case (Fig. 2), the Inglis formula gives $A=0.03$, whereas our method gives $A=0.116$, nearly four times the Inglis result. If we now change the single-particle spacings only slightly (Fig. 3), we do not expect the spectrum to undergo a drastic change. However, the Inglis formula now gives negative result for A, i.e., $A=-0.007$, whereas our method gives $A=0.114$, a sensible result. Quite generally, the Inglis formula is not applicable to odd-A or odd-odd nuclei, so that the comparison in the case of ^{21}Ne is rather unfair.

Note that in Figs. 2 and 3 some unoccupied orbitals come below some occupied orbitals. Also, the degeneracy between $|K\rangle$ and $|-K\rangle$ orbitals maintained in an even-even nucleus is removed. These phenomena have been discussed before[3] and we will not go into these details again here. We want to emphasize, however, that although in the $K=\frac{3}{2}$ band there is no real gap between occupied and unoccupied levels, the HF description is still valid. This is because there is no low-lying two-particle–two-hole state with $K=\frac{3}{2}$, the only K value which can connect to the ground state. Indeed, by explicitly calculating second-order corrections to the ground state in a Rayleigh-Schrödinger expansion we verified that the HF description for the $K=\frac{3}{2}$ band in ^{21}Ne should be as valid as in the $K=0$ band in ^{20}Ne, where a significant gap appears between occupied and unoccupied orbitals.

VI. CONCLUSION

We have suggested a method for calculating the moment of inertia which, unlike the Inglis method, can be used for any nucleus, even-even, odd-A, or odd-odd. By calculating higher-order corrections in many cases in the s-d shell, we have verified that these are small. Although the examples shown used intrinsic HF states, the method can be readily extended to cases where Hartree-Fock-Bogoliubov correlations exist. Equations (2) and (3) can be also used when the intrinsic state is, for example, a Tamm-Dancoff state. Our only restriction is that we do not allow for any change of the intrinsic state from one spin state to another but, in view of the past investigations,[2,3] this is not a severe restriction in the deformed region of the s-d shell.

ACKNOWLEDGMENTS

Part of this work was done when one of the authors (S.D.G.) was a summer visitor at Chalk River Nuclear Laboratories. He would like to thank Dr. L. G. Elliott, Dr. G. C. Hanna, and H. H. Clayton for the hospitality extended to him.

PHYSICAL REVIEW VOLUME 164, NUMBER 4 20 DECEMBER 1967

Morse Function and Velocity-Dependent Nucleon-Nucleon Potentials*

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(Received 12 July 1967)

We investigate a static soft-core potential (the Morse potential) and three representative velocity-dependent N-N potentials. First we study the problem of adjusting potentials to maintain the $1S$ state bound near zero energy, and we test two integral criteria. For the Morse potential we derive a convenient analytical expression for the S-wave phase shifts. By fitting experimental phase shifts in the energy range $E_{\rm lab}\leq 400$ MeV for elastic nucleon-nucleon scattering, we evaluate the potential parameters for the $^1S_0$ and $^3S_1$ states. We then examine and compare a phenomenological and two-meson-theoretic one-boson-exchange potentials in S states which are representative of a number of N-N potentials in current use. We point to ways of characterizing these potentials which are helpful in assessing their scattering consequences. We show that different mechanisms embodied in such potentials can account equally well for the negative S-wave phase shifts observed at higher energies.

1. INTRODUCTION

THE nucleon-nucleon interaction is S states has and continues to be a fundamental topic of nuclear research. The development of effective-range theory[1–4] which indicated that any reasonable two-parameter potential can fit the low-energy data (i.e., under 10 MeV) was an important early clarification of the problem. Then the introduction of a hard core by Jastrow,[5] to account for high-energy data up to about 350 MeV where π-meson production begins, had a great impact not only upon our view of the N-N interaction but also later in the development of the Brueckner–Hartree-Fock[6,7] treatment of the nuclear many-body problem.

* Supported in part by the U. S. Air Force Office of Scientific Research Grant No. 902-66.
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In recent years a number of studies have indicated that short-range interactions with soft cores can equally well account for the behavior of S-wave phase shifts at high energies, in particular the fact that these phase shifts become negative. The numerical solutions of Bystritskii *et al.*,[8] using a combination of an attractive Yukawa interaction in conjunction with a repulsive Yukawa of slightly shorter range, provides an example of a static soft-core interaction which fits the experimental S-wave phase shifts throughout the energy range. A number of studies carried out several years ago have shown that phenomenological velocity-dependent potentials can also account for the S-wave phases.[9,10]

Perhaps the most important recent development in our fundamental understanding of the N-N interaction has been the successful fitting of the experimental phase shifts with one-boson-exchange potentials (OBEP) drawn from meson field theory.[11–16] Unfortunately, most of these studies have neglected the S-wave phases which are of primary importance in nuclear applications.

The present study originated as an effort to guide the development of OBEP models which could simultaneously give good S and higher partial-wave phase shifts over a broad energy range. The success in this regard of regularized, almost relativistic, N-N potentials of Green and Sawada[17] was in part a consequence of the clarifications engendered by the present study. The clarifications pointed to the necessity of maintaining the ^1S state near the zero binding energy condition and the ^3S state in a weakly bound condition (constraints long recognized in phenomenological studies) while pursuing OBEP parameter variations intended to improve the fit of the high-energy and higher partial-wave phase shifts. They also pointed to the importance of employing regularized potentials whose nonsingular properties near $r=0$ are the key elements in the success of the Green-Sawada studies with S waves. The major purpose of this paper is to elucidate the large degree of variability in soft core versus velocity dependence inherent in the high-energy S-wave phase shifts. For simplicity we do not include the tensor coupling of the $^3D_1$ to $^3S_1$ potentials, which while important in precision fits, is not consequential to the gross physical aspects under investigation.

In this study we have made considerable use of the Morse potential[18] as an analytic repulsive soft-core potential whose characteristics can be conveniently specified by three parameters. The Morse potential has been used extensively as a repulsive core atom-atom potential in studies of the bound states of diatomic molecules. Its great advantages are a simple energy eigenvalue formula and its relatively simple analytic wave functions. In a recent study[19] we have utilized the Morse function to treat the problem of a nucleon bound in a velocity-dependent nuclear potential. The practicality of the Morse function in this nuclear context led us to consider it for this nucleon-nucleon problem. In Sec. 3 we treat the S-wave scattering problem for a static Morse potential and present an analytic formula for phase shifts which can be used to fit the experimental S-wave N-N scattering data over the entire energy region. In Sec. 4 we examine the S-wave problem for a phenomenological velocity-dependent potential and two meson-theoretic velocity-dependent potentials. Here again the Morse function can serve as a convenient approximate effective potential provided that its parameters are varied with energy. In the conclusion we discuss the physical mechanisms implicit in the various static soft-core and velocity-dependent potentials which we have examined.

Before going to the central topics of this paper, let us first consider the general problem of maintaining an S state bound near zero energy.

2. MAINTAINING THE 1S STATE AT ZERO BINDING ENERGY

From effective-range theory it is known that low-energy S waves are sensitive to some over-all aspect of the potential. It would be valuable to know what aspect of the potential should be preserved while testing various models against the higher-energy phase shifts. Another way of viewing this is to say we should examine

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only models which preserve the scattering length and the effective range. Only for such models is it meaningful to interpret the information embodied in S waves at higher energies in terms of the middle range and inner range of the potentials.

As measured by the number of states it can bind, the nuclear force is a very weak potential, one which barely binds the ^3S state and barely misses binding the ^1S state. In view of this it seems worthwhile to seek invariants of a potential function which will insure $E(1S)=0$. Then it would be clear that this would represent the ^1S state rather well and that with slight strengthening it could be used to represent the ^3S state.

The relationship between an interaction protential and the number of bound states has been studied many times.[20–24] Here we concentrate our attention on adjusting the parameters of a potential to achieve one bound S state at or near zero energy.

Effective-range theory[1–4] shows that S-wave phase shifts at low energies for any potential are given by

$$k\cot\delta_0=-a_s^{-1}+\tfrac{1}{2}r_0k^2, \tag{2.1}$$

where a_s is the Fermi scattering length evaluated at zero energy and r_0 is the effective range. We wish to consider the case when a_s is large or infinite. It is possible to find such potentials by solving the scattering problem for the phase shifts and representing them by Eq. (2.1). With some trial and error it is simple to bracket the combination of well parameters which lead to large or infinite a_s. It is interesting to note that when $a_s=\infty$

$$r_0=2\int_0^\infty[1-u_0^2(r)]dr,$$

where $u_0(r)$ is a function which vanishes at the origin and goes to unity outside the range of the potential. If, following Bethe,[4] we take for u_0 the function $1-e^{-\beta r}$, it follows immediately that $\beta=3/r_0$. This gives a wave function that can be used in conjunction with a known effective range as an approximate zero-energy wave function for dealing with various potentials including velocity-dependent potentials.

For application to the low-energy N-N problem it would be desirable to have simple criteria on potential parameters for any reasonable well which would insure that we are in the neighborhood of $E(1S)=0$. A problem of this type has been studied previously by one of us (AESG) [25] in connection with the adjustment of realistic shell-model potentials to achieve $2S$, $3S$, and $4S$ neutron size resonances at $A=12$, 55, and 160. Then in exploring various central potentials with uniform interior and exponentially diffuse tails it was helpful to use the equivalent square well strength

$$S=2\int_0^\infty U(r)rdr,\quad U(r)=-2\mu V(r)/\hbar^2, \tag{2.2}$$

where $V(r)$ is the potential and μ is the reduced mass, as a measure of the binding power of a potential. In a larger perspective we are now concerned with an extrapolation of the shell-model size resonance problem to the problem of maintaining the $1S$ size resonance very close to $A=2$, i.e., the deuteron.

If we use Eq. (2.2) to calculate the equivalent square well strength, then the criterion to achieve $E(1S)=0$ may be written as

$$I_1=S^{1/2}=\tfrac{1}{2}\pi=1.571. \tag{2.3}$$

A similar criterion proposed by Jost and Pais[21] based on scattering considerations is equivalent to the condition $I_1=\sqrt{2}=1.414$, which is a bit weak for the square well.

An alternative criterion comes out of the old semiclassical phase-space quantization techniques and is implicit in the application of the WJKB approximation. Here one examines

$$I_2=\int_{r_p}^\infty[U(r)]^{1/2}dr, \tag{2.4}$$

where r_p is the classical turning point. If we impose the condition $I_2=\frac{1}{2}\pi$, again we obtain the exact quantum-mechanical criterion for $E(1S)=0$ for the square well.

In applications to various wells both criteria serve to relate a natural well distance parameter a to a well depth parameter V_0. Thus we define

$$\epsilon_0=(2\mu a^2V_0/\hbar^2)^{1/2}. \tag{2.5}$$

Now the critical values of ϵ_0, for various types of potential forms (to be denoted by ϵ_0^*) which will lead to $E(1S)=0$ may be determined exactly for a number of analytically solvable potentials or may be established by numerical analyses. These values are listed in column 3 of Table I. Sources of these constants are indicated by the reference numbers in column 4. Here A denotes analytically solvable and DG denotes this work. In columns 5 and 6 we indicate the corresponding critical values of I_1 and I_2 which will insure $E(1S)=0$ on the basis of the critical ϵ_0^* listed in column 3. For the case of the Morse function and the hard-core square well, the lower limit of Eq. (2.2) has been replaced by r_p. In column 5 the choice $r_c/a=\frac{1}{4}$ has been used as representative for the square well with a core and the parameters for the $^1S_0$ state discussed in Sec. 3 have been used as representative for the Morse function. One notes from this table that the I_1^* values are fairly consistently in the range 1.57 to 2 for all of the func-

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TABLE I. Critical values of I_1 and I_2 for various potential forms required on the basis of quantum-mechanical determinations of critical values of ϵ_0.

Potential	Functional form	ϵ_0^*	Ref.	I_1^*	I_2^*
Square well	$V=-V_0$ $(r<a)$ $=0$ $(r>a)$	$\pi/2$	A	$1\epsilon_0^*=1.571$	$1\epsilon_0^*=1.571$
Exponential	$V=-V_0e^{-r/a}$	1.202	A	$\sqrt{2}\epsilon_0^*=1.700$	$2\epsilon_0^*=2.404$
Yukawa	$V=-V_0(a/r)e^{-r/a}$	1.297	3	$\sqrt{2}\epsilon_0^*=1.834$	$(2\pi)^{1/2}\epsilon_0^*=3.252$
Gaussian	$V=-V_0e^{-(r/a)^2}$	1.640	1,3	$1\epsilon_0^*=1.640$	$(\frac{1}{2}\pi)^{1/2}\epsilon_0^*=2.055$
Square well with core	$V=\infty$ $(r<r_c)$ $V=-V_0$ $(r_c<r<a+r_c)$ $V=0$ $(r>a+r_c)$	$\pi/2$	A	$[1+2(r_c/a)]^{1/2}\epsilon_0^*\approx1.93$	$\epsilon_0^*=1.571$
Morse	$V=V_0z(z-2)$ $z=e^{-(r-r_m)/a}$	$\frac{1}{2}$	DG	$[6+4\{(r_m/a)-\ln2\}]^{1/2}\epsilon_0^*\approx1.73$	$\pi\epsilon_0^*=1.571$

tional forms considered. The I_2^* values are more sensitive to the well shapes tending to values greater than 2 for attractive-core potentials but taking on values from 2 to 1.571 for coreless or repulsive-core potentials. For applications to the N-N interaction problem either I_1 or I_2 may be used, provided that the numerical constraint imposed is taken from a potential form similar in shape to the potential or effective potential under study.

3. REPRESENTATION OF $^1S_0$ AND $^3S_1$ INTERACTION BY THE MORSE FUNCTION

In this section we show that the $^1S_0$ and $^3S_1$ nucleon-nucleon interaction in the energy range $E_{\rm lab}\lesssim400$ MeV are well described by a simple, three-parameter, static, soft-core potential, the so-called Morse potential[18];

$$V_m(r;\,V_0,a_m,r_m)=V_0\{\exp[-2(r-r_m)/a_m]-2\exp[-(r-r_m)/a_m]\}. \quad (3.1)$$

For this potential, Schrödinger's equation has the positive-energy solution

$$G(r)=e^{-Z/2}[Z^{i\kappa}{}_1F_1(\tfrac{1}{2}-\epsilon_0+\kappa i;\,1+2\kappa i;\,Z)-\alpha Z^{-i\kappa}\,{}_1F_1(\tfrac{1}{2}-\epsilon_0-\kappa i;\,1-2\kappa i;\,Z)], \quad (3.2)$$

where G/r is the S-state wave function,

$$\alpha=Z_0{}^{2\kappa i}{}_1F_1(\tfrac{1}{2}-\epsilon_0+\kappa i;\,1+2\kappa i;\,Z_0)/{}_1F_1(\tfrac{1}{2}-\epsilon_0-\kappa i;\,1-2\kappa i;\,Z_0),$$
$$Z=2\epsilon_0\exp[-(r-r_m)/a_m], \quad (3.3)$$
$$Z_0=2\epsilon_0\exp[r_m/a_m],\quad \kappa=ka_m,$$
$$\epsilon_0=(MV_0a_m{}^2/\hbar^2)^{1/2},\quad \text{and}\quad {}_1F_1(a;\,c;\,Z)$$

is the confluent hypergeometric function. The S-wave phase shift is readily obtained now by comparing the asymptotic form of (3.2), viz.,

$$Z^{i\kappa}=\alpha Z^{-i\kappa},\quad \text{with}\quad e^{-i(kr+\delta_0)}-e^{i(kr+\delta_0)}.$$

At large Z_0,

$${}_1F_1(a;\,c;\,Z)\to e^ZZ^{a-c}\Gamma(c)/\Gamma(a)$$

and (3.4)

$$\alpha\to\Gamma(1+2\kappa i)\Gamma(\tfrac{1}{2}-\epsilon_0-\kappa i)/[\Gamma(1-2\kappa i)\Gamma(\tfrac{1}{2}-\epsilon_0+\kappa i)],$$

where $\Gamma(Z)$ is the usual γ function. It then follows that the S-wave phase shift takes on the simple, analytic form, for large Z_0,

$$\delta_0{}^m(k;\,V_0,a_m,r_m)=\arg\Gamma(1+2ka_mi)-\arg\Gamma(\tfrac{1}{2}-\epsilon_0+ka_mi)-k[r_m+a_m\ln(2\epsilon_0)]+n\pi, \quad (3.5)$$

where n is an integer. The Morse parameters V_0, a_m, and r_m are obtained by fitting $\delta_0{}^m(k)$ to the experimentally determined data on n-p and n-n scattering in the $l=0$ states. We may relate the parameters r_m and a_m to the scattering length a_s and the effective-range parameter r_0 (Table II), by expanding $k\cot\delta_0{}^m$ in powers of k^2. When the first two coefficients are compared with the effective-range formula (2.1), we obtain

$$a_s=r_m-a_m\phi_1/(\tfrac{1}{2}-\epsilon_0)$$

and

$$r_0=\tfrac{2}{3}a_s+2a_m{}^3\phi_2/[3a_s{}^2(\tfrac{1}{2}-\epsilon_0)^3],$$

where ϕ_1 and ϕ_2 are the slowly varying functions (see Fig. 1)

$$\phi_1=1+(\tfrac{1}{2}-\epsilon_0)/(\tfrac{3}{2}-\epsilon_0)-(\tfrac{1}{2}-\epsilon_0)[\gamma+1+\ln(2\epsilon_0)+f(x)],$$
$$\phi_2=1+[(\tfrac{1}{2}-\epsilon_0)/(\tfrac{3}{2}-\epsilon_0)]^3-(\tfrac{1}{2}-\epsilon_0)^3[8\zeta(3)+\tfrac{1}{2}d^2f/dx^2],$$
$$x=(\tfrac{1}{2}-\epsilon_0),$$

TABLE II. Parameters for the S-state Morse functions.

Sytsem	a_s (F)	r_0 (F)	V_0 (MeV)	a_m (F)	r_m (F)	z_0
$^1S_0$ of n-p[a]	-23.678 ± 0.028	2.44 ± 0.11	61.99	0.3957	0.9365	10.3
$^1S_0$ of n-n[a]	-17 ± 1	2.84 ± 0.03	40.38	0.4799	1.0531	8.5
$^3S_1$ of n-p[b]	$+5.397\pm0.011$	1.727 ± 0.013	119.49	0.3408	0.8668	14.7

[a] See Ref. 26.
[b] See Ref. 27.

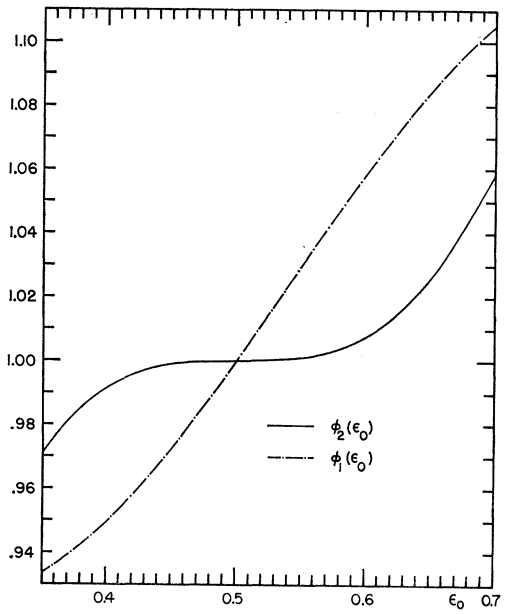

FIG. 1. Functions $\phi_1(\epsilon_0)$ and $\phi_2(\epsilon_0)$ used in evaluating Morse parameters from effective-range parameters.

and

$$f=\sum_{n=2}^{\infty}(-1)^n[\zeta(n)-1]x^{n-1}.$$

Here γ is Euler's constant and $\zeta(n)$ is the Riemann zeta function. The third parameter ϵ_0 (i.e., V_0) is now adjusted so that δ_0 fits the experimental data above 10 MeV. Specifically, in the $^1S_0$ state it is chosen so that $\delta_0=0$ when $E_{\rm lab}=253$ MeV for both the *n-p* and the *n-n* case, and in the $^3S_1$ state such that $\delta_0=29.6°$ when $E_{\rm lab}=142$ MeV. We find that by fitting the ^3a_s and $^3r_0$ parameters for the $^3S_1$ state the same potential gives the correct binding energy of the deuteron (2.22 MeV) if the latter is assumed to be a pure $^3S_1$ state. The values of the Morse parameters so obtained, for *n-p* and *n-n* elastic scattering are given in Table II. Figure 2 is a plot of $\delta_0{}^m$ versus energy for *n-p* and *n-n* scattering together with the experimentally determined phase shifts.[26–28] Figures 3 give the corresponding potentials. It will be noted that there is a substantial difference between the *n-p* and *n-n* potential curves in the $^1S_0$ state, suggesting a violation of charge independence. This is a reflection primarily of the large differences in the experimental values of the effective-range parameters for *n-p* and *n-n*, as is evident from Table II. To what extent this apparent contradiction of charge independence of nuclear forces persists depends on how well the presently available experimental data will hold up under further investigation. In particular, there are no high-energy *n-n* data (we have used the *n-p*, *p-p* high energy data in fitting the *n-n* curve) while the low-energy data (usually parametrised in effective-range form) are very imprecise and available only from indirect, three- (or more) body experiment. There is

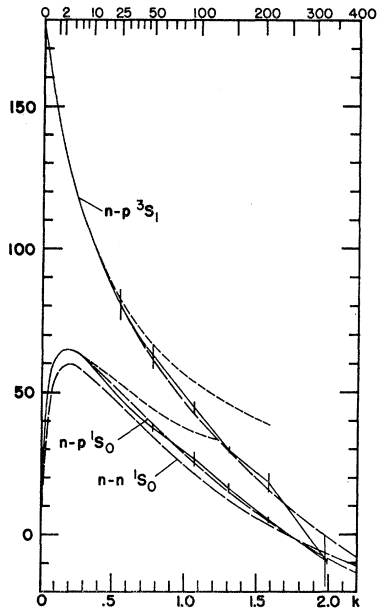

FIG. 2. Morse potential fits (long dashed lines) to experimental *S*-wave phase shifts (solid lines). Above 20 MeV the experimental *n-p* and *p-p* phase-shift analysis of Arndt and MacGregor are used (Ref. 28). Short dashed curves are effective-range parametrization of low-energy experimental data (Refs. 26 and 27). These are coincident with Morse potential prediction at low energy.

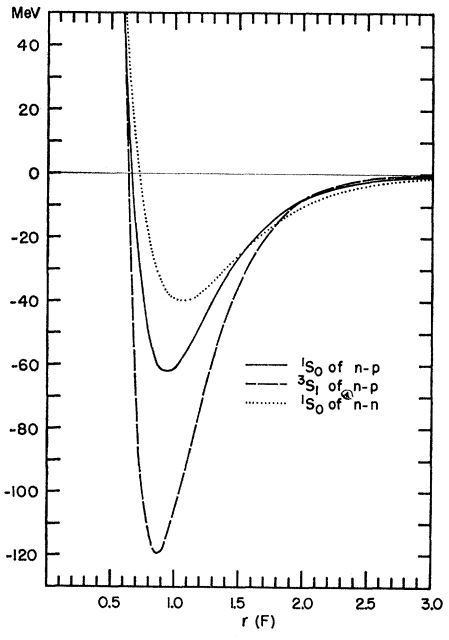

FIG. 3. Morse function parametrization of *n-p* and *n-n* potential in *s* states.

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also some ambiguity in the value of the effective-range parameter r_0 for the $^1S_0$ of n-p [26,27].

The integrals I_1 and I_2 may be evaluated analytically for the Morse function. The results are

$$I_1=\epsilon_0^*[6-4\ln 2+4(r_m/a_m)]^{1/2}\approx 1.73$$

and

$$I_2=\pi\epsilon_0^*=1.571,$$

where for I_1 we have used the parameters for the $^1S_0$ state as representative of the case $E(1S)=0$. We see that both I_1 and I_2 are close to the square-well values.

4. S-WAVE STUDIES WITH VELOCITY-DEPENDENT MESON POTENTIALS

For the purposes of this study we first consider phenomenological velocity-dependent potentials, which lead to good $^1S_0$ and $^3S_1$ phase shifts. We examine in particular the work of Herndon *et al*,[29] which not only fits these phases quite well but serves quite well in three- and four-body problems. We compare these with our static Morse potentials and with the meson-theoretic potentials of Green and Sawada.[16,17] Our purpose is also to investigate the nature of the effective repulsion provided by these potentials.

In this connection we come to the problem of how to compare different velocity-dependent potentials. Since we have two degrees of freedom, the central term and the momentum-dependent term, a variety of radial dependences might have equivalent effects upon the phase shifts. Let us assume that we wish to compare velocity-dependent potentials, which may be presented in the symmetrized form

$$V(r,p)=V_c(r)+M^{-1}[p^2w(r)+w(r)p^2]. \quad (4.1)$$

We may now compare separately $V_c(r)$ and $w(r)$. Of course, when two potentials have similar functions $V_c(r)$ and $w(r)$ they should act somewhat similarly upon the phase shifts. Alternatively, we might write $V(r,p)$ in the form

$$V(r,p)=V_0(r)+a^2(\nabla V_\Delta\cdot\nabla+V_\Delta\nabla^2), \quad (4.2)$$

where a is a suitable unit of length and

$$V_0=V_c(r)-E_0a^2\nabla^2w(r) \quad \text{and} \quad V_\Delta=-2E_0w(r) \quad (4.3)$$

or

$$V_c=V_0-\tfrac{1}{2}a^2\nabla^2V_\Delta \quad \text{and} \quad w=-\tfrac{1}{2}V_\Delta/E_0 \quad (4.4)$$

and $E_0=\hbar^2/Ma^2$. Then two equivalent acting potentials may be compared by separately comparing $V_c(r)$ and $w(r)$ or $V_0(r)$ and $V_\Delta(r)$.

A third possibility, if one knows the true wave function, is to introduce an effective potential given by

$$V_e(r,\psi)=V_0(r)+a^2\psi^{-1}(\nabla V_\Delta\cdot\nabla\psi+V_\Delta\nabla^2\psi). \quad (4.5)$$

This would be the closest thing for comparison to a static potential since this is what is acting upon the true wave function. A problem arises with this representation since the potential becomes infinite at the zeros of the wave function. Such zeros occur at high energies for S waves in the nucleon-nucleon problems. It should, however, be clear that these infinities are not troublesome precisely because the wave function vanishes at the same point. Thus this type of respresentation probably provides a realistic way of displaying the difference between a velocity-dependent potential and a static potential. Unfortunately, the use of this representation is complicated by the necessity of generating an accurate wave function. For the zero-energy solution we can use the wave function $\psi_0=1-(r/a_s)-\exp(-3r/r_0)$.

A fourth mode of representation is to make use of the transformation which eliminates the first derivative in the Schrödinger equation. In this case one arrives at an effective potential given by

$$V_{\rm eff}(r,E)=(1-V_\Delta/E_0)^{-1}\{V_0(r)-\tfrac{1}{2}a^2\nabla^2V_\Delta-a^2k^2V_\Delta -a^2(\nabla V_\Delta)^2/[4(E_0-V_\Delta)]\}.+l(l+1)E_0a^2/r^2. \quad (4.6)$$

This potential must be considered only in relation to a "false" wave function given by

$$\chi_l(r)=(1-V_\Delta/E_0)^{1/2}G_l(r), \quad (4.7)$$

where G_l/r is the true radial wave function. For scattering problems, we may use this false wave function since in the asymptotic limit the true and the false wave functions agree. Accordingly, one can extract the phase shifts from false wave functions.

Figure 4 shows the various representations V_c, w, V_0, $V_e(r,\psi_0)$, $V_{\rm eff}(r,E)$ of Herndon's[29] velocity-dependent potential $V(r,p)$ for the $^1S_0$ state of the p-p system together with our Morse potential for the n-p system in a similar state. The effect of the velocity dependence, in this case, is to provide an effective repulsive soft core which grows with the energy, but is weak at low energies.

Figure 5 represents a six-parameter OBEP model of Green and Sawada[17] involving π, η, vector ω, ρ, and scalar (ω_s,ρ_s) meson exchanges with derivative coupling on the ρ. This model fits all phase-shift data quite well. In this case the effective energy-dependent potentials exhibit clearly a soft repulsive core nature. The effective potentials are very similar to our Morse function potential provided that the parameters vary with energy.

Figure 6 is a similar plot of a modified relativistic OBEP of Green and Sawada.[16,17] involving only pion, vector meson (ω), and scalar meson (s) exchanges. This one-parameter model gives an excellent description of the S-wave phase shifts and all other partial phase shifts except the $^1P_1$ wave. The effective potentials in this case are quite unlike those of Figs. 4 and 5. The effect of the velocity dependence here seems to be to "squeeze out" the attraction which is apparent in the inner region. An actual repulsion begins to manifest itself only at high energies.

An examination of Figs. 5 and 6 reveals a great contrast in the form of the effective, energy-dependent

[29] R. C. Herndon, E. W. Schmid, and Y. C. Tang, Nucl. Phys. **42**, 113 (1963).

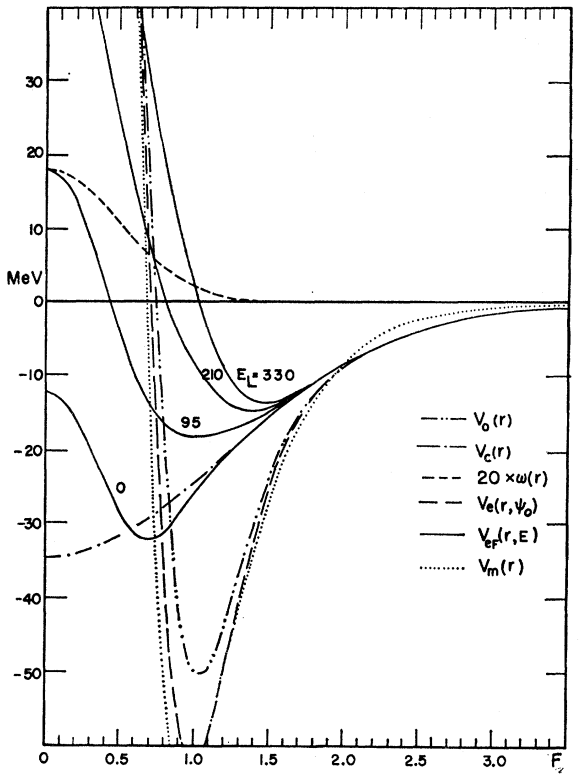

FIG. 4. Various representations of the phenomenological velocity-dependent N-N $^1S_0$ potential of Herndon *et al.* (Ref. 29).

component $V_{\text{eff}}(r,E)$ in these two cases. Although the energy-independent component $V_0(r)$ and the coefficient of the momentum-dependent term $w(r)$ in $V(r,p)$ are qualitatively similar, their quantitative values show considerable difference. Thus, the model of Fig. 6 has a substantially stronger velocity-dependent coefficient $w(r)$ by about a factor of 2.5 while the central term $V_0(r)$ is larger only by about 1.4. In this sense the modified relativistic OBEP model of Fig. 6 is more strongly velocity-dependent than that of Fig. 5. These differences lead to entirely different effective potentials representing different repulsions which account for the negative S-wave phase shifts observed at higher energies. These two examples of velocity-dependent OBEP's indicate that the observed N-N elastic scattering phase shifts can be accounted for by entirely different physical mechanisms in the inner and intermediate regions of $r \lesssim 1.5$ F (the outer region is, of course, dominated by the one-pion-enchange contribution). In particular, it is evident from Fig. 6 that it is not necessary that the effective potentials have a repulsive core in the inner region, as is the case in both the phenomenological model of Herndon *et al.*[29] (Fig. 4) and the meson-theoretical model of Fig. 5.

It should be clear that the $V_0(r)$ representation shows more vividly the gross equivalence of velocity dependence to repulsive core than does the $V_c(r)$ representation. However, the "effective potential" representation $V_{\text{eff}}(r,E)$ exhibits most clearly how the energy dependence manifests itself effectively by reducing the strength of the well and strengthening the repulsive core.

5. DISCUSSION AND CONCLUSIONS

We find from the foregoing study that for a potential to describe low-energy N-N interactions in S states it is sufficient, in first approximation, that it bind the system in the $^1S_0$ state at or near zero energy. With this condition as a point of departure, the fits to S-wave data can then be improved by requiring that the potential reproduce the observed scattering length and effective range. Clearly, from old work,[1-4] a great variety of S-wave potentials can be used. In the present work we also show that, for a phenomenological Morse potential, it is simply necessary to take the well-strength parameter ϵ_0 to be $\frac{1}{2}$ to insure an S state of zero binding (Table I). When the effective-range parameters are reproduced (Table II), this well-strength parameter is not much different from the critical value of $\frac{1}{2}$, being 0.4841 for $^1S_0$ of n-p, 0.5787 for $^3S_1$ of n-p, and 0.4738 for $^1S_0$ of n-n.

To reproduce the observed negative S-wave phase shifts at higher energies with a static potential it is necessary to incorporate a repulsion into the potential. Here the Morse function serves as a far more realistic point of departure than the square-well, exponential,

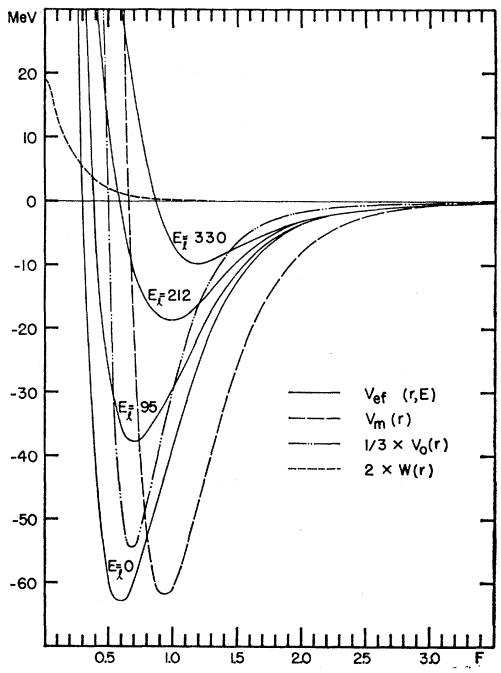

FIG. 5. Effective energy-dependent N-N potentials in $^1S_0$ state for OBEP model of Green and Sawada (Ref. 13) (Fig. 4 of Ref. 17). Dashed curve is the static soft core (Morse potential for the same state).

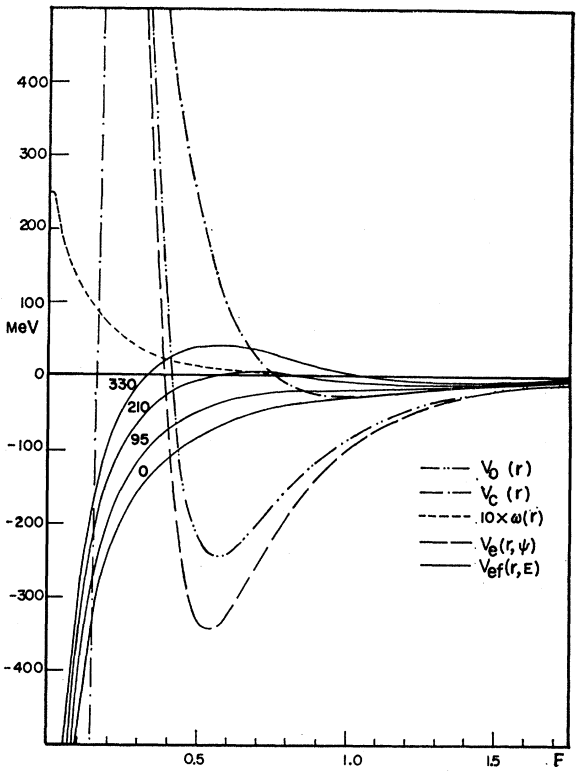

FIG. 6. One-parameter modified relativistic OBEP of Green and Sawada (Ref. 17) involving π (pesudoscalar), ω (vector), and s (scalar) meson exchange (Fig. 3 of Ref. 17).

Gaussian, or Hulthén potential used in early studies. Our studies clearly show that the soft repulsive core of the Morse function, which goes at the origin to 5.7 GeV for the $^1S_0$ state of n-p, 16.3 GeV for the $^3S_1$ state of n-p, and 2.5 GeV for the $^1S_0$ state of n-n, can do just as well as an infinite hard core.

To achieve a zero-energy scattering length approximately with velocity-dependent potentials or with various combinations of soft-core and velocity-dependent potentials, we may again use our integral criteria using either the true potential with the effective-range wave function or the false effective potential at zero energy. Since this latter quantity can be studied in the absence of a detailed knowledge of the wave function, the integral of this function can be evaluated either numerically or by planimeter. Our studies indicate that I_1 or I_2 for velocity-dependent potential functions are close to the invariant estimates for static potentials.

Good high-energy behavior for the S-wave phase shifts can also be obtained by using a velocity-dependent potential in conjunction with a coreless static component or by using various combinations of soft-core and velocity-dependent potentials. Looked at from the standpoint of the $V_c(r)$, $\omega(r)$ representation, the phenomenological potential of Herndon *et al.*[29] appears to be a coreless potential. However, looked at from the standpoint of the $V_0(r)$,$V_\Delta(r)$, $V_e(r,\psi_0)$, or $V_{\rm eff}(r,E)$ representations, this potential displays a core. Hence it should be clear why such a velocity-dependent potential acts in a similar way as a static repulsive-core potential.

Most of OBEP in current use, including several models of Green and Sawada (e.g., Fig. 5), have a similar character to the potential of Herndon *et al.*, as indicated by the V_0 representation or the $V_{\rm eff}(r,E)$. On the other hand, several of the Green-Sawada models, particularly their almost relativistic models (e.g., Fig. 6) which also do well on S-waves, have an entirely different character. For these the effective potentials $V_{\rm eff}(r,E)$ are coreless. At low energies they are near the strength needed to bind an S state. However, they become increasingly weaker as the center-of-mass energy increases. The $V_0(r)$ and $V_e(r,\psi_0)$ representations are still suggestive of repulsive-static-core-type potentials. Kiang[30] has also pointed out that velocity-dependent potentials which can reproduce S-wave phase shifts may actually contribute an attraction.

These indications as to the variablility of combinations of soft core with velocity dependence which are successful with regard to S-wave suggest that it may be difficult to distinguish physically which type of velocity-dependent potentials actually characterize the N-N interaction. Possibly such effects as the deuteron magnetic moment and the quadrupole moment would help distinguish between them. In any case, the details of the N-N potential in the very innermost region ($r<0.3$ F) can probably not be determined unambiguously from experimental evidence or from field-theoretical OBEP models as presently constituted.[11–17] At such small internucleon separations the effects of nucleon and perhaps boson structure will surely have to be considered in detail.[31] Alternatively, the possibility exists that applications of the N-N interaction to the nuclear many-body problem might help resolve the ambiguity. In this connection we might note that Köhler and McCarty,[32] using the highly velocity-dependent N-N potential represented by Fig. 6, obtain a binding energy of 7.2 MeV/nucleon for O^{16} as compared to the experimental value of 8 MeV. This contrasts with the results of their calculations with hard- and soft-core-type potentials which give much too little binding energies.

ACKNOWLEDGMENTS

We would like to thank Dr. S. K. Dutta and Dr. T. Sawada for their assistance and Dr. I. Ulehla, Dr. Eugene Guth, Dr. H. S. Köhler, and Dr. Gregory Briet for helpful conversations and communications.

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[32] H. S. Köhler (private communication).