Three-Body Correlations in Reaction-Matrix Calculations*

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An improved method of treating three-body cluster correlations in reaction-matrix calculations is presented and applied to nuclear matter. A three-particle reaction matrix is defined which allows three particles to interact with each other in all possible ways after specifying the particles involved in the initial and terminal interactions. The method is similar to that proposed by Bethe but, since all off-energy-shell terms can be treated separately, full advantage can be taken of the Faddeev formalism as applied to the three-nucleon system. The Pauli exclusion operator is included in calculating on-energy-shell reaction-matrix elements but is neglected in the off-energy-shell terms. Numerical calculations are presented for two spin-independent, s-state, separable potentials, one of which contains a hard-shell repulsive term. Higher-order terms are found to be important for the hard-shell potential but not for the simple attractive potential. In both cases the total three-body correlation energy is found to be small.

I. INTRODUCTION

THE basic idea behind the Brueckner-Goldstone[1] theory of nuclear matter is the rearrangement of the perturbation series for the energy into an expansion in terms of the two-body reaction matrix t.[2] The reaction matrix is defined by the integral equation

$$t=v-v(Q/e)t, \tag{1}$$

where v is the nucleon-nucleon potential, Q forbids scattering into occupied states, and the energy denominator e is the difference in energy of the total system before and after the virtual excitation. Matrix elements of t remain finite even when the two-body potential is singular, so it is reasonable to hope that a perturbation expansion in terms of t will converge. The terms involved in the revised expansion are conveniently represented by a series of diagrams involving t-matrix interactions of increasing order. The first-order diagram shown in Fig. 1(a) gives the major contribution to the energy, and since the t matrix sums all ladder diagrams involving repeated v interactions, the second-order diagram is automatically included in first order.

It is also possible to include some of the third-order diagrams in the first order by modifying the definition of the propagator appearing in the definition of the t matrix. Thus the contribution of the hole-bubble diagram shown in Fig. 1(b) can be absorbed into the first-order diagram by a suitable definition of the single-particle potential energy of occupied (hole) states.[3] The hole-bubble interaction is evaluated on the energy shell[3,4] and is found to be strongly attractive. Similarly, a potential energy for excited (particle) states can be defined to cancel the particle-bubble diagrams of Fig. 1(c). However, the particle-bubble diagram, which involves the excitation of three nucleons and must be evaluated off the energy shell, is found to be strongly repulsive due to the effect of the hard core. If the particle-bubble diagrams (hereafter referred to simply as bubble diagrams) are included in the first-order calculation according to the prescription of Bethe, Brandow, and Petscheck,[4] they give rise to a positive contribution to the energy per particle of $\sim$5 to 10 MeV.[5]

However, Rajaraman[6] showed that the other third-order diagrams, including the various exchange terms, are of comparable magnitude to the third-order bubble diagram and thus cannot be neglected. Rajaraman also showed that all third-order diagrams, even those corresponding to genuine three-body clusters, can be cancelled by slightly modifying the BBP definition of the potential energy in intermediate states. Further investigation by the same author[7] of fourth- and higher-order terms (which now constitute the only correction to the revised first-order calculation) was carried out to study the rate of convergence of the Brueckner-Goldstone series. He found that there are many higher-order diagrams whose contributions are comparable to the third order. The important higher-order diagrams are those which involve only three hole lines. Each hole line implies an integration over the hole momentum from 0 to k_F. Thus a diagram involving two hole lines (first order) is proportional to k_F^6 (or ρ^2, where k_F is Fermi momentum and ρ is density); while the diagrams involving three hole lines are proportional to the cube of the density. A diagram of any order which involves

FIG. 1. First and third-order t-matrix diagrams discussed in text.

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[1] K. A. Brueckner, in *The Many-Body Problem*, edited by B. S. DeWitt (Dunod Cie, Paris, 1960), pp. 47–166; J. Goldstone, Proc. Roy. Soc. (London) **A239**, 267 (1957).
[2] Here we use the notation of the t matrix instead of Bethe's g or Brueckner's K. We reserve the symbols g and K for defining some other quantities in this paper.
[3] K. A. Brueckner and D. T. Goldman, Phys. Rev. **117**, 207 (1960).
[4] H. A. Bethe, B. H. Brandow, and A. G. Petschek, Phys. Rev. **129**, 225 (1963); referred to in text as BBP.
[5] E. E. Brown, G. T. Schappert, and C. W. Wong, Nucl. Phys. **56**, 191 (1964).
[6] R. Rajaraman, Phys. Rev. **129**, 265 (1963).
[7] R. Rajaraman, Phys. Rev. **131**, 1244 (1963).

only three hole lines is therefore proportional to the same power of the density as the important third-order diagrams.

The density dependence is not the only reason for the importance of the higher-order diagrams. An nth order diagram involving three hole lines will contain $n-3$ extra factors of te^{-1} as compared to a third-order diagram. These factors must be integrated over a set of intermediate particle states and the importance of fourth- and higher-order diagrams arises from the fact that te^{-1} integrated over particle states gives a factor ~ 1. This estimate of Rajaraman[7] is based on the assumption of a hard-core potential. Thus, for a hard-core potential of radius c, an appropriate expansion parameter is neither the density nor the t matrix alone but rather the product of the interaction strength (e.g., c^3) and the density. Rajaraman's work shows that the entire series of diagrams involving three hole lines must be summed at once and not simply order by order. Thus the Brueckner-Goldstone expansion for the energy should be further rearranged into a series in terms of the number of interacting particles rather than in the number of successive t matrix interactions.

Bethe,[8] extending these ideas further, has proposed a method for summing the three-body cluster interactions to all orders of perturbation theory by applying the Faddeev[9] technique. He obtained an approximate solution to the resulting Bethe-Faddeev equation and found that the total three-body correlation energy was small. Bethe did not calculate the correlation energy directly. Instead, he modified the potential energies of the particle states to try to include all three-particle correlations and then estimated the effect of the new spectrum on the total binding energy. He estimated the binding energy found by including only the third-order bubble diagram in defining particle potentials was too low by ~ 2 to 8 MeV.

Brandow[10] recently suggested extending the above ideas still further and completely rearranging the linked-cluster perturbation series in terms of "compact clusters." His theory requires that the single-particle potentials be defined only in terms of those diagrams which can be evaluated entirely on the energy shell. Thus, while Bethe still prefers to consider the three-body cluster terms as contributing to the potential energy of particle states, Brandow's theory requires a separate calculation of the three-body terms. Brandow's theory, however, is essentially equivalent to Bethe's as far as the graphs included in two-body and three-body clusters are concerned. Thus, if it can be shown that the total three-body contribution is negligible it will not be necessary to extend Brandow's theory beyond the third-order except, possibly, to calculate some higher-order terms associated with the rearrangement energy.[3,10–12]

The three-body diagrams are obviously very important and more careful treatments of their effects are needed. In this paper we present an alternate method of summing all three-body cluster interactions which we believe to be simpler in interpretation, more symmetrical in form and easier to use in both nuclear matter and finite-nucleus calculations. The central quantity in our theory is a three-body reaction matrix which differs in definition from that of Bethe and allows us to specify both the initial and terminal t matrices in the required ladders of t-matrix interactions. Our three-body reaction matrix can thus be used as the middle interaction in third-order diagrams in calculating the three-body correlation energy. The important point here is that the three-body reaction matrix can then be calculated entirely off the energy shell and our formalism and calculations become identical to those used in solving the usual Faddeev equations.

Our formalism is discussed in detail in the next section where the integral equations for the three-body reaction matrix are presented. The expression for the three-body cluster energy is also presented and we show that our solution reduces to that of Bethe when expressed in terms of three-body "defect" wave functions. However, the power of the present formalism lies in the fact that our equations maintain the compactness of Faddeev's equations, which enables us to evaluate the integral equations by replacing them with matrix equations. Thus if we know the off-energy-shell matrix elements of the two-body reaction matrix we can solve for matrix elements of our three-body operator. Since we have specified the initial and terminal t-matrix interactions involved in our three-body operator we can work directly with these three-body matrix elements rather than with three-body wave functions.

The calculations are outlined in Sec. III. We work entirely in momentum space using separable two-body t matrices which we have evaluated using the Yamaguchi[13] potential and a modified version of the Puff[14] potential. Both of the potentials used are separable, S-state, spin- and isospin-independent potentials which have been used extensively in three-nucleon calculations. The Yamaguchi potential has been fitted to low-energy nucleon-nucleon scattering data and contains no repulsive term. The Puff potential, however, has been fitted to high-energy data and has a repulsive term consisting of an infinite hardshell. We can thus compare directly the effect of the strong repulsive term in the higher-order diagrams.

Finally, in Sec. IV we present the results obtained and compare them with previous calculations. We consider in detail the dependence of the results on the

[8] H. A. Bethe, Phys. Rev. **138**, B804 (1965).
[9] L. D. Faddeev, Zh. Eksperim. i Teor. Fiz. **39**, 1459 (1960) [English transl.: Soviet Phys.—JETP **12**, 1014 (1961)].
[10] B. H. Brandow, Phys. Rev. **152**, 863 (1966).
[11] H. S. Köhler, Phys. Rev. **137**, B1145 (1965); **138**, B831 (1965).
[12] K. S. Masterson, Jr., and A. M. Lockett, Phys. Rev. **129**, 776 (1963).
[13] Y. Yamaguchi, Phys. Rev. **95**, 1628 (1954); **95**, 1635 (1954).
[14] R. D. Puff, Ann. Phys. (N.Y.) **13**, 317 (1961).

different potentials used. The effect of the Pauli operator, which we have included in the on-energy-shell matrix elements, is also discussed.

II. FORMALISM

The series of Goldstone diagrams involving three hole lines can be summed to all orders in t matrix interaction by introducing a three-particle reaction matrix T defined by the matrix equation

$$T=t-KT, \tag{2}$$

where T is the 3×3 matrix

$$\begin{pmatrix} T^{11} & T^{12} & T^{13} \\ T^{21} & T^{22} & T^{23} \\ T^{31} & T^{32} & T^{33} \end{pmatrix}, \tag{3}$$

and the two-body t is a diagonal matrix

$$\begin{pmatrix} t_1 & 0 & 0 \\ 0 & t_2 & 0 \\ 0 & 0 & t_3 \end{pmatrix}. \tag{4}$$

The interpretation of superscripts and subscripts is given below. The kernel of integral equation (2) is given by

$$K=\begin{pmatrix} 0 & t_1e^{-1}Q & t_1e^{-1}Q \\ t_2e^{-1}Q & 0 & t_2e^{-1}Q \\ t_3e^{-1}Q & t_3e^{-1}Q & 0 \end{pmatrix}, \tag{5}$$

where e^{-1} is a three-particle energy denominator

$$e^{-1}=[E_i+E_j+E_k-E_l-E_m-E_n]^{-1}. \tag{6}$$

In Eq. (6), E_i, E_j, and E_k denote single-particle energies in excited states, and E_l, E_m, E_n are energies of states inside the Fermi sea. From now on we shall always use (i,j,k) and (l,m,n) for states above and below the Fermi sea, respectively. With these definitions we note that the energy denominator in Eq. (6) is always positive definite.

The above equations are modified Faddeev integral equations for three-particle scattering. The most important point in Faddeev's generalization of the Lippman-Schwinger equation is that he avoids the non-uniqueness of the solutions of these equations. Here also in Eq. (2) we maintain the property of compactness which is a necessary feature of the Faddeev formalism. This fact is obvious in Eq. (5), where the kernel has zero diagonal elements thus eliminating the occurrence of δ-function singularities. This compactness property enables one to evaluate integral equations by replacing them with matrix equations.[15]

An individual matrix element of T can be written as

$$T^{ij}=t_i\delta_{ij}-t_i\sum_k \frac{Q}{e}(1-\delta_{ik})T^{kj}, \tag{7}$$

[15] C. Lovelace, Phys. Rev. **135**, B1225 (1964).

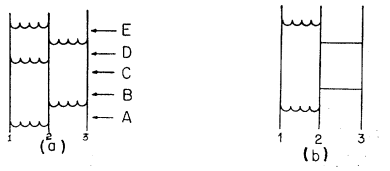

FIG. 2. An nth order diagram containing T^{11}.

where the two-body t matrix element obeys the integral equation

$$t_i=v_i-v_i(Q/e)t_i. \tag{8}$$

The Pauli operator Q forbids scattering into occupied states. However, matrix elements of T^{ij} are evaluated only between momentum states above the Fermi sea so we can probably neglect the effect of the Pauli operator in evaluating T^{ij}. We examine this point further in Sec. IV but from here on we drop the operator Q in Eqs. (2)–(8). It should be emphasized that the energy denominator e occurring in Eq. (8) is the three-body excitation energy and not the usual two-body energy. Thus, Eq. (8) requires that the two-body reaction matrix be evaluated off the energy shell.

The superscripts or subscripts i, j, k in the above equations are particle indices. Thus t_i is the two-body reaction matrix in which particle i is not involved, and T^{ij} is the three-body reaction matrix not involving j and i in initial and final interactions, respectively. This fact becomes transparent upon iteration of Eq. (6). Finally the factor $(1-\delta_{ik})$ ensures that the same pair of particles cannot interact twice in succession.

Our definition (6) differs from that of Bethe only in the fact that we specify the initial and final t-matrix interactions rather than just the final one. This definition enables us to write down the three-particle cluster contribution to the binding energy as

$$W=\sum_{l,m,n}\langle \mathbf{k}_l,\mathbf{k}_m,\mathbf{k}_n|\, t_3\frac{Q}{e}\Big\{T^{11}\frac{Q}{e}(t_2+t_3)+T^{12}\frac{Q}{e}(t_1+t_3) +T^{13}\frac{Q}{e}(t_1+t_2)\Big\}|\,\mathbf{k}_l,\mathbf{k}_m,\mathbf{k}_n\rangle, \tag{9}$$

where l, m, n involves summation over spin and isospin variables and integration over the momenta of three particles. In other words, matrix elements of T^{ij} occurring in the middle are to be evaluated entirely off the energy shell, whereas the two-body operators $t_i(Q/e)$ or $(Q/e)t_i$ are to be evaluated on the energy shell and refer to two-body excitation only in the initial or final state.

A typical three-body diagram involving T^{ij}'s is shown in Fig. 2(a) where, following the convention established by Rajaraman,[6] the hole lines have purposely been left open. In Fig. 2(a), the initial (bottom) interaction lifts particles 1 and 2 out of the Fermi sea. This initial interaction is followed by an arbitrary

number of t-matrix interactions beginning and ending with t_1. Finally nucleons 1 and 2 are returned to the Fermi sea.

There are an arbitrary number of t interactions between lines B and C. However, the interaction just below C cannot be t_2 and the interaction just above line B cannot be t_1. That part of the diagram between A and E represents T^{11}, while the subdiagram between A and D can be identified as one term of T^{13}. In Fig. 2(b) we represent the diagram of Fig. 2(a) using a condensed notation.

Figure 2(b) is one of six distinct (topologically) diagrams entering into the calculation of the three-body cluster contribution. These are shown in Fig. 3, where we have defined the final interaction as t_3 and the second-to-last interaction as t_1. Therefore, as discussed by Bethe,[8] we must integrate independently over the momenta of all three particles in summing all possible three-body diagrams. In this manner, summing all topologically distinct three-body cluster diagrams, we get Eq. (9). Thus to evaluate the three-body correlation contribution, we will need to introduce two sets of intermediate states in (9), and only the T^{ij} part of (9) will have to be evaluated off the energy shell.

In the above discussion we have not taken into account the effect of exchange terms or spin-isospin statistics. We show later that these effects can be taken into account by a statistical factor. Neglecting the statistical factor, there are 2^{n-2} distinct diagrams in nth order which enter into the energy calculation. The six diagrams shown in Fig. 3 reduce to the two third-order and four fourth-order diagrams shown in Fig. 4 when only the lowest order (in t) terms are retained for T^{11}, T^{12}, and T^{13}.

We shall now show how our equations reduce to those of Bethe[8] when we impose identical assumptions. We introduce unperturbed and perturbed three-body wave functions by

$$T^{ij}\phi = t_i\psi^{ij} \tag{10}$$

or, in obvious matrix notation, by

$$T\Phi = t\Psi . \tag{11}$$

The substitution of (11) into (2) or (7) leads to the matrix equation

$$\Psi = \Phi - e^{-1}t\Psi , \tag{12}$$

where, now

$$e^{-1} = \begin{bmatrix} 0 & e^{-1} & e^{-1} \\ e^{-1} & 0 & e^{-1} \\ e^{-1} & e^{-1} & 0 \end{bmatrix} . \tag{13}$$

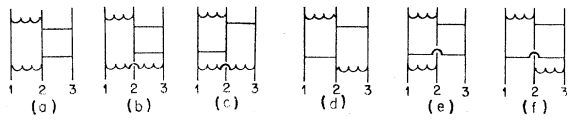

FIG. 3. All diagrams involved in calculating the three-body cluster energy.

Next, in the spirit of Bethe[8] and Moszkowski,[16] we replace the integral operators $e^{-1}t$ by functions of the interparticle distances $\xi(r)$. Thus Eq. (12) becomes a set of simultaneous equations given by

$$\Psi = \Phi - \xi\Psi , \tag{14}$$

where

$$\xi = \begin{bmatrix} 0 & \xi_1 & \xi_1 \\ \xi_2 & 0 & \xi_2 \\ \xi_3 & \xi_3 & 0 \end{bmatrix} . \tag{15}$$

It may be noted here that in Eq. (14) we require only one set of the ξ_i's in solving for the ψ^{ij}'s, since we are working entirely off the energy shell.

Solving Eq. (14) we thus obtain

$$\psi^{ii} = (1-\xi_j\xi_k)D^{-1}\phi , \tag{16}$$

and

$$\psi^{ij} = -\xi_j(1-\xi_k)D^{-1}\phi , \tag{17}$$

where the denominator D is the determinant

$$D = 1-\xi_1\xi_2-\xi_2\xi_3-\xi_3\xi_1+2\xi_1\xi_2\xi_3 . \tag{18}$$

In order to evaluate the energy we must also define a function $\eta(r)$ to replace the integral operator $e^{-1}t$ when the latter is evaluated on the energy shell. The evaluation of the function $\xi(r)$ differs from that of $\eta(r)$ in an essential way. These functions were evaluated by Bethe using the reference-spectrum approximation of BBP. The parameter γ_3^2 in the reference-spectrum energy denominator used in solving for $\xi(r)$ is not the same as the two-body γ_2^2. The former contains an extra term $\frac{3}{4}k_K^2$ because of $\xi(r)$ must be evaluated off the energy shell. [See Ref. 8 Eq. (3.10)].

Substituting the above results into Eq. (9) we find the three-body cluster energy given by

$$W = \sum_{l,m,n} \langle \mathbf{k}_l,\mathbf{k}_m,\mathbf{k}_n | D^{-1}\eta_3\{(1-\xi_2\xi_3)(\eta_2+\eta_3) - \xi_2(1-\xi_3)(\eta_1+\eta_3) - \xi_3(1-\xi_2)(\eta_1+\eta_2)\} \times | \mathbf{k}_l,\mathbf{k}_m,\mathbf{k}_n \rangle . \tag{19}$$

This expression is equivalent to that obtained by Bethe, however, the derivation here enables us to handle on-energy-shell terms and off-energy-shell terms separately.

III. CALCULATIONS

In this section we present calculations using the above formalism. The calculations are done entirely in momentum space, and the notations and kinematics are given in Appendix A. We work in momentum space since, as suggested by Faddeev, the coupled T-matrix equations can be expressed and solved in terms of three equivalent sets of momentum variables.

Thus in momentum space, assuming that total momentum is conserved [and dropping the factor

[16] S. A. Moszkowski, Phys. Rev. **140**, B283 (1965).

$\delta(K)]$, Eq. (7) becomes

$$T^{ii}(\mathbf{k}_{jk},\mathbf{k}_i;\,\mathbf{p}_{jk},\mathbf{p}_i;\,\gamma_3^2)=t_i(\mathbf{k}_{jk},\mathbf{p}_{jk};\,\gamma_3^2+\tfrac{3}{4}k_i^2)\delta(\mathbf{k}_i-\mathbf{p}_i)-\int d^3q\frac{t_i(\mathbf{k}_{jk},\,-\mathbf{q}-\tfrac{1}{2}\mathbf{k}_i;\,\gamma_3^2+\tfrac{3}{4}k_i^2)}{k_i^2+q^2+\mathbf{k}_i\cdot\mathbf{q}+\gamma_3^2}$$
$$\times[T^{ji}(\mathbf{k}_i+\tfrac{1}{2}\mathbf{q},\,\mathbf{q};\,\mathbf{p}_{jk},\,\mathbf{p}_i;\,\gamma_3^2)+T^{ki}(\mathbf{k}_i+\tfrac{1}{2}\mathbf{q}_j-\mathbf{q};\,\mathbf{p}_{jk},\,\mathbf{p}_i;\,\gamma_3^2)] \quad (20)$$

and

$$T^{ji}(\mathbf{k}_{ki},\mathbf{k}_j;\,\mathbf{p}_{jk},\mathbf{p}_i;\,\gamma_3^2)=-\int d^3q\frac{t_j(\mathbf{k}_{ki},\,-\mathbf{q}-\tfrac{1}{2}\mathbf{k}_j;\,\gamma_3^2+\tfrac{3}{4}k_j^2)}{k_j^2+q^2+\mathbf{k}_j\cdot\mathbf{q}+\gamma_3^2}$$
$$\times[T^{ki}(\mathbf{k}_j+\tfrac{1}{2}\mathbf{q},\,\mathbf{q};\,\mathbf{p}_{jk},\,\mathbf{p}_i;\,\gamma_3^2)+T^{ii}(\mathbf{k}_j+\tfrac{1}{2}\mathbf{q},\,-\mathbf{q};\,\mathbf{p}_{jk},\,\mathbf{p}_i;\,\gamma_3^2)]. \quad (21)$$

The steps involved in these equations are given in Appendix B. The γ_3^2 is given in Eq. (A16) and is the three-body extension of the two-body γ_2^2 given in Eq. (A18). Thus the fact to be noticed here is that the two-body t matrices involved in Eqs. (20) and (21) are not only calculated with respect to the three-body γ_3^2 but with $\gamma_3^2+\frac{3}{4}k_i^2$, i.e., involving the momentum of the third particle whose effect occurs as a spectator in shifting the energy denominator. Once the t-matrix form is assumed these equations can be solved. Equations (20) and (21) are exact except for two approximations. We have used the reference spectrum method for the hole-state energies and we neglect the Pauli operator in excited states.

With the above definition of T^{ij} in momentum space, we get a total contribution to the three-particle correlation energy (below we shall drop writing γ_3^2 explicitly) of

$$W=\int_0^{k_F}d^3k_1d^3k_2d^3k_3\delta(K)\int_0^{\infty}d^3qd^3q'\Bigg\{t_3(\mathbf{k}_{12},\,\mathbf{q}+\tfrac{1}{2}\mathbf{k}_3;\,\gamma_2^2)\frac{Q(|\mathbf{q}+\frac{1}{2}\mathbf{k}_3|,\,\bar{K}_{ij},\,K_F)}{(\mathbf{q}+\frac{1}{2}\mathbf{k}_3)^2+\gamma_2^2}[T^{11}(\mathbf{k}_3+\tfrac{1}{2}\mathbf{q},\,\mathbf{q};\,\mathbf{q}',\,\tfrac{1}{2}\mathbf{q}'+\mathbf{k}_3)$$
$$+T^{12}(\mathbf{k}_3+\tfrac{1}{2}\mathbf{q},\,\mathbf{q};\,\mathbf{q}',\,\tfrac{1}{2}\mathbf{q}'+\mathbf{k}_3)]\frac{Q(|\mathbf{q}'+\frac{1}{2}\mathbf{k}_3|,\,\bar{K}_{ij},\,K_F)}{(\mathbf{q}'+\frac{1}{2}\mathbf{k}_3)^2+\gamma_2^2}t_3(\mathbf{q}'+\tfrac{1}{2}\mathbf{k}_3,\,\mathbf{k}_{12};\,\gamma_2^2)+t_3(\mathbf{k}_2+\tfrac{1}{2}\mathbf{k}_3,\,\mathbf{q}+\tfrac{1}{2}\mathbf{k}_3;\,\gamma_2^2)$$
$$\times\frac{Q(|\mathbf{q}+\frac{1}{2}\mathbf{k}_3|,\,\bar{K}_{ij},\,K_F)}{(\mathbf{q}+\frac{1}{2}\mathbf{k}_3)^2+\gamma_2^2}[T^{11}(\mathbf{k}_3+\tfrac{1}{2}\mathbf{q},\,\mathbf{q};\,\mathbf{q}',\,\tfrac{1}{2}\mathbf{q}'+\mathbf{k}_2)+T^{13}(\mathbf{k}_3+\tfrac{1}{2}\mathbf{q},\,\mathbf{q};\,\mathbf{q}',\,\tfrac{1}{2}\mathbf{q}'+\mathbf{k}_2)]$$
$$\times\frac{Q(|\mathbf{q}'+\frac{1}{2}\mathbf{k}_2|,\,\bar{K}_{ij},\,K_F)}{(\mathbf{q}'+\frac{1}{2}\mathbf{k}_2)^2+\gamma_2^2}t_2(\mathbf{q}'+\tfrac{1}{2}\mathbf{k}_2,\,\mathbf{k}_3+\tfrac{1}{2}\mathbf{k}_2;\,\gamma_2^2)+t_3(\mathbf{k}_1+\tfrac{1}{2}\mathbf{k}_3,\,\mathbf{q}+\tfrac{1}{2}\mathbf{k}_3;\,\gamma_2^2)$$
$$\times\frac{Q(|\mathbf{q}+\frac{1}{2}\mathbf{k}_3|,\,\bar{K}_{ij},\,K_F)}{(\mathbf{q}+\frac{1}{2}\mathbf{k}_3)^2+\gamma_2^2}[T^{12}(\mathbf{k}_3+\tfrac{1}{2}\mathbf{q},\,\mathbf{q};\,\mathbf{q}',\,\tfrac{1}{2}\mathbf{q}'+\mathbf{k}_1)+T^{13}(\mathbf{k}_3+\tfrac{1}{2}\mathbf{q},\,\mathbf{q};\,\mathbf{q}',\,\tfrac{1}{2}\mathbf{q}',+\mathbf{k}_1)]$$
$$\times\frac{Q(|\mathbf{q}'+\frac{1}{2}\mathbf{k}_1|,\,\bar{K}_{ij},\,K_F)}{(\mathbf{q}'+\frac{1}{2}\mathbf{k}_1)^2+\gamma_2^2}t_1(\mathbf{q}'+\tfrac{1}{2}\mathbf{k}_1,\,\mathbf{k}_3+\tfrac{1}{2}\mathbf{k}_1;\gamma_2^2)\Bigg\}. \quad (22)$$

In the above expression we have dropped minus signs occurring before the momentum variables as we are going to use expressions (22) for S-state calculations and hence dependence of $-k$ can be put k. Otherwise Eq. (22) is exact and has no approximation and the sign can be restored back by use of Appendix A. Again in arriving at Eq. (22), the intermediate steps involved are similar to those given in Appendix B.

In order to solve Eqs. (20) and (21) we need expressions for off-energy-shell two-body reaction matrices. Our approach in solving these equations has been motivated by the extensive applications of Faddeev's formalism to the three-nucleon system using separable potentials. The simplicity offered by separable potentials in solving the integral equations has led us to apply them to nuclear-matter calculations. Furthermore, as Lovelace[15] has pointed out, near bound states or resonances the two-body scattering matrix can always be factorized in off-shell variables and thus

$$t(\mathbf{k},\mathbf{p},\gamma^2)=\sum_l\frac{g_l(k)g_l(p)}{k^2+\gamma^2P_l(\hat{\mathbf{k}}\cdot\hat{\mathbf{p}})}. \quad (23)$$

This is, in fact, another reason to use separable potentials since the kernel of the integral equations remains compact and enables one to numerically evaluate the integral equations using matrix methods.

However, a common shortcoming of most separable potentials is the purely attractive nature of the force.

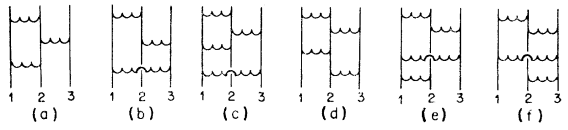

FIG. 4. Third- and fourth-order diagrams involved in the energy calculation.

One cannot expect reasonable results in a nuclear-matter three-body correlation calculation if a separable potential is used which is fitted only to low-energy scattering data. It is only because of the hard-core nature of the force that higher-order diagrams are of comparable magnitude to those of third order. On the other hand, there is every reason to expect that a separable potential which does fit nucleon-nucleon scattering data at all energies and does yield a reasonable saturation density and energy for nuclear matter in a first-order reaction-matrix calculation, will yield a reasonable value for the three-body correlation energy of nuclear matter.

In a previous investigation[17] we calculated the first-order (in t) binding energy per particle and saturation density for a number of separable potentials in current use. We used the reference-spectrum approximation for hole-state energies and assumed kinetic energies only in particle states. In the present work we investigate the cluster energy first with the Yamaguchi average (triplet and singlet) s-state force which is fitted to low-energy data only. This potential has a simple attractive structure and gives rise to a first-order energy per particle of -15 MeV at $k_F=1.36$ F^{-1}, but does not yield reasonable saturation properties. From the above arguments we would expect a very small three-body correlation energy.

The second part of our analysis involves calculations with a modified version of the Puff potential which contains an infinite hard shell of radius $r_c=0.45$ F. The original Puff potential used in our earlier paper has separate attractive singlet and triplet s states and the same hard shell. This potential yielded an energy per particle of -19 MeV at a saturation density corresponding to $k_F=1.6$ F^{-1} and an energy per particle of ~-16 MeV at $k_F=1.36$ F^{-1}. In the present calculation we retain the same hard-shell repulsion and use an average s-state attractive force chosen to yield a binding energy per particle of ~16 MeV at $k_F=1.36$ F^{-1}. We expect the Puff potential to yield results equivalent to those of a local s-state interaction containing a hard shell or hard-core repulsion.

A. Yamaguchi Potential

This is an attractive average s-state force whose momentum-space matrix elements are given by

$$\langle \mathbf{k}|v|\mathbf{k}'\rangle=-\lambda g(k)g(k'), \tag{24}$$

which gives rise to a two-body reaction matrix

$$\langle \mathbf{k}|t|\mathbf{p}\rangle=t(\mathbf{k},\mathbf{p},\gamma_2^2)=g(k)g(p)D(\gamma_2^2), \tag{25}$$

where

$$D(\gamma_2^2)=-\lambda\Big/\left\{1-\lambda\int d^3q\frac{g^2(q)}{q^2+\gamma^2}\right\}. \tag{26}$$

[17] B. S. Bhakar and R. J. McCarthy, Nucl. Phys. (to be published).

The parameter λ is taken to be 0.3519 F^{-3} while the function $g(k)$ is given by

$$g(k)=(k^2+\beta^2)^{-1},$$

where $\beta=1.4488$ F^{-1}. Equation (26) is given in terms of the two-body parameter γ_2^2. The same expression is valid when γ_2^2 is replaced by $\gamma_3^2+\frac{3}{4}k_k^2$ as is required in the three-body reaction matrix equation.

Substitution of Eq. (25) in Eqs. (20) and (21) yields

$$T^{ii}(\mathbf{k}_{jk},\mathbf{k}_i;\,\mathbf{p}_{jk},\mathbf{p}_i;\,\gamma_3^2)=g(k_{jk})g(p_{jk})D(\gamma_3^2+\tfrac{3}{4}k_i^2)\times[A(\mathbf{k}_i,\mathbf{p}_i)+B(\mathbf{k}_i,\mathbf{p}_i)], \tag{27}$$

and

$$T^{ji}(\mathbf{k}_{ki},\mathbf{k}_j;\,\mathbf{p}_{jk},\mathbf{p}_i;\,\gamma_3^2)=g(k_{ki})g(p_{jk})D(\gamma_3^2+\tfrac{3}{4}k_j^2)B(\mathbf{k}_j,\mathbf{p}_i),$$

where the spectator functions A and B obey the integral equations

$$A(\mathbf{k},\mathbf{p})=\delta(\mathbf{k}-\mathbf{p})+\int d^3qK(\mathbf{k},\mathbf{q})A(\mathbf{q},\mathbf{p}),$$

and

$$B(\mathbf{k},\mathbf{p})=-K(\mathbf{k},\mathbf{p})-\int d^3q\Big[K(\mathbf{k},\mathbf{q})+2\int d^3q'K(\mathbf{k},\mathbf{q}')K(\mathbf{q}',\mathbf{q})\Big]B(\mathbf{q},\mathbf{p}). \tag{28}$$

The kernel of the integral equations is given by

$$K(\mathbf{k},\mathbf{p})=\frac{g(\mathbf{k}+\frac{1}{2}\mathbf{p})g(\mathbf{p}+\frac{1}{2}\mathbf{k})}{k^2+p^2+\mathbf{k}\cdot\mathbf{p}+\gamma_3^2}D(\gamma_3^2+\tfrac{3}{4}p^2). \tag{29}$$

An iterative procedure was adopted to replace the part of Eq. (28) involving $K(\mathbf{k},\mathbf{p})$. Thus the $\delta(\mathbf{k}-\mathbf{p})$ part of $A(\mathbf{k},\mathbf{p})$ is kept as such so as to enable us to calculate separately the third-order diagrams contribution to W. Finally the values of γ_2^2 and γ_3^2 were taken as given in Appendix A.

B. Puff's Potential

The potential in momentum space is again separable and is given by

$$\langle \mathbf{k}|v|\mathbf{k}'\rangle=-\lambda g(k)g(k')+\lambda_{\text{hard shell}}h(k)h(k'), \tag{30}$$

where $\lambda_{\text{hard shell}}$ in the limit goes to infinity. The parameter λ of the attractive terms is given as 6.35 F^{-3} while the functions $g(k)$ and $h(k)$ are defined by

$$g(k)=(k^2+\beta^2)^{-1},$$

and

$$h(k)=(\sin k\cdot r_c)/k,$$

where $\beta=2.2785$ F^{-1} and $r_c=0.45$ F. Using this potential the two-body scattering matrix is given by

$$\langle \mathbf{k}|t|\mathbf{p}\rangle=g(k)g(p)D_1(\gamma_2^2)+h(k)h(p)D_2(\gamma_2^2)+\{g(k)h(p)+h(k)g(p)\}D_3(\gamma_2^2), \tag{31}$$

where

$$D_1(\gamma_2^2)=-\lambda H_2(\gamma_2^2)\{H_2(\gamma_2^2)(1-\lambda H_1(\gamma_2^2))+\lambda H_3^2(\gamma_2^2)\}^{-1},$$

$$D_2(\gamma_2^2)=\{1-\lambda H_1(\gamma_2^2)\}\{H_2(\gamma_2^2)(1-\lambda H_1(\gamma_2^2))+\lambda H_3^2(\gamma_2^2)\}^{-1},$$

and

$$D_3(\gamma_2^2)=\lambda H_3(\gamma_2^2)\{H_2(\gamma_2^2)(1-\lambda H_1(\gamma_2^2))+\lambda H_3^2(\gamma_2^2)\}^{-1}. \quad (32)$$

The functions $H_1(\gamma_2^2)$, $H_2(\gamma_2^2)$, and $H_3(\gamma_2^2)$ are given by the following integrals:

$$H_1(\gamma_2^2)=\int d^3q\frac{g^2(q)}{q^2+\gamma^2},$$

$$H_2(\gamma_2^2)=\int d^3q\frac{h^2(q)}{q^2+\gamma^2},$$

and

$$H_3(\gamma_2^2)=\int d^3q\frac{g(q)h(q)}{q^2+\gamma^2}. \quad (33)$$

Taking expression (30) for the two-body t matrix and replacing this in Eqs. (20) and (21) yields the expressions

$$T^{ii}(\mathbf{k}_{jk},\mathbf{k}_i;\,\mathbf{p}_{jk},\mathbf{p}_i;\,\gamma_3^2)=g(k_{jk})g(p_{jk})A_1(\mathbf{k}_i,\mathbf{p}_i)+h(k_{jk})h(p_{jk})B_2(\mathbf{k}_i,\mathbf{p}_i)+g(k_{jk})h(p_{jk})A_2(\mathbf{k}_i,\mathbf{p}_i)+h(k_{jk})g(p_{jk})B_1(\mathbf{k}_i,\mathbf{p}_i),$$

and

$$T^{ji}(\mathbf{k}_{ki},\mathbf{k}_j;\,\mathbf{p}_{jk},\mathbf{p}_i;\,\gamma_3^2)=g(k_{ki})g(p_{jk})C_1(\mathbf{k}_j,\mathbf{p}_i)+h(k_{ki})h(p_{jk})E_2(\mathbf{k}_j,\mathbf{p}_i)+g(k_{ki})h(p_{jk})C_2(\mathbf{k}_j,\mathbf{p}_i)+h(k_{ki})g(p_{jk})E_1(\mathbf{k}_j,\mathbf{p}_i), \quad (34)$$

where the spectator functions (A's, B's, C's, and E's) satisfy the two sets of coupled integral equations

$$A_1(\mathbf{k},\mathbf{p})=D_1(\gamma_3^2+\tfrac{3}{4}k^2)\delta(\mathbf{k}-\mathbf{p})-2\int d^3q\times[K_1(\mathbf{k},\mathbf{q})C_1(\mathbf{q},\mathbf{p})+K_2(\mathbf{k},\mathbf{q})E_1(\mathbf{q},\mathbf{p})], \quad (35a)$$

$$B_1(\mathbf{k},\mathbf{p})=D_3(\gamma_3^2+\tfrac{3}{4}k^2)\delta(\mathbf{k}-\mathbf{p})-2\int d^3q\times[K_3(\mathbf{k},\mathbf{q})C_1(\mathbf{q},\mathbf{p})+K_4(\mathbf{k},\mathbf{q})E_1(\mathbf{q},\mathbf{p})], \quad (35b)$$

$$C_1(\mathbf{k},\mathbf{p})=-\int d^3q[K_1(\mathbf{k},\mathbf{q})\{C_1(\mathbf{q},\mathbf{p})+A_1(\mathbf{q},\mathbf{p})\}+K_2(\mathbf{k},\mathbf{q})\{E_1(\mathbf{q},\mathbf{p})+B_1(\mathbf{q},\mathbf{p})\}], \quad (35c)$$

$$E_1(\mathbf{k},\mathbf{p})=-\int d^3q[K_3(\mathbf{k},\mathbf{q})\{C_1(\mathbf{q},\mathbf{p})+A_1(\mathbf{q},\mathbf{p})\}+K_4(\mathbf{k},\mathbf{q})\{E_1(\mathbf{q},\mathbf{p})+B_1(\mathbf{q},\mathbf{p})\}]. \quad (35d)$$

Similar equations are obtained for A_2, B_2, C_2, and E_2 if one makes the replacements $D_1\rightarrow D_3$ and $D_3\rightarrow D_2$ in the inhomogeneous terms, and changes the subscripts $1\rightarrow 2$ in A, B, C, and E. The kernels of these integral equations are given by

$$K_1(\mathbf{k},\mathbf{q})=GG(\mathbf{k},\mathbf{q})D_1(\gamma_3^2+\tfrac{3}{4}k^2)+HG(\mathbf{k},\mathbf{q})D_3(\gamma_3^2+\tfrac{3}{4}k^2),$$

$$K_2(\mathbf{k},\mathbf{q})=GH(\mathbf{k},\mathbf{q})D_1(\gamma_3^2+\tfrac{3}{4}k^2)+HH(\mathbf{k},\mathbf{q})D_3(\gamma_3^2+\tfrac{3}{4}k^2),$$

$$K_3(\mathbf{k},\mathbf{q})=GG(\mathbf{k},\mathbf{q})D_3(\gamma_3^2+\tfrac{3}{4}k^2)+HG(\mathbf{k},\mathbf{q})D_2(\gamma_3^2+\tfrac{3}{4}k^2),$$

$$K_4(\mathbf{k},\mathbf{q})=HH(\mathbf{k},\mathbf{q})D_2(\gamma_3^2+\tfrac{3}{4}k^2)+GH(\mathbf{k},\mathbf{q})D_3(\gamma_3^2+\tfrac{3}{4}k^2). \quad (36)$$

Because of the following expressions:

$$GG(\mathbf{k},\mathbf{q})=GG(\mathbf{q},\mathbf{k})=\frac{g(\mathbf{k}+\frac{1}{2}\mathbf{q})g(\mathbf{q}+\frac{1}{2}\mathbf{k})}{k^2+q^2+\mathbf{k}\cdot\mathbf{q}+\gamma_3^2},$$

$$HH(\mathbf{k},\mathbf{q})=HH(\mathbf{q},\mathbf{k})=\frac{h(\mathbf{k}+\frac{1}{2}\mathbf{q})h(\mathbf{q}+\frac{1}{2}\mathbf{k})}{k^2+q^2+\mathbf{k}\cdot\mathbf{q}+\gamma_3^2},$$

and

$$HG(\mathbf{k},\mathbf{q})=GH(\mathbf{q},\mathbf{k})=\frac{h(\mathbf{k}+\frac{1}{2}\mathbf{q})g(\mathbf{q}+\frac{1}{2}\mathbf{k})}{k^2+q^2+\mathbf{k}\cdot\mathbf{q}+\gamma_3^2}, \quad (37)$$

we notice that the spectator functions are not all independent but they have certain symmetries which result in much saving of computer time. Thus here we have

$$A_1(\mathbf{k},\mathbf{p})=A_1(\mathbf{p},\mathbf{k}),$$
$$B_2(\mathbf{k},\mathbf{p})=B_2(\mathbf{p},\mathbf{k}),$$
$$C_1(\mathbf{k},\mathbf{p})=C_1(\mathbf{p},\mathbf{k}),$$
$$E_2(\mathbf{k},\mathbf{p})=E_2(\mathbf{p},\mathbf{k}),$$

and further

$$E_1(\mathbf{k},\mathbf{p})=C_2(\mathbf{p},\mathbf{k}),$$
$$B_1(\mathbf{k},\mathbf{p})=A_2(\mathbf{p},\mathbf{k}).$$

Another simplification which appears is the fact that in Eq. (35) we have essential coupling only between $C_1(\mathbf{k},\mathbf{p})$ and $E_1(\mathbf{k},\mathbf{p})$, which becomes clear if one eliminates $A_1(\mathbf{k},\mathbf{p})$ and $B_1(\mathbf{k},\mathbf{p})$ from the last two equations. Thus one has to solve only two coupled equations, which we solved by iteration after taking the θ_{kq} angle average of the kernels in Eq. (36). Once the matrices for $C_1(\mathbf{k},\mathbf{p})$ and $E_1(\mathbf{k},\mathbf{p})$ are known, the structure of $A_1(\mathbf{k},\mathbf{p})$ and $B_1(\mathbf{k},\mathbf{p})$ is given in terms of them. Thus it appears as if there are two sets of four coupled integral equations, but in fact one has only two sets of two coupled equations. Another point which we would like to emphasize is that the averaging of the kernels has an impact only on the spectator functions and hence most of the three-particle T matrix is given in analytic form. This is very useful for the evaluation of W. The latter appears to have several three-dimensional

TABLE I. Three-body cluster contributions to the energy per particle of nuclear matter. Units in MeV. See text for description of individual terms.

	W_B	W_R	W_1	W_2	W_3	W
Yamaguchi	−0.241	−0.129	−0.021	−0.013	−0.023	−0.427
Puff	+1.09	+1.00	−0.096	−0.059	−1.10	+0.835

integrals, but most of them are quite independent and involve at most one angle θ between two vectors which can be evaluated analytically or by numerical methods. Moreover, the magnitude integration is also of decoupled form. Therefore, in numerical calculations, most parts can be evaluated in one do-loop. For the integrations we used the Gaussian quadrature method with 20 points and the integrals involving 0 to ∞ integration over magnitude were checked before making an upper-limit cutoff.

Equations (27) and (34) can now be used in Eq. (22) to find the three-body correlation energy for the Yamaguchi and Puff potentials, respectively. Since we have explicitly separated out the δ-function term in T^{11} we can calculate separately the two third-order diagrams and the contribution of all fourth- and higher-order diagrams. Thus (22) can be expressed as

$$W=W_B+W_R+W_1+W_2+W_3, \tag{38}$$

where W_B and W_R are the contributions of the third-order bubble and ring diagrams while W_i is the contribution of all fourth- and higher-order diagrams having t_i as the initial interaction.

We now consider the effect of exchange terms and spin-isospin statistics. In calculating the energy we must sum over all distinct diagrams, and over all possible initial and final states. It is sufficient to antisymmetrize one wave function and we choose the initial state. Thus the energy is given in Eq. (22) by six sums of the form

$$W=\sum_{l,m,n}\langle \mathbf{l},\mathbf{m},\mathbf{n}|\,t(Q/e)T(Q/e)t\,|\mathbf{lmn}-\mathbf{mln}-\mathbf{lnm}-\mathbf{nml}+\mathbf{mnl}+\mathbf{nlm}\rangle. \tag{39}$$

There is no factor of $\frac{1}{6}$ appearing since we are summing only over topologically distinct diagrams (i.e., those diagrams which cannot be related to each other by any exchange). The sum over the momenta l, m, and n converts into an integral over the three momenta. As mentioned previously, we must integrate these three momenta independently from 0 to k_F. The spatial contribution of each of the six terms in Eq. (39) will be the same since we are considering only s-state interactions. There are 64 possible spin-isospin states available for the three particles, each of which contributes to the direct term. Each of the single-exchange terms (which appear with a minus sign) carries a weight factor of 16 since the spin-isospin states of the exchanged nucleons must be equal. Similarly, the double-exchange terms carry a weight factor of +4. Therefore, since the spatial contributions of all the terms are equal, we can rewrite Eq. (39) as

$$W=24\int_0^{k_F}d^3k_1\int_0^{k_F}d^3k_2\int_0^{k_F}d^3k_3\times\langle \mathbf{k}_1\mathbf{k}_2\mathbf{k}_3|\,\underset{e}{Q}\,t\,\underset{e}{Q}\,T\,t\,|\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3\rangle. \tag{40}$$

Equation (40) actually gives the contribution of $g(Q/e)T(Q/e)g$ to the energy per unit volume and must be divided by the appropriate density to give the energy per particle.

IV. DISCUSSION OF RESULTS

The results obtained with the Yamaguchi and Puff potentials are shown in Table I. The total three-body cluster energy is found to be small for both potentials, but for different reasons. For the Yamaguchi case, the third-order diagrams dominate, as was expected, but the third-order terms are themselves very small. Since the potential has no repulsive term all the diagrams are attractive and no cancellations are contributing to the small results for W_1, W_2, and W_3. The contributions of the bubble and ring diagrams are of comparable magnitude to previous estimates[18,19] obtained using simple, s-state, attractive forces.

Results obtained with the modified Puff potential are of much greater interest. The third-order diagrams are repulsive due to the combined effect of the hard shell and the Pauli operator. However, the higher-order terms are also important and give an attractive contribution. Thus the sum of all higher-order terms cancels a large part of the repulsive third-order terms.

The final results for the Puff potential are due to a delicate balance between attractive and repulsive terms and this balance depends a great deal on the effect of the Pauli operator Q shown explicitly in Eq. (9). A third-order diagram involves the operator $t(Q/e)t(Q/e)t$, where the middle t operator is evaluated off the energy shell and without the Pauli operator. Both energy denominators are positive-definite while the matrix elements of the three t operators can be positive or negative depending on the relative momenta involved. The Pauli operator Q discriminates against small relative momenta in intermediate states and thus causes matrix elements of t to be positive on the average. When the Pauli operator is neglected, the third-order terms become attractive and the higher-order terms become repulsive.

A further cancellation appears in summing the higher-order diagrams. Because of the Pauli operator each t matrix element is, on the average, repulsive. Thus the sign of an nth order term is determined solely by the

[18] K. A. Brueckner, Phys. Rev. **100**, 36 (1955).
[19] H. A. Bethe, Phys. Rev. **103**, 1353 (1956).

sign obtained when Eq. (7) is iterated and the over-all attractive contributions of W_1, W_2, and W_3 seem to be determined by the fact that the fourth-order diagrams are attractive and the successive terms in the alternating series are of decreasing magnitude.

Our results seem to confirm Bethe's conclusion[8] that the total three-body cluster contribution is small. However, the individual terms do not agree with previous estimates obtained using hard-core or hard-shell potentials. For example, the contribution of the bubble diagram above is much smaller than the 5 to 10 MeV per particle obtained in most previous estimates[5,20,21] and is closer to the results obtained by Brueckner and Masterson[22] who did not treat off-energy-shell effects correctly. However, the reason for the discrepancy does not lie in our off-energy-shell treatment but rather in the fact that the hard-shell potential acts only in s states rather than in all partial waves. An exact comparison of results is rather difficult here since we evaluate the bubble-diagram contribution directly rather than introducing intermediate-state potentials.

On the other hand, the contribution of the ring diagram found here is larger than previous estimates.[19,23] Our result for this term might be reduced somewhat if our force contained Serber exchange characteristics but our present result confirms the estimate of Rajaraman[6] that all third-order terms are of comparable magnitude.

We have not considered higher partial waves, spin-dependent forces, nor tensor forces. These effects could modify our quantitative results a great deal and would probably make our over-all result attractive. However, we would still expect the total three-body contribution to be small. Therefore it seems more worthwhile to consider first the case of a soft core or velocity-dependent potential since, at this time, it appears that hard-core potentials cannot yield the correct binding energy per particle for nuclear matter or for finite nuclei.

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APPENDIX A: NOTATION AND KINEMATICS

Let the system of three particles be characterized by masses m_1, m_2, m_3, and momenta p_1, p_2, p_3, where subscripts 1, 2, 3 denote particles. Sometime we denote these particles by (i,j,k) subscripts which take values 1, 2, 3 cyclicly, or we use (l,m,n) for particles inside the Fermi sea. This system can be characterized in momentum space by three equivalent sets of variables which are well suited for the c.m. coordinate system.

$$\mathbf{K}=\mathbf{P}_1+\mathbf{P}_2+\mathbf{P}_3, \quad \mathbf{K}_{ij}=\mathbf{P}_i+\mathbf{P}_j$$
$$\mathbf{k}_{ij}=(m_j\mathbf{P}_i-m_i\mathbf{P}_j)/m_{ij}, \tag{A1}$$

and

$$\mathbf{k}_k=\{m_k(\mathbf{P}_i+\mathbf{P}_j)-m_{ij}\mathbf{P}_k\}/M,$$

where

$$M=m_i+m_j+m_k, \quad m_{ij}=m_i+m_j.$$

The inverse relations are given by

$$\mathbf{P}_i=\frac{m_i}{M}\mathbf{K}+\frac{m_i}{m_{ij}}\mathbf{k}_k+\mathbf{k}_{ij},$$

$$\mathbf{P}_j=\frac{m_j}{M}\mathbf{K}+\frac{m_j}{m_{ij}}\mathbf{k}_k-\mathbf{k}_{ij},$$

and

$$\mathbf{P}_k=\frac{m_K}{M}\mathbf{K}_k-\mathbf{k}_k. \tag{A2}$$

The three coordinate systems thus defined are related to each other and one can transform from one to the other using the relations

$$\mathbf{k}_{jk}=-\frac{m_k}{m_{jk}}\mathbf{k}_{ij}+\frac{m_kM}{m_{jk}m_{ij}}\mathbf{k}_k,$$

and

$$\mathbf{k}_i=-\mathbf{k}_{ij}-\frac{m_i}{m_{ij}}\mathbf{k}_k. \tag{A3}$$

The total

$$\text{K.E.}=\frac{K^2}{2M}+\frac{k_{ij}^2}{2\mu_{ij}}+\frac{k_k^2}{2\mu_k},$$

where the reduced masses

$$\mu_{ij}=\frac{m_im_j}{m_{ij}} \quad \text{and} \quad \mu_k=\frac{m_km_{ij}}{M}. \tag{A4}$$

Sometimes it is convenient to work with pairs of variables $\mathbf{k}_i$, $\mathbf{k}_j$, $\mathbf{k}_k$ which have the property

$$\mathbf{k}_i+\mathbf{k}_j+\mathbf{k}_k=0. \tag{A5}$$

This is true whether you have c.m. momentum zero or not. In terms of these variables the above variables can be put as

$$\mathbf{k}_{ij}=-\mathbf{k}_i-\frac{m_i}{m_{ij}}\mathbf{k}_k=\mathbf{k}_j+\frac{m_{jk}}{m_{ij}}\mathbf{k}_k \tag{A6}$$

and total

$$\text{K.E.}=\frac{k_i^2}{2\mu_{ki}}+\frac{\mathbf{k}_i\cdot\mathbf{k}_j}{m_k}+\frac{k_j^2}{2\mu_{jk}}. \tag{A7}$$

These expressions become simple in the case of equal-

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[21] S. A. Coon and J. Dabrowski, Phys. Rev. **140**, B287 (1965).
[22] K. A. Brueckner and K. S. Masterson, Jr., Phys. Rev. **128**, 2267 (1962).
[23] H. S. Köhler, Ann. Phys. (N. Y.) 12, 444 (1961).

mass particles. Thus

$$\mathbf{k}_{ij}=\tfrac{1}{2}(\mathbf{P}_i-\mathbf{P}_j),\qquad \mathbf{k}_k=\tfrac{1}{3}(\mathbf{P}_i+\mathbf{P}_j)-\tfrac{2}{3}\mathbf{P}_k, \tag{A8}$$

and

$$\mathbf{k}_{ij}=-\tfrac{1}{2}\mathbf{k}_{ki}+\tfrac{3}{4}\mathbf{k}_j=-\tfrac{1}{2}\mathbf{k}_{jk}-\tfrac{3}{4}\mathbf{k}_i,\qquad \mathbf{k}_k=-\mathbf{k}_{ki}-\tfrac{1}{2}\mathbf{k}_j=\mathbf{k}_{jk}-\tfrac{1}{2}\mathbf{k}_i. \tag{A9}$$

Similarly we have

$$\mathbf{k}_{ij}=-\mathbf{k}_i-\tfrac{1}{2}\mathbf{k}_k=\mathbf{k}_j+\tfrac{1}{2}\mathbf{k}_k. \tag{A10}$$

The Jacobian of the transformations among different coordinate representations is unity, and

$$\text{Total K.E.}=(1/m)[k_{ij}^2+\tfrac{3}{4}k_k^2]=(1/m)[k_i^2+\mathbf{k}_i\cdot\mathbf{k}_j+k_j^2]. \tag{A11}$$

Further we assume the total momentum is conserved since nuclear matter is expected to have translational invariance. Thus a total-momentum δ function will be assumed to be factored out. The Hilbert space for the three-particle system consists of square-integrable functions of two-momentum vectors $|\mathbf{p},\mathbf{q}\rangle$ normalized to one.

The three-particle energy denominator is

$$e=E_i+E_j+E_K-E_l-E_m-E_n, \tag{A12}$$

where i, j, k refer to particles in excited states, and l, m, n to particles inside the Fermi sea (again cyclic notation is maintained).

The single-particle energy for states above the Fermi sea is given by

$$E_i=\tfrac{1}{2}k_i^2, \tag{A13}$$

where, in our notation, $\hbar=m=1$. We use the reference spectrum approximation for states below the Fermi sea so

$$E_l=(k_l^2/2m^*)+A, \tag{A14}$$

where the reference-spectrum parameters m^* and A must be found from self-consistent two-body calculations. Thus the three-particle energy denominator can be put in momentum space as

$$\langle\mathbf{k}_{ij},\mathbf{k}_k|e^{-1}|\mathbf{k}_{ij}',\mathbf{k}_k'\rangle=\delta(\mathbf{k}_{ij}-\mathbf{k}_{ij}')\delta(\mathbf{k}_k-\mathbf{k}_k')\times[k_{ij}^2+\tfrac{3}{4}k_k^2+\gamma_3^2]^{-1}, \tag{A15}$$

where

$$\gamma_3^2=-3A+\tfrac{1}{6}K^2(1-1/m^*)-(k_{lm}^2+\tfrac{3}{4}k_n^2)/m^*\approx-3A+(3k_F^2/10)(1-3/m^*). \tag{A16}$$

We list here also the expressions for the one-energy-shell energy denominators and the corresponding γ_2^2 for the two-body case:

$$\langle\mathbf{k}_{ij},\mathbf{k}_k|e^{-1}|\mathbf{k}_{ij}',\mathbf{k}_k'\rangle=\delta(\mathbf{k}_{ij}-\mathbf{k}_{ij}')\delta(\mathbf{k}_k-\mathbf{k}_k')[k_{ij}^2+\gamma_2^2]^{-1}, \tag{A17}$$

and

$$\gamma_2^2=-2A+\tfrac{1}{4}K_{lm}^2(1-1/m^*)-k_{lm}^2/m^*\approx-2A+(3k_F^2/10)(1-2/m^*). \tag{A18}$$

In the expressions for γ_2^2 and γ_3^2 we have used the following momentum averages:

$$\langle K^2\rangle=(9/5)k_F^2,\quad \langle k_{lm}^2\rangle=\tfrac{3}{10}k_F^2,\quad \langle k_n^2\rangle=\tfrac{2}{5}k_F^2,$$

and

$$\langle K_{lm}^2\rangle=(6/5)k_F^2. \tag{A19}$$

However, we do not average over the third-particle momentum k_k^2 occurring in Eq. (A15). Besides the three-particle system, we shall also use two-particle subsystems of it. The corresponding operators in three-particle Hilbert space are given by

$$\langle\mathbf{k}_{ij},\mathbf{k}_k|\,t_k(\gamma_3^2)\,|\mathbf{p}_{ij},\mathbf{p}_k\rangle=\delta(\mathbf{k}_k-\mathbf{p}_k)t_k(\mathbf{k}_{ij},\mathbf{p}_{ij};\gamma_3^2+\tfrac{3}{4}k_k^2), \tag{A20}$$

and most of the time we use the notation for the three-particle reaction matrix as

$$\langle\mathbf{k}_{ij},\mathbf{k}_k|\,T^{kj}|\,\mathbf{p}_{ki},\mathbf{p}_j\rangle=T^{kj}(\mathbf{k}_{ij},\mathbf{k}_k;\,\mathbf{p}_j,\mathbf{p}_{ki};\,\gamma_3^2). \tag{A21}$$

This notation is convenient when writing down the integral equations.

Finally, we list here the reference-spectrum parameters used in defining γ_2^2 and γ_3^2. We have

$$A=-2.255\ \text{fm}^{-2},\quad m^*=0.5995;$$

and

$$A=-2.346\ \text{fm}^{-2},\quad m^*=0.5373,$$

for the Yamaguchi and Puff calculations, respectively.

APPENDIX B: INTERMEDIATE STEPS

The reaction matrix for two-particle correlations is given in momentum space by

$$\langle\mathbf{k}|\,t_i(\gamma_2^2)\,|\mathbf{k}'\rangle=\langle\mathbf{k}|\,v\,|\mathbf{k}'\rangle-\int d^3p\frac{\langle\mathbf{k}|\,v\,|\mathbf{p}\rangle Q(p,K,k_F)\langle\mathbf{p}|\,t_i(\gamma_2^2)\,|\mathbf{k}'\rangle}{p^2+\gamma_2^2}, \tag{B1}$$

where γ_2^2 is given by Eq. (A18) and we use the angle-averaged Pauli operator defined by

$$Q(p,K,k_F)=0\quad\text{if}\quad(p^2+K^2)^{1/2}<k_F,\qquad=1\quad\text{if}\quad|\mathbf{p}-\mathbf{K}|>k_F,\qquad=\frac{p^2+K^2-k_F^2}{2pk_F},\quad\text{otherwise.} \tag{B2}$$

Here K is the two-body center-of-mass momentum instead of the usual three-body c.m. momentum. We use the average values of K and K^2 in evaluating the Pauli operator.

Here below we give details of the steps involved in obtaining integral equations for the three-particle reaction matrix. Thus Eq. (7) in momentum space becomes

$$\langle \mathbf{k}_{jk},\mathbf{k}_i|\,T^{ii}|\,\mathbf{p}_{jk},\mathbf{p}_i\rangle=\langle \mathbf{k}_{jk}|\,t_i(\gamma_3^2+\tfrac{3}{4}k_i^2)|\,\mathbf{p}_{jk}\rangle\delta(\mathbf{k}_i-\mathbf{p}_i)$$
$$-\langle \mathbf{k}_{jk},\mathbf{k}_i|\,t_i(\gamma_3^2)e^{-1}\{T^{ji}+T^{ki}\}|\,\mathbf{p}_{jk},\mathbf{p}_i\rangle. \quad \text{(B3)}$$

Similarly one can write down an expression for T^{ji}. The most important advantage of Faddeev's approach is used here, i.e., the wave function has been expressed in terms of momentum variables which are appropriate for two interacting particles and a third particle. Introducing intermediate states the second term in (B3) becomes

$$\int_{\text{over all primed variables}}\langle \mathbf{k}_{jk},\mathbf{k}_i|\,t_i(\gamma_3^2)|\,\mathbf{p}_{jk}',\mathbf{p}_i'\rangle$$
$$\times[\langle \mathbf{p}_{jk}',\mathbf{p}_i'|\,e^{-1}|\,\mathbf{p}_{ki}'',\mathbf{p}_j''\rangle\langle \mathbf{p}_{ki}'',\mathbf{p}_j''|\,T^{ji}|\,\mathbf{p}_{jk},\mathbf{p}_i\rangle$$
$$+\langle \mathbf{p}_{jk}',\mathbf{p}_i'|\,e^{-1}|\,\mathbf{p}_{ij}'',\mathbf{p}_k''\rangle\langle \mathbf{p}_{ij}'',\mathbf{p}_k''|\,T^{ki}|\,\mathbf{p}_{jk},\mathbf{p}_i\rangle].$$

This term can be simplified by making use of the three δ functions occurring in

$$\langle \mathbf{k}_{jk},\mathbf{k}_i|\,t_i(\gamma_3^2)|\,\mathbf{p}_{jk}',\mathbf{p}_i'\rangle=\delta(\mathbf{k}_i-\mathbf{p}_i)t_i(\mathbf{k}_{jk},\mathbf{p}_{jk}';\gamma_3^2+\tfrac{3}{4}k_i^2),$$

and

$$\langle \mathbf{p}_{jk}',\mathbf{p}_i'|\,e^{-1}|\,\mathbf{p}_{ki}'',\mathbf{p}_j''\rangle$$
$$=\delta(\mathbf{p}_{jk}'-\mathbf{p}_{jk}'')\delta(\mathbf{p}_i'-\mathbf{p}_i'')[p_{jk}'^2+\tfrac{3}{4}p_i'^2+\gamma_3^2]^{-1}. \quad \text{(B4)}$$

These δ functions lead to the following expression after performing integrations:

$$\int d^3p_j'\frac{t_i(\mathbf{k}_{jk},\,-\mathbf{p}_j'-\tfrac{1}{2}\mathbf{k}_i;\,\gamma_3^2+\tfrac{3}{4}k_i^2)}{k_i^2+p_j'^2+\mathbf{k}_i\cdot\mathbf{p}_j'+\gamma_3^2}$$
$$\times T^{ji}(\mathbf{k}_i+\tfrac{1}{2}\mathbf{p}_j',\,\mathbf{p}_j';\,\mathbf{p}_i,\,\mathbf{p}_{jk};\,\gamma_3^2)$$
$$+\int d^3\mathbf{p}_k'\frac{t_i(\mathbf{k}_{jk},\,\mathbf{p}_k'+\tfrac{1}{2}\mathbf{k}_i;\,\gamma_3^2+\tfrac{3}{4}k_i^2)}{k_i^2+p_k'^2+\mathbf{k}_i\cdot\mathbf{p}_k'+\gamma_3^2}$$
$$\times T^{ki}(-\mathbf{k}_i-\tfrac{1}{2}\mathbf{p}_k,\,\mathbf{p}_k';\,\mathbf{p}_i,\,\mathbf{p}_{jk};\,\gamma_3^2).$$

In the last step we have made use of the following transformations:

$$\mathbf{p}_{jk}'=-\tfrac{1}{2}\mathbf{p}_{ij}'+\tfrac{3}{4}\mathbf{p}_k'=-\tfrac{1}{2}\mathbf{p}_{ki}'-\tfrac{3}{4}\mathbf{p}_j',$$
$$\mathbf{p}_i'=\mathbf{p}_{ki}'-\tfrac{1}{2}\mathbf{p}_j'=-\mathbf{p}_{ij}'-\tfrac{1}{2}\mathbf{p}_k',$$

and

$$\mathbf{p}_{jk}'=-\mathbf{p}_j'-\tfrac{1}{2}\mathbf{k}_i,\quad \mathbf{p}_{ki}'=\mathbf{k}_i+\tfrac{1}{2}\mathbf{p}_j'$$

in the first term and

$$\mathbf{p}_{jk}'=\mathbf{p}_k'+\tfrac{1}{2}\mathbf{k}_i;\quad \mathbf{p}_{ij}'=-\mathbf{k}_i-\tfrac{1}{2}\mathbf{p}_k'$$

in the second term. These lead to our expressions (20) and (21) in the text when the integration variable is replaced by $\mathbf{q}$.

A similar procedure has been adopted throughout the calculations for the reduction of the reaction-matrix equations and also in the evaluation of W.