84 (1963).

transformation

$$\langle\alpha|f\rangle\to(\langle\alpha|f\rangle)'=\langle\alpha|f\rangle+g(\alpha)e^{-\frac{1}{2}|\alpha|^2}. \quad (10)$$

This vector invariance, under the transformation (10) of its expansion coefficients, can be extended to arbitrary operator representations in terms of the coherent states. For this purpose we multiply now the null vector (9) by an arbitrary, not necessarily null, bra vector $\langle h|$, expressed in terms of the coherent states,

$$\int|\alpha\rangle g(\alpha)e^{-\frac{1}{2}|\alpha|^2}d^2\alpha\times\int h(\beta,\beta^*)\langle\beta|d^2\beta$$
$$=\int|\alpha\rangle g(\alpha)e^{-\frac{1}{2}|\alpha|^2}h(\beta,\beta^*)\langle\beta|d^2\alpha d^2\beta=0. \quad (11)$$

$h(\beta,\beta^*)$ is an absolutely integrable function of β and β^*. The arbitrary operator representation (3) is invariant now under the transformation

$$\langle\alpha|\hat{A}|\beta\rangle\to(\langle\alpha|\hat{A}|\beta\rangle)'$$
$$=\langle\alpha|\hat{A}|\beta\rangle+g(\alpha)e^{-\frac{1}{2}|\alpha|^2}h(\beta,\beta^*). \quad (12)$$

When $h(\beta,\beta^*)=h(\beta^*)e^{-1/2|\beta|^2}$, there is a formal symmetry between the variables α and β^* in Eqs. (11) and (12). This corresponds to the multiplication of two null vectors.

The invariances (10) and (12) of the matrix elements $\langle\alpha|f\rangle$ and $\langle\alpha|\hat{A}|\beta\rangle$ are mathematical consequences of the overcompleteness of the coherent states. They do not seem to imply any physical conservation laws. Useful representation of vectors and operators have the property of uniqueness. To ensure this property, Glauber chooses the weight coefficients $f(\alpha^*)$ and $A(\alpha^*,\beta)$ of the arbitrary vector and operator expansions

$$|f\rangle=\frac{1}{\pi}\int f(\alpha^*)e^{-\frac{1}{2}|\alpha|^2}|\alpha\rangle d^2\alpha, \quad (13)$$

$$\hat{A}=\frac{1}{\pi^2}\int|\alpha\rangle A(\alpha^*,\beta)\langle\beta|e^{-\frac{1}{2}|\alpha|^2-\frac{1}{2}|\beta|^2}d^2\alpha d^2\beta \quad (14)$$

as entire functions of α^* and α^*, β, respectively.[4] He calls the resulting unique expansion in the case of the density operator the R representation.

There is a class of operators $\hat{A}$ which also have a diagonal representation of the form

$$\hat{A}=\int P(\alpha)|\alpha\rangle\langle\alpha|d^2\alpha. \quad (15)$$

$P(\alpha)$ is a weight functional in the complex α plane. The fact that the use of this representation for the density operator leads to an equivalence of descriptions between the classical and quantum-mechanical versions of optical-coherence theory was pointed out by Sudarshan,[5] and has since been the subject of several publications.[6–11]

[5] E. C. G. Sudarshan, Phys. Rev. Letters **10**, 277 (1963).

Section II of this paper is devoted to a discussion of some elements of the theory of generalized functions which are germane to the proof in Sec. III of the existence and uniqueness of the diagonal representation. It is shown that every bounded operator, hence every density operator, and every unbounded operator which can be expressed as a polynomial in the boson creation and annihilation operators has a unique diagonal representation, where the weight functional $P(\alpha)$ is a generalized function in the space Z'. The uniqueness of this representation implies that an invariance relation of the form, Eq. (12), derived for the double-integral representation does not exist for the weight functional $P(\alpha)$.

In Sec. IV a formula for $P(\alpha)$ derived in Sec. III is used for the computation of the diagonal representation of the thermal and coherent radiation fields. The physical significance of the existence of the diagonal representation for the density operator is examined in Sec. V in relation to the question of the correspondence between classical and quantum-mechanical models of physical phenomena.

II. DISTRIBUTIONS AND THEIR FOURIER TRANSFORMS

We review now some elements of the theory of generalized functions which are relevant to the existence and uniqueness theorems of the diagonal representation discussed in the next section. In particular, we show how it is possible to define the Fourier transform of an arbitrary distribution by considering linear functionals defined on a space of entire functions. We first review some basic concepts and definitions. Further details and proofs of the results stated here can be found in Refs. 12–15.

A. Definitions

A generalized function f is a continuous linear functional which maps each test function φ of the linear

[6] J. R. Klauder, J. McKenna, and D. G. Currie, J. Math. Phys. **6**, 733 (1965).
[7] C. L. Mehta and E. C. G. Sudarshan, Phys. Rev. **138**, B274 (1965).
[8] R. J. Glauber, *Quantum Optics and Electronics, Les Houches, 1964* (Gordon and Breach, Science Publishers, New York, 1965), p. 64.
[9] R. J. Glauber, *Physics of Quantum Electronics* (McGraw-Hill Book Company, New York, 1966), p. 788.
[10] J. R. Klauder, Phys. Rev. Letters **16**, 534 (1966).
[11] R. Bonifacio, L. M. Narducci, and E. Montaldi, Phys. Rev. Letters **16**, 1125 (1966).
[12] A. H. Zemanian, *Distribution Theory and Transform Calculus* (McGraw-Hill Book Company, New York, 1965), pp. 192–205.
[13] H. Bremmerman, *Distributions, Complex Variables, and Fourier Transforms* (Addison-Wesley Publishing Company, Reading, Massachusetts, 1965), Chap. 8.
[14] I. M. Gelfand and G. E. Shilov, *Generalized Functions* (Academic Press Inc., New York, 1965), Vol. I. The spaces D' and D are denoted here by K' and K.
[15] L. Ehrenpreis, Am. J. Math. **76**, 883 (1954); Ann. Math. **63**, 129 (1956).

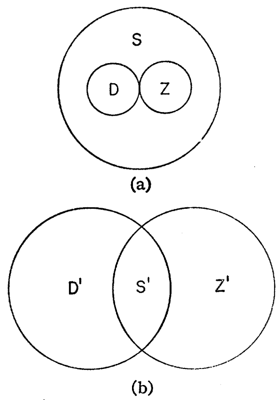

FIG. 1. (a) Relationship of the test function spaces D, Z, and S; (b) Relationship of the generalized function spaces D', Z', and S'.

space Φ onto the complex number $\langle f,\Phi\rangle$. Linearity and continuity of the generalized function f imply

$$\langle f,\alpha\varphi_1+\beta\varphi_2\rangle=\alpha\langle f,\varphi_1\rangle+\beta\langle f,\varphi_2\rangle \tag{16}$$

for any two φ_1, $\varphi_2\in\Phi$ and complex numbers α,β and

$$\lim_{n\to\infty}\langle f,\varphi_n\rangle=\langle f,\lim_{n\to\infty}\varphi_n\rangle=\langle f,\varphi\rangle. \tag{17}$$

The set $\{\varphi_n\}$ is a sequence of elements in Φ which converge to $\varphi\in\Phi$. The test functions $\varphi(x)$ satisfy certain growth and smoothness conditions and form a linear space. In general, particular values cannot be assigned to the generalized function f at any isolated point x.

B. D, D'; S, S'; and Z, Z' Spaces

The class of infinitely differentiable test functions of bounded support defines the linear space D. All the continuous linear functionals defined on D form the space of distributions D'.[14]

S denotes the space of infinitely differentiable test functions which together with their derivatives vanish at infinity more rapidly than any negative power of $|x|$. S' is the linear space of continuous linear functionals defined on S and termed *tempered distributions*. If $f(x)$ is a locally integrable function of polynomial growth, the linear functional $\int f(x)\varphi(x)dx$ converges for all $\varphi\in S$ and hence defines a tempered distribution. Clearly $D\subset S$, therefore every continuous linear functional defined on S is also a continuous linear functional on D; i.e., $S'\subset D'$, *every tempered distribution is a distribution.* The space of tempered distributions includes the Dirac function, its derivatives, and all their finite linear combinations.

The entire test functions $\psi(z)$ of the complex variable $z=u+iv$, which satisfy the inequalities

$$|z^n\psi^{(m)}(z)|\leq c_{n,m}e^{a|v|};\quad n,m=0,1,2,\cdots \tag{18}$$

$a,c_{n,m}$ are constants which may depend on ψ, n, and m, constitute the linear test function space Z. Z' is the space of generalized functions defined on Z. Since $\psi(z)$ is an entire function, it cannot vanish along any finite interval of the real axis without being identically zero over the whole complex plane. Hence the spaces D (test functions of compact support) and Z have only the zero element in common [Fig. 1(a)]. However, it follows from the definitions of the spaces S and Z that if $\psi(z)\in Z$, then $\psi(u)\in S$; that is, Z consists of all entire analytic functions of exponential type which decrease rapidly for real values of their argument, and hence $Z\subset S$, where it is understood that the independent variable for the test functions of Z is real. Elements of Z' are sometimes called *ultradistributions*[12] since the spaces D' (distributions) and Z' intersect but neither is contained in the other. From the fact that $Z\subset S$, it follows that $S'\subset Z'$, i.e., every tempered distribution is in Z' [Fig. 1(b)].

C. Fourier Transforms

When test function φ of the real variable x lies in S, the Fourier transform of $\varphi,\tilde{\varphi}$ is also in S. An analogous result holds for the tempered distributions or linear functionals defined on S, i.e., the spaces S and S' are closed under the Fourier transformation. For example, the Dirac function is a tempered distribution. Its Fourier transform is a constant which can be viewed again as a tempered distribution in the S' space.

The closure property of the spaces S and S' under Fourier transformation cannot be extended to arbitrary elements of D and D'. Consider the case of a single real variable x. Let $\varphi(x)$ be an arbitrary test function in D of bounded support $|x|<a$. Its Fourier transform, $\tilde{\varphi}(u)\equiv\psi(u)$, can be extended to an entire function of $z=(u+iv)$ over the complex z plane. That is, the function

$$\psi(z)=\int_{-a}^{+a}\varphi(x)e^{ixz}dx \tag{19}$$

is analytic for all finite z. Integrating (19) by parts n times with respect to x and differentiating m times with respect to z, we obtain

$$(-iz)^n\psi^{(m)}(z)=\int_{-a}^{+a}\frac{d^n}{dx^n}[(ix)^m\varphi(x)]e^{ixz}dx, \tag{20}$$

so that for all z

$$|z^n\psi^{(m)}(z)|\leq\int_{-a}^{+a}\left|\frac{d^n}{dx^n}[x^m\varphi(x)]|e^{-xv}\right|dx$$
$$\leq c_{n,m}e^{a|v|},\quad n,m=0,1,2,\cdots, \tag{21}$$

where

$$c_{n,m}=\int_{-a}^{+a}\left|\frac{d^n}{dx^n}[x^m\varphi(x)]\right|dx. \tag{22}$$

The converse is also true; that is, every entire function $\psi(z)$ which satisfies (18 or 21) for every n,m is the Fourier transform of some infinitely differentiable function $\varphi(x)$ of bounded support $|x|<a$. The Fourier transform thus establishes a one-to-one correspondence

between the D and Z spaces which preserves linear operations and convergence. A similar mapping between the linear functionals defined on these test function spaces can be established using the following generalization of Parseval's theorem

$$\langle \tilde{f},\varphi\rangle = 2\pi\langle f,\tilde{\varphi}\rangle \tag{23}$$

to define the Fourier transform of an arbitrary distribution f in D'. When $\varphi\in Z$, $\tilde{\varphi}\in D$ and the functional $\tilde{f}\in Z'$ assigns to each φ the same number that $f\in D'$ assigns to $\tilde{\varphi}$. The Fourier transformation establishes then a one-to-one mapping between the spaces D' and Z'. For example, the function

$$f(x)=e^{x^2}=\sum_0^\infty \frac{x^{2n}}{n!}$$

lies in D'. Hence, the Fourier transform of $f(x)$

$$\tilde{f}(u)=\sum_0^\infty \frac{\tilde{x}^{2n}}{n!}=\sum_0^\infty \frac{(-1)^n}{n!}\delta^{(2n)}(u)\in Z'. \tag{24}$$

This is a particular example of the general result; every ultradistribution has the infinite series representation[14]

$$f(z+c)=\sum_{n=0}^\infty f^{(n)}(z)\frac{c^n}{n!}, \tag{25}$$

where c is an arbitrary, complex constant.

D. Multipliers

When the generalized function $f\in\Phi'$, the product of f with the generalized function g is defined by the relation

$$\langle gf,\varphi\rangle=\langle f,g\varphi\rangle, \tag{26}$$

where φ belongs to the corresponding test function space Φ. The right-hand side of Eq. (26) is meaningful only when the product $(g\varphi)\in\Phi$. g is then called a multiplier for the space Φ.

The following functions are multipliers for the following spaces:

(1) for D, all infinitely differentiable functions with arbitrary support;

(2) for S, all infinitely differentiable functions of polynomial growth;

(3) for Z, all entire analytic functions $\psi(z)$ which satisfy the inequality

$$|\psi(z)|\le ce^{a|v|}(1+|z|)^b,\quad v=\mathrm{Im}z, \tag{27}$$

for some constants a, b, and c.

III. EXISTENCE AND UNIQUENESS OF THE DIAGONAL REPRESENTATION

The diagonal coherent-state matrix elements of Eq. (15) lead to the following convolution integral equation

$$\begin{aligned}A(\alpha^*,\alpha)&=\langle\alpha|\hat{A}|\alpha\rangle\\&=e^{-|\alpha|^2}\sum_{n,m=0}^\infty\frac{\langle n|\hat{A}|m\rangle(\alpha^*)^n\alpha^m}{(n!m!)^{1/2}}\\&=\int P(\beta)e^{-|\alpha-\beta|^2}d^2\beta\equiv P(\alpha)*e^{-|\alpha|^2}.\end{aligned} \tag{28}$$

The existence of a diagonal representation $P(\alpha)$ for an arbitrary operator $\hat{A}$ reduces then to the inversion of the integral equation (28) and its solution for $P(\alpha)$, in an appropriate generalized function space. Equivalently, when $\hat{A}=\hat{\rho}$, is the density operator, the normally ordered characteristic function

$$\chi_N(\gamma)=\langle\exp(\gamma\hat{a}^\dagger)\exp(-\gamma^*\hat{a})\rangle$$

must have the Fourier transform $P(x,y)$ [7,9]

$$\begin{aligned}\chi_N(\gamma)&=\langle\exp(\gamma\hat{a}^\dagger)\exp(-\gamma^*\hat{a})\rangle\\&=\mathrm{Tr}\{\hat{\rho}\exp(\gamma\hat{a}^\dagger)\exp(-\gamma^*\hat{a})\}\\&=\int P(x,y)e^{i(ux+vy)}dxdy,\end{aligned} \tag{29}$$

where $\alpha=(x+iy)$, $\gamma=\frac{1}{2}i(u+iv)$.

An alternative criterion for the existence of a diagonal representation of the density matrix operator ρ has been suggested.[16] It states that if $\hat{\rho}$ has only diagonal matrix elements in the bilinear coherent-state representation

$$\langle\alpha|\hat{\rho}|\beta\rangle=\langle\alpha|\hat{\rho}|\alpha\rangle\delta^{(2)}(\alpha-\beta), \tag{30}$$

then $\hat{\rho}$ can be represented in the diagonal form (15) with the weight functional $P(\alpha)$ given by

$$P(\alpha)=\frac{1}{\pi^2}\langle\alpha|\hat{\rho}|\alpha\rangle. \tag{31}$$

The last equation makes it clear, however, that this criterion is not valid. The weight functionals $P(\alpha)$ and $(1/\pi)\langle\alpha|\hat{\rho}|\alpha\rangle$, for normal and antinormal ordering, are not equivalent; in general they have entirely different mathematical properties.[7,17,18] They are related by the integral equation (28), with $\hat{A}=\hat{\rho}$. The difficulty with this criterion is that although formally Eq. (15) follows from Eq. (3), when Eq. (30) is satisfied, the latter condition is never true. An arbitrary operator cannot be diagonalized in any overcomplete representation.[19] Moreover, in the case of the coherent states the matrix element $\langle\alpha|\hat{\rho}|\beta\rangle$ is, within an exponential factor, an entire analytic function of α^* and β,[4] and hence can not be equal to zero for $\alpha\neq\beta$, as implied by Eq. (30), unless it is identically zero.

We next consider the question of the generality and uniqueness of the diagonal representation, i.e., the class

[16] P. L. Kelley and W. H. Kleiner, Phys. Rev. **136**, A316 (1964).
[17] Reference 8, p. 178.
[18] Y. Kano, J. Math. Phys. **6**, 1913 (1965).
[19] J. R. Klauder (private communication).

of operators $\hat{A}$ for which such a representation is possible and is unique.[20] We shall prove the following theorem[21]: Every bounded operator *and* every unbounded operator which is a polynomial in the creation and annihilation operators has a diagonal representation with a weight functional $P(\alpha)$ which lies in the generalized function space $Z'(R_2)$.[22]

Proof: We first establish bounds on the matrix elements $A_{nm}=\langle n|A|m\rangle$ for the two classes of operators under consideration. If $\hat{A}$ is bounded, we have by the Schwarz inequality

$$|\langle n|\hat{A}|n\rangle|^2\leq\langle n|n\rangle\langle n|\hat{A}^\dagger\hat{A}|n\rangle=\|\hat{A}|n\rangle\|^2\leq C, \quad (32)$$

where C is a constant.

If $\hat{A}$ is a polynomial in the creation and annihilation operators, it can always be expressed in the normally ordered form

$$\hat{A}=\sum_{j,k=0}^{2N,2M} c_{jk}(\hat{a}^\dagger)^j\hat{a}^k, \quad (33)$$

and hence

$$\langle n|\hat{A}|m\rangle=\sum_{j,k=0}^{2N,2M} c_{jk}\left(\frac{n!m!}{(n-j)!(m-k)!}\right)^{1/2}\delta_{n-j,m-k}$$
$$\leq C'n^Nm^M,\ \text{for finite constants } C',N,M. \quad (34)$$

In both cases, $\hat{A}$ bounded or a polynomial in $\hat{a}$ and $\hat{a}^\dagger$, it is easily seen that the double power series in Eq. (28)

$$A(\alpha,\alpha^*)=A'(x,y)=e^{-|\alpha|^2}\sum_{n,m=0}^{\infty}\frac{A_{nm}(\alpha^*)^n\alpha^m}{(n!m!)^{1/2}} \quad (35)$$

is absolutely convergent over the entire finite α plane. $A'(x,y)$, considered then as a generalized function, is a regular distribution; i.e., $A'(x,y)\subset D'(R_2)$. Considering the growth properties of the series (35), we can also prove the stronger result that $A'(x,y)\subset S'(R_2)$ and hence also $\subset Z'(R_2)$. Although S' is a proper subspace of both Z' and D', $S'\subset D'$ and $S'\subset Z'$, the spaces D' and Z' intersect, but neither is contained in the other (see Sec. II). Hence a proof[23] that a particular generalized function of the form

$$f(u)=\sum_{n=0}^{\infty}c_n\delta^{(n)}(u),\quad \delta^{(n)}(u)=\left(\frac{d}{du}\right)^n\delta(u), \quad (36)$$

where an infinite number of the expansion coefficients c_n are nonzero is not a distribution in D', does not necessarily imply that such a sum does not converge in the space Z'. For example, the translated δ function defined by

$$\delta(z+c)=\sum_{n=0}^{\infty}\frac{c^n}{n!}\delta^{(n)}(z),\quad z=(u+iv) \quad (37)$$

lies in Z' for every complex constant c. In general, the sum (36) converges in Z' if the coefficients c_n satisfy a growth condition that can be deduced by noting that for every $\Psi(z)$ which belongs to the test function space Z, the dual of Z', we have

$$\langle\sum_{n=0}^{\infty}c_n\delta^{(n)}(z),\Psi(z)\rangle=\langle\sum_{n=0}^{\infty}c_n\delta^{(n)}(u+iv),\Psi(u+iv)\rangle$$
$$=\langle\sum_{n=0}^{\infty}c_n\delta^{(n)}(u),\Psi(u+iv-iv)\rangle=\langle\sum_{n=0}^{\infty}c_n\delta^{(n)}(u),\Psi(u)\rangle$$
$$=\langle\delta(u),\sum_{n=0}^{\infty}(-1)^nc_n\Psi^{(n)}(u)\rangle=\sum_{n=0}^{\infty}(-1)^nc_n\Psi^{(n)}(0), \quad (38)$$

where the second equality follows from the shifting property of ultradistributions.[12] Since $|\Psi^{(n)}(0)|\leq ca^n$, where c and a are constants [see Eq. (21)], c_n must be such that the series $\sum c_nz^n=g(z)$ be an entire function of z. This is then the sufficient condition for the c_n in order that the sum Eq. (36) lie in Z'.

The generalized function $A(\alpha,\alpha^*)\subset S'(R_2)$ when its representation (35) grows no faster than a polynomial as $|\alpha|\to\infty$. Since

$$|A(\alpha,\alpha^*)|\leq C'e^{-r^2}\sum_{n=0}^{\infty}\frac{n^Nr^n}{(n!)^{1/2}}\sum_{m=0}^{\infty}\frac{m^Mr^m}{(m!)^{1/2}},\quad r=|\alpha|, \quad (39)$$

we must show that the series

$$\sum_{n=0}^{\infty}\frac{n^Nr^n}{(n!)^{1/2}}\leq Q(r)e^{r^2/2}, \quad (40)$$

where $Q(r)$ is a polynomial in r. For this purpose we decompose the series $\sum_{n=0}^{\infty}r^n/(n!)^{1/2}$ into sums of even and odd terms:

$$\sum_{n=0}^{\infty}\frac{r^n}{(n!)^{1/2}}=\sum_{n=0}^{\infty}\frac{r^{2n}}{[(2n)!]^{1/2}}+\sum_{n=0}^{\infty}\frac{r^{2n+1}}{[(2n+1)!]^{1/2}}. \quad (41)$$

Using Stirling's formula for $n!$,

$$n!\underset{n\to\infty}{\longrightarrow}n^ne^{-n}(2\pi n)^{1/2}, \quad (42)$$

we have

$$\frac{r^{2n}}{[(2n)!]^{1/2}}\underset{n\to\infty}{\longrightarrow}\frac{r^{2n}}{(2n)^ne^{-n}(4\pi n)^{1/4}}=\frac{(\pi n)^{1/4}(\frac{1}{2}r^2)^n}{n^ne^{-n}(2\pi n)^{1/2}}$$
$$=\frac{(\pi n)^{1/4}(\frac{1}{2}r^2)^n}{n!}, \quad (43)$$

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[20] While this work was in progress, a note by Bonifacio *et al.* has appeared (Ref. 11) which states that the diagonal representation is unique. The generalized function space within which their result holds is not specified.

[21] The statement of this theorem is due to J. R. Klauder.

[22] Although there are many important states for which the weight function is a tempered distribution; e.g., the coherent, chaotic, and Fock states, it also has been shown that the space S' is not large enough to include the weight functional associated with an arbitrary state. For specific examples of states for which $P(\alpha)\not\subset S'$, see Refs. 6 and 9.

[23] K. E. Cahill, Phys. Rev. **138**, B1566 (1965).

and hence

$$\sum_{n=0}^{\infty}\frac{r^{2n}}{[(2n)!]^{1/2}}<\sum_{n=1}^{\infty}\frac{2n(\frac{1}{2}r^2)^n}{n!}=r\frac{\partial}{\partial r}\sum_{n=0}^{\infty}\frac{(\frac{1}{2}r^2)^n}{n!}=r^2e^{r^2/2}. \quad (44)$$

Using the same order of magnitude arguments for the odd series, one obtains

$$\sum_{0}^{\infty}\frac{r^n}{(n!)^{1/2}}<r^2e^{r^2/2}. \quad (45)$$

Multiplying both sides of inequality (45) by the differential operator $(r\partial/\partial r)^N$, we obtain

$$\left(r\frac{\partial}{\partial r}\right)^N\sum_{0}^{\infty}\frac{r^n}{(n!)^{1/2}}=\sum_{0}^{\infty}\frac{n^Nr^n}{(n!)^{1/2}}<Q_{2N+2}(r)e^{r^2/2}, \quad (46)$$

where $Q_{2N+2}(r)$ is a $(2N+2)$th order polynomial in r. Therefore, from Eqs. (39) and (46), we have

$$|A(\alpha,\alpha^*)|<C'e^{-r^2}Q_{2N+2}(r)e^{r^2/2}P_{2M+2}(r)e^{r^2/2}=CS_{2N+2M+2}(r), \quad (47)$$

where C is a constant and $P_{2M+2}(r)$ and $S_{2N+2M+4}(r)$ are polynomials of order $(2M+2)$ and $(2N+2M+4)$ in r, respectively. $A(\alpha,\alpha^*)=A'(x,y)$ is thus a function of polynomial growth and hence $\subset S'(R_2)$ and $Z'(R_2)$.

There exists a solution of the integral equation (28) in the generalized function space Z'. For if

$$A'(x,y)=P(x,y)*e^{-(x^2+y^2)}\subset Z'(R_2), \quad (48)$$

then, by taking the Fourier transform of both sides, we obtain

$$\tilde{A}(u,v)=\pi\tilde{P}(u,v)e^{-(u^2+v^2)/4}\subset D'(R_2). \quad (49)$$

The function $e^{(u^2+v^2)/4}$ is a multiplier (see Sec. II D) in the test function space $D(R_2)$ and hence

$$\tilde{P}(u,v)=\frac{1}{\pi}\tilde{A}(u,v)e^{(u^2+v^2)/4}=\frac{1}{\pi}\tilde{A}(u,v)\sum_{n,m=0}^{\infty}\frac{(\frac{1}{2}u)^{2n}(\frac{1}{2}v)^{2m}}{n!m!}. \quad (50)$$

Taking the inverse transform, we see that the generalized function $P(x,y)$ lies in the space $Z'(R_2)$ in accordance with the statement above.

$$P(x,y)=\frac{1}{\pi}A'(x,y)*\sum_{n=0}^{\infty}\frac{(-1)^n\delta^{(2n)}(x)}{4^nn!}*\sum_{m=0}^{\infty}\frac{(-1)^m\delta^{(2m)}(y)}{4^mm!}=\frac{1}{\pi}\sum_{n,m=0}^{\infty}\frac{(-1)^n(-1)^m}{4^{(n+m)}n!m!}\frac{\partial^{(2n)}}{\partial x^{(2n)}}\frac{\partial^{(2m)}}{\partial y^{(2m)}}\times A(x,y)\subset Z'(R_2). \quad (51)$$

Since $e^{(u^2+v^2)/4}$ is *not* a multiplier in the test function spaces S and Z, it is, in general, impossible to invert the integral equation (28) and obtain a solution $P(x,y)$ which lies in the generalized function spaces S' and D'.

We prove now that the solution $P(x,y)\subset Z'$ is unique. From Eq. (48) it is clear that this is equivalent to the uniqueness of the solution $P(x,y)=0$ of the homogeneous integral equation

$$P(x,y)*e^{-(x^2+y^2)}=0. \quad (52)$$

Since $e^{-(u^2+v^2)/4}$ is a multiplier in the test function space D, the product $\tilde{P}(u,v)e^{-(u^2+v^2)/4}$ where $\tilde{P}(u,v)\subset D'$ is defined and also lies in D'. It is therefore meaningful to define the convolution Eq. (52) in terms of the product of the individual Fourier transforms

$$P(x,y)*e^{-(x^2+y^2)}=\pi F^{-1}[\tilde{P}(u,v)e^{-(u^2+v^2)/4}]=0. \quad (53)$$

The last equation has the unique solution $\tilde{P}(u,v)=0$ since the Gaussian does not vanish anywhere in the finite (u,v) plane, and hence $P(x,y)\subset Z'$ is also unique.

This result clarifies the conjecture that the only solution of Eq. (52) is $P(x,y)=0$ whether $P(x,y)$ is tempered or not.[24] We have proven that the solution of Eq. (52) is unique in Z' as well as S'. It is not meaningful, however, when $P(x,y)\subset D'$; in the latter case $\tilde{P}(u,v)\subset Z'$ and since $e^{-(u^2+v^2)/4}$ is *not* a multiplier in the test function space Z the convolution integral is, in general, not defined.

IV. WEIGHT FUNCTIONALS FOR THERMAL AND COHERENT RADIATION FIELDS

The results of the previous section, besides delineating the constraints under which the diagonal representation exists and is unique, are of importance from the practical point of view in that they provide for an approach to the computation of $P(x,y)$. This is facilitated when the power series (35) can be put in closed form and the Fourier transforms of $A'(x,y)$ and $\tilde{P}(u,v)$ can be easily found. We shall demonstrate this method by considering the examples of the unimodal radiation field in thermal equilibrium at a temperature T and the unimodal coherent radiation field. The density operator of the thermal field is given by[25]

$$\hat{\rho}=\frac{\exp(-\beta\hat{a}^\dagger\hat{a})}{\mathrm{Tr}\exp(-\beta\hat{a}^\dagger\hat{a})}=\exp(-\beta\hat{a}^\dagger\hat{a})(1-e^{-\beta}),\quad \beta=\hbar\omega/KT, \quad (54)$$

and the average number of photons equals

$$\langle n\rangle=\mathrm{Tr}(\hat{\rho}\hat{a}^\dagger\hat{a})=e^{-\beta}/(1-e^{-\beta}). \quad (55)$$

The diagonal matrix representation of the density operator in the terms of the photon number states

$$\langle n|\hat{\rho}|m\rangle=e^{-\beta n}(1-e^{-\beta})\delta_{n,m}=\frac{e^{-\beta(n+1)}}{\langle n\rangle}\delta_{n,m} \quad (56)$$

[24] L. Schwartz, *Théorie des Distributions* (Hermann and Cie, Paris, France, 1951), Vol. II, p. 138.

[25] A. Messiah, *Quantum Mechanics* (John Wiley & Sons, Inc., New York, 1961), Vol. I, p. 337.

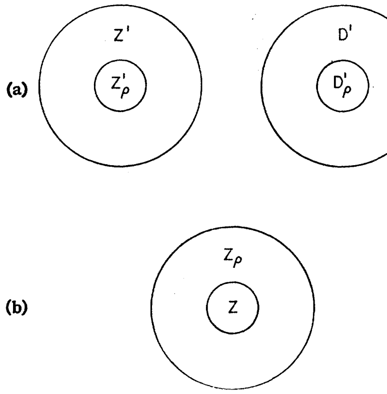

FIG. 2. (a) Relationship of the generalized function spaces Z_ρ', Z', D_ρ', and D'. (b) Relationship of the test function spaces Z_ρ and Z.

reduces the power series, Eq. (35), to the closed form

$$A'(x,y)=A'(r)=\frac{e^{-r^2e^{-\beta}}}{\langle n\rangle}\sum_{n=0}^{\infty}\frac{(r^2e^{-\beta})^n}{n!}$$

$$=\frac{e^{-\beta}}{\langle n\rangle}\exp\left(-\frac{e^{-\beta}}{\langle n\rangle}r^2\right), \quad (57)$$

with Fourier transform

$$\tilde{A}(u,v)=\frac{e^{-\beta}}{\langle n\rangle}\frac{\pi\langle n\rangle}{e^{-\beta}}\exp\left(-\frac{(u^2+v^2)\langle n\rangle}{4e^{-\beta}}\right)$$

$$=\pi\exp\left(-\frac{u^2+v^2}{4(1-e^{-\beta})}\right). \quad (58)$$

The Fourier transform of the weight function $P(x,y)$ for the thermal field follows then from Eq. (50):

$$\tilde{P}(u,v)=\exp\left[\left(-\frac{(u^2+v^2)}{4}\right)\left(\frac{1}{1-e^{-\beta}}-1\right)\right]$$

$$=\exp\left(-\frac{(u^2+v^2)\langle n\rangle}{4}\right), \quad (59)$$

with $P(x,y)$ defined by its inverse transform

$$P(x,y)=\frac{1}{\pi\langle n\rangle}\exp\left(-\frac{(x^2+y^2)}{\langle n\rangle}\right), \quad (60)$$

in agreement with the result obtained by Glauber using an argument based upon a quantum-mechanical version of the central limit theorem.[26,27]

For a coherent radiation field, $\hat{\rho}=|\beta\rangle\langle\beta|$, and its matrix representation in the photon number states is

$$\langle n|\hat{\rho}|m\rangle=\frac{e^{-|\beta|^2}\beta^n(\beta^*)^m}{(n!m!)^{1/2}}. \quad (61)$$

[26] Reference 4, p. 2780.

[27] J. E. Moyal, Proc. Cambridge Phil. Soc. 45, 99 (1949).

The series expansion, Eq. (35), can again be put in the closed form

$$A'(x,y)=e^{-(|\alpha|^2+|\beta|^2)}\sum_{n=0}^{\infty}\frac{(\alpha^*\beta)^n}{n!}\sum_{m=0}^{\infty}\frac{(\alpha\beta^*)^m}{m!}=e^{-|\alpha-\beta|^2} \quad (62)$$

with its Fourier transform

$$\tilde{A}(u,v)=\pi e^{-(u^2+v^2)/4}e^{-i(us+vt)}, \quad \beta=(s+it). \quad (63)$$

Equation (50) leads now to

$$\tilde{P}(u,v)=e^{-i(us+vt)}, \quad (64)$$

and the diagonal representation of the coherent field has the weight function

$$P(x,y)=\delta(x-s)\delta(y-t)=\delta^{(2)}(\alpha-\beta), \quad (65)$$

in agreement with the known result.[3]

V. DIAGONAL REPRESENTATION OF THE DENSITY OPERATOR AND THE Z_ρ, Z_ρ', AND D_ρ' SPACES

We have seen that every bounded operator, and every unbounded operator which is a polynomial in $\hat{a}$ and $\hat{a}^\dagger$ has a diagonal representation where the weight function $P(x,y)$ and its Fourier transform $\tilde{P}(u,v)$ are, in general, functionals in the spaces Z' and D', respectively. For example, the weight function associated with the diagonal representation of the unit operator is $P(x,y)=1/\pi$ with Fourier transform $\tilde{P}(u,v)=(1/\pi)\delta(u)\delta(v)$. For applications in quantum optics, however, we are primarily concerned with the diagonal representation of one particular operator, the density operator, which describes the statistical state of the radiation field. Equation (29) shows that the weight function $P(x,y)$ associated with this operator has an important property which differentiates it from the weight function associated with arbitrary bounded and polynomial operators. The Fourier transform $\tilde{P}(u,v)$ is not an arbitrary *distribution* in D' such as $\delta(u)\delta(v)$, but equals the normally ordered characteristic *function*, $\chi_N(\gamma)=\langle\exp(-\gamma\hat{a}^\dagger)\exp(-\gamma^*\hat{a})\rangle$. This function exists for all $\gamma=\frac{1}{2}i(u+iv)$, is bounded by $e^{|\gamma|^2/2}$, and uniquely specifies the state of the system. The weight functionals associated density operators thus belong to that *subspace of* Z' which is mapped by the Fourier transformation onto the *subspace* of D' comprising continuous functions of γ bounded by $e^{|\gamma|^2/2}$. We denote these subspaces by Z_ρ' and D_ρ', respectively. [See Fig. 2(a).] Since Z_ρ' is contained in Z', the dual test function space associated with Z_ρ', denoted by Z_ρ, is correspondingly larger than Z [Fig. 2(b)]. Hence, the mapping[7]

$$\chi_N(u,v)=\langle P(x,y),e^{iux+ivy}\rangle \quad (66)$$

is *always* valid in the case of density operators for all u,v in spite of the fact that the function $f(x,y)=e^{iux+ivy}$ which belongs to Z_ρ does *not* lie in Z (see Sec. II). Because it is the space on which the functionals in Z_ρ' are defined, Z_ρ has been called the "natural" test function space for the diagonal representation of the density operator.[28]

The importance of these considerations stems from the fact that the operators $\exp(\gamma\hat{a}^\dagger)\exp(\gamma^*\hat{a})$ are simply related to the unitary Weyl operators, $\exp(\gamma\hat{a}^\dagger-\gamma^*\hat{a})$,[27] linear combinations of which lead to all bounded operators. Since the mean value mapping is a linear one, this implies that we can compute the mean value of every *bounded* operator in the diagonal representation from the mean value of the Weyl operator. The latter quantity is the characteristic function

$$\chi(\gamma)=\langle\exp(\gamma\hat{a}^\dagger-\gamma^*\hat{a})\rangle.$$

The identity

$$\exp(\gamma\hat{a}^\dagger)\exp(-\gamma^*\hat{a})=e^{|\gamma|^2/2}\exp(\gamma\hat{a}^\dagger-\gamma^*\hat{a}) \quad (67)$$

leads to the simple relation between the Fourier transform of $P(x,y)$ and the mean value of the Weyl operator.

$$\chi_N(\gamma)=e^{|\gamma|^2/2}\chi(\gamma). \quad (68)$$

There is also an alternate approach to the problem of the diagonal representation of the density operator ρ in which the weight functional $P(x,y)\subset Z'_\rho$ is defined as the limit of an infinite sequence of well-behaved weight functions $\{P_n(x,y)\}$.[10] It can be shown that the resulting infinite sequence of diagonal density operators $\{\hat{\rho}_n\}$ converges to $\hat{\rho}$ in the sense of the trace-class norm. This implies that the difference between the mean value of an arbitrary *bounded* operator $\hat{O}$ calculated with $\hat{\rho}$ and $\hat{\rho}_n$ respectively can be made arbitrarily small, i.e., for any $\epsilon>0$ there exists an integral N such that

$$|\mathrm{Tr}(\hat{\rho}\hat{O})-\mathrm{Tr}(\hat{\rho}_n\hat{O})|\langle\epsilon\|\hat{O}\| \quad \text{for} \quad n>N, \quad (69)$$

where

$$\hat{\rho}_n=\int P_n(\alpha)|\alpha\rangle\langle\alpha|d^2\alpha, \quad (70)$$

and $P_n(\alpha)$ is an infinitely differentiable function of rapid decrease. Although this result does not guarantee that the mean value of *unbounded* operators such as $\hat{a}^\dagger\hat{a}$ can be generally approximated in the manner of Eq. (69) it does apply to all unitary operators, projection operators, and those operators whose mean value define the photoelectric counting distribution. For example, the probability that m photons are present in a single mode whose statistical properties are specified by the density operator $\hat{\rho}$ can be approximated to an arbitrary accuracy by the integral

$$P(m)=\mathrm{Tr}(\hat{\rho}|m\rangle\langle m|)=\int P_n(\alpha)\frac{e^{-|\alpha|^2}}{m!}|\alpha|^{2m}d^2\alpha. \quad (71)$$

One final remark is in order. The preceding considerations demonstrate the fact that the overcompleteness of the coherent states makes it possible to calculate the mean value of every bounded operator to an arbitrary degree of accuracy using the diagonal representation of the density operator. The usefulness of this representation in quantum optics stems from the fact that it facilitates the calculation of mean values of normally ordered operators in a form which bears a close formal resemblance to the classical ensemble averages which arise in the classical theory of optical coherence where the radiation field is treated as a classical stochastic variable. If an arbitrary operator $\hat{O}$ has the normally-ordered power series expansion

$$\hat{O}=f(\hat{a}^\dagger,\hat{a})=\sum_{n,m}c_{nm}(\hat{a}^\dagger)^n\hat{a}^m, \quad (72)$$

where c_{nm} are arbitrary c-number expansion coefficients, then using the diagonal representation for ρ we have

$$\langle\hat{O}\rangle=\mathrm{Tr}(\rho f(\hat{a}^\dagger,\hat{a}))=\int P(\alpha)\sum_{n,m}c_{nm}(\alpha^*)^n\alpha^m d^2\alpha$$
$$=\int P(\alpha)f(\alpha,\alpha^*)d^2\alpha. \quad (73)$$

The diagonal representation thus accomplishes a formal, symbolic equivalence between the classical and quantum *descriptions* of the optical field.[5] This formal similarity however, does not imply an equivalence between the predictions of classical and quantum *theory* for arbitrary fields. There are states of the quantized field which have no classical analog no matter how great the photon excitation; e.g., the Fock states $|n\rangle$. Such states lead to predictions of phenomena, such as anticorrelation effects[29] in photoelectric coincidence-counting experiments, which are not accounted for by the classical theory.

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[28] J. R. Klauder and E. C. G. Sudarshan, Fundamentals of Quantum Optics (W. A. Benjamin, Inc., New York, to be published).

[29] M. M. Miller and E. A. Mishkin, Phys. Letters **24A**, 188 (1966).