Statistical Properties of the Scattering Amplitude and Cross Sections Using the Statistical Collision Matrix

NAZAKAT ULLAH
Tata Institute of Fundamental Research, Bombay, India
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The statistical collision matrix is used to study the statistical distribution of the random part of the partial scattering amplitude and the partial fluctuation cross section. Using simplifying assumptions, explicit expressions are obtained for the lower moments of the partial scattering amplitude for the inelastic processes. It is shown that the real and imaginary parts of the fluctuating scattering amplitude have a Gaussian distribution with the same variance. Using ensemble averages of the resonance parameters of the statistical collision matrix, an expression is obtained for the partial reaction fluctuation cross section in the region of overlapping resonances. A remark is made about the partial elastic fluctuation cross section.

I. INTRODUCTION

THE statistical theory has been used to study the average properties and the fluctuations of the nuclear collision cross sections for the reactions which go through the formation of a compound nucleus.[1] The main interest of the various models developed by Ericson[2] and Brink *et al.*[3] has been to explain the fluctuations in the cross sections and the calculation of correlation functions. In both models the random part of the scattering amplitude is taken to be Gaussian. In the Ericson model this assumption is justified by making use of the central-limit theorem, while in the Brink model it is one of the basic assumptions. The statistical properties of the random part of the scattering amplitude can be studied much more accurately if we start from some theory which can predict the statistical distributions of the partial-width amplitudes which are the central quantities entering into the pole expansion of the scattering amplitude. The need for such a study increases when we are also interested in studying the average cross sections, or the fluctuating part of the cross sections in the case of overlapping resonances.

We feel that since R-matrix theory[4] is ideally suited for a statistical description of the compound nucleus, it should provide a natural way to describe the average values and the mean-square fluctuations of the cross sections. Recently a formalism has been developed by Moldauer[5] for the statistical treatment of the cross sections. This formalism makes use of a statistical collision matrix, which is defined in terms of the eigenstates of a complex boundary-value problem. The advantage of using this type of formalism is that the statistical properties of the parameters of collision matrix can be easily studied using the random-matrix hypothesis.[6] The purpose of the present paper is to use the knowledge of the statistical distribution of the parameters of the collison matrix, together with some simplifying assumptions, to study the statistical properties of the random part of the scattering amplitude and the average cross sections in the region of overlapping resonances.

The statistical collision matrix is given by[5]

$$U_{cc'}{}^S(E,E_0)=U_{cc'}{}^0(E_0)-i\sum_\mu \frac{g_{\mu c}g_{\mu c'}}{E-\mathcal{E}_\mu+\frac{1}{2}i\Gamma_\mu}, \quad (1)$$

where E_0 is a specified total energy and the channel label c specifies the coupling scheme $\{\alpha slJM\}$, α being the target and projectile internal state, s the channel spin, l the orbital angular momentum, J the total angular momentum, and M the Z projection of J. $\mathcal{E}_\mu-\frac{1}{2}i\Gamma_\mu$ is the complex energy of the state μ. The quantities $g_{\mu c}$ are related to the complex amplitudes $\theta_{\mu c}$,[5] which are proportional to the overlap integral of the channel wave function and the compound-nucleus wave function. The term $U_{cc'}(E_0)$ gives rise to the major part of the direct interactions.

The statistical properties of the cross section, like its average value at E_0, are obtained by averaging it over the ensemble of random matrix functions of which U^S is an element. The ergodic theorem[7] tells us that such averages are equal to ordinary energy averages of the appropriate functions of $U^S(E,E_0)$ over an energy interval which is large compared to the mean resonance spacing D and the average total width Γ.

We rewrite expression (1) as

$$U_{cc'}{}^S=U_{cc'}{}^0-i\langle g_{\mu c}g_{\mu c'}\rangle_\mu \sum_\mu \frac{1}{E-\mathcal{E}_\mu+\frac{1}{2}i\Gamma_\mu} -i\sum_\mu \frac{a_{\mu cc'}}{E-\mathcal{E}_\mu+\frac{1}{2}i\Gamma_\mu}, \quad (2)$$

[1] For a brief review and earlier references see J. P. Bondorf, in *Proceedings of the IX Summer Meeting of Nuclear Physicists* (Hercegnovi, 1964), Vol. II p. 133, (Publication No. 156, NORDITA, Copenhagen ϕ, Denmark).

[2] T. Ericson, Ann. Phys. (N. Y.) **23**, 390 (1963).

[3] D. M. Brink and R. O. Stephen, Phys. Letters **5**, 77 (1963); D. M. Brink, R. O. Stephen, and N. W. Tanner, Nucl. Phys. **54**, 577 (1964).

[4] A. M. Lane and R. G. Thomas, Rev. Mod. Phys. **30**, 257 (1958).

[5] P. A. Moldauer, Phys. Rev. **135**, B642 (1964).

[6] Nazakat Ullah, Phys. Rev. **154**, 893 (1967).

[7] A. M. Yaglom, *An Introduction to the Theory of Stationary Random Functions*, translated by R. A. Silverman (Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1962).

where

$$a_{\mu cc'}=g_{\mu c}g_{\mu c'}-\langle g_{\mu c}g_{\mu c'}\rangle_\mu\,,$$

and $\langle\ \rangle_\mu$ denotes an ensemble average over the resonance parameters of the statistical collision matrix, which is defined in terms of the eigenfunctions of a complex boundary-value problem.[5] The ensemble averages of the quantities involving $g_{\mu c}$ can be worked out using the formalism described in Ref. 6. We remark here that the statistical treatment of the complex boundary-value problem[6] using the random-complex-orthogonal matrix has the further advantage that certain relations among the average values of the parameters of the statistical collision matrix, arising from the imposition of the unitarity condition, are also satisfied.

II. RANDOM PART OF THE SCATTERING AMPLITUDE

The statistical properties of any random quantity which is a function of $U^S(E,E_0)$ will be studied by calculating its various moments. In principle it is possible to calculate such moments using the definition of $U^S(E,E_0)$ and the formalism described in Ref. 6, but this will lead to quite complicated results. To bring out the main points of the present paper, we now make the following simplifying assumptions. (i) We consider the case of large Γ/D, for which the number of open exit channels is large and therefore we can approximately take $\Gamma_\mu=\Gamma$, the average width.[1] (ii) For $c\neq c'$, the quantities $g_{\mu c}$ are uncorrelated.[5]

Let us consider the case of inelastic scattering; then the fluctuating part of the partial scattering amplitude $U_{cc'}{}^{S\text{ fl.}}$, using assumption (ii), can be written as

$$U_{cc'}{}^{S\text{ fl.}}=-i\sum_\mu\frac{g_{\mu c}g_{\mu c'}}{E-\mathcal{E}_\mu+\frac{1}{2}i\Gamma_\mu}\,. \tag{3}$$

The moments of $U_{cc'}{}^{S\text{ fl.}}$ will be calculated by first averaging it over an energy interval, the width of which is allowed to grow beyond all bounds, and then using the resonance statistics. The energy averages will be calculated using a technique developed by Moldauer[5] and will be denoted by $\langle\ \rangle_{\text{av}}$. We first consider the real part of $U_{cc'}{}^{S\text{ fl.}}$ and write[5]

$$\langle\text{Re}U_{cc'}{}^{S\text{ fl.}}\rangle_{\text{av}}=\lim\frac{1}{W}\times\int_W\text{Re}\left[-i\sum_\mu\frac{g_{\mu c}g_{\mu c'}}{E-\mathcal{E}_\mu+\frac{1}{2}i\Gamma_\mu}\right]dE\,, \tag{4}$$

where W is the averaging interval. The integral in expression (4) is evaluated by considering the contour integral of

$$\text{Re}U_{cc'}{}^{S\text{ fl.}}=\frac{i}{2}\left[-\sum_\mu\frac{g_{\mu c}g_{\mu c'}}{E-\mathcal{E}_\mu+\frac{1}{2}i\Gamma_\mu}+\sum_\mu\frac{g_{\mu c}{}^*g_{\mu c'}{}^*}{E-\mathcal{E}_\mu-\frac{1}{2}i\Gamma_\mu}\right]\,, \tag{5}$$

along a rectangle in the upper half-plane of the complex E plane with its base of length W along the real axis and of large enough height so that along the top of the rectangle $\text{Re}U_{cc'}{}^{S\text{ fl.}}$ has reached its limiting constant value[5] which is zero because of our assumption (ii). This gives us[5]

$$\langle\text{Re}U_{cc'}{}^{S\text{ fl.}}\rangle_{\text{av}}=\lim\frac{2\pi i}{W}\sum R_\mu{}^+\,, \tag{6}$$

where $R_\mu{}^+$ is the residue of any pole of $\text{Re}U_{cc'}{}^{S\text{ fl.}}$ which lies in the upper half-plane and whose real coordinate is $\mathcal{E}_\mu$. The first term in expression (5) has no pole in upper-half plane and the second term gives us

$$\langle\text{Re}U_{cc'}{}^{S\text{ fl.}}\rangle_{\text{av}}=-\lim\frac{\pi}{W}\sum_\mu g_{\mu c}{}^*g_{\mu c'}{}^*=-\frac{\pi}{D}\langle g_{\mu c}{}^*g_{\mu c'}{}^*\rangle_\mu\,. \tag{7}$$

This is zero, because of our assumption (ii) and so

$$\langle\text{Re}U_{cc'}{}^{S\text{ fl.}}\rangle_{\text{av}}=0\,. \tag{8}$$

For the calculation of

$$\langle(\text{Re}U_{cc'}{}^{S\text{ fl.}})^2\rangle_{\text{av}}\,,$$

we see from expressions (5) and (6) that

$$\left(\sum_\mu\frac{g_{\mu c}g_{\mu c'}}{E-\mathcal{E}_\mu+\frac{1}{2}i\Gamma_\mu}\right)^2$$

does not have any poles in upper half-plane, also the contribution of

$$\left(\sum_\mu\frac{g_{\mu c}{}^*g_{\mu c'}{}^*}{E-\mathcal{E}_\mu-\frac{1}{2}i\Gamma_\mu}\right)^2$$

is zero and the remaining term gives

$$\langle(\text{Re}U_{cc'}{}^{S\text{ fl.}})^2\rangle_{\text{av}}=\lim\frac{\pi i}{W}\times\sum_{\xi_\mu\text{ in }W}\sum_\nu\frac{g_{\mu c}{}^*g_{\mu c'}{}^*g_{\nu c}g_{\nu c'}}{(\mathcal{E}_\mu-\mathcal{E}_\nu)+\frac{1}{2}i(\Gamma_\mu+\Gamma_\nu)}\,.$$

This finally gives us[5]

$$\langle(\text{Re}U_{cc'}{}^{S\text{ fl.}})^2\rangle_{\text{av}}=\frac{\pi}{D}\left\langle\frac{|g_{\mu c}|^2|g_{\mu c'}|^2}{\Gamma_\mu}\right\rangle_\mu+\frac{\pi^2}{D^2}\left\langle g_{\mu c}{}^*g_{\mu c'}{}^*g_{\nu c}g_{\nu c'}\Phi_0\!\left(\frac{\Gamma_\mu+\Gamma_\nu}{2D}\right)\right\rangle_{\mu\neq\nu}\,, \tag{9}$$

where the function Φ_0 is defined in Ref. 5. Using the assumptions (i) and (ii), expression (9) becomes

$$\langle(\mathrm{Re}U_{cc'}{}^{S\ \mathrm{fl.}})^2\rangle_{\mathrm{av}}=\frac{\pi}{D}\frac{\langle|g_{\mu c}|^2|g_{\mu c'}|^2\rangle_\mu}{\Gamma}. \quad (10)$$

For the third-order moment of $\mathrm{Re}U_{cc'}{}^{S\ \mathrm{fl.}}$, we find using expressions (5) and (6) that there is a contribution from the term

$$Q=\left[\sum_\nu \frac{g_{\nu c}g_{\nu c'}}{E-\mathcal{E}_\nu+\frac{1}{2}i\Gamma_\nu}\right]^2\left[\sum_\mu \frac{g_{\mu c}{}^*g_{\mu c'}{}^*}{E-\mathcal{E}_\mu-\frac{1}{2}i\Gamma_\mu}\right],$$

which can be written as

$$\lim\frac{3\pi}{4W}\left[-\sum_{\xi_\mu \text{ in } W}\frac{|g_{\mu c}|^2|g_{\mu c'}|^2g_{\mu c}g_{\mu c'}}{\Gamma_\mu{}^2}+2\sum_{\xi_\mu \text{ in } W}\sum_{\neq\nu}\frac{|g_{\mu c}|^2|g_{\mu c'}|^2g_{\nu c}g_{\nu c'}}{(i\Gamma_\mu)[(\mathcal{E}_\mu-\mathcal{E}_\nu)+\frac{1}{2}i(\Gamma_\mu+\Gamma_\nu)]}\right.$$
$$\left.+\sum_{\xi_\mu \text{ in } W}\sum_{\neq\nu}\frac{g_{\mu c}{}^*g_{\mu c'}{}^*g_{\nu c}{}^2g_{\nu c'}{}^2}{[(\mathcal{E}_\mu-\mathcal{E}_\nu)+\frac{1}{2}i(\Gamma_\mu+\Gamma_\nu)]^2}+\underset{\text{(all different)}}{\sum_{\xi_\lambda \text{ in } W}\sum_\mu\sum_\nu}\frac{g_{\mu c}g_{\mu c'}g_{\nu c}g_{\nu c'}g_{\lambda c}{}^*g_{\lambda c'}{}^*}{[(\mathcal{E}_\lambda-\mathcal{E}_\mu)+\frac{1}{2}i(\Gamma_\lambda+\Gamma_\mu)][(\mathcal{E}_\lambda-\mathcal{E}_\nu)+\frac{1}{2}i(\Gamma_\lambda+\Gamma_\nu)]}\right],$$

and a similar contribution from the complex conjugate Q^*. Under the assumptions (i) and (ii) we find

$$\langle(\mathrm{Re}U_{cc'}{}^{S\ \mathrm{fl.}})^3\rangle_{\mathrm{av}}=-\frac{3\pi}{2D}\frac{\langle|g_{\mu c}|^2|g_{\mu c'}|^2\ \mathrm{Re}(g_{\mu c}g_{\mu c'})\rangle_\mu}{\Gamma^2}. \quad (11)$$

Similarly the fourth-order moment in our approximation is given by

$$\langle(\mathrm{Re}U_{cc'}{}^{S\ \mathrm{fl.}})^4\rangle_{\mathrm{av}}=\frac{3\pi^2}{D^2}\left[\frac{\langle|g_{\mu c}|^2|g_{\mu c'}|^2\rangle_\mu}{\Gamma}\right]^2. \quad (12)$$

Higher moments of $\mathrm{Re}U_{cc'}{}^{S\ \mathrm{fl.}}$ can also be written down in the same fashion. However, these higher moments are not needed if we use the principle of moments, which says that by identifying the lower moments of the two distributions we bring them to approximate equality.[8]

The ensemble averages of the quantities involving $g_{\mu c}$ can be obtained using the formalism of Ref. 6. Let us consider the complex amplitude $\theta_{\mu c}$,[5] the real and imaginary parts of which will be denoted by $\theta_{\mu c}{}^r$, $\theta_{\mu c}{}^i$, respectively, and write the joint probability

$$P(\theta_{1c}{}^r,\theta_{1c}{}^i,\theta_{1c'}{}^r,\theta_{1c'}{}^i).$$

This probability is given by[6]

$$P(\theta_{1c}{}^r,\theta_{1c}{}^i,\theta_{1c'}{}^r,\theta_{1c'}{}^i)=\frac{(N-2)(N-3)}{4\pi^2N^2|\Sigma|}\left[\int\rho_N(\lambda)[\lambda(1+\lambda)]^{(1/2)(N-3)}d\lambda\right]^{-1}\int d\lambda\ \rho_N(\lambda)[\lambda(1+\lambda)]^{(1/2)(N-5)}$$
$$\times\left[1-\frac{1}{N}\left[\frac{1}{1+\lambda}\{(\Sigma^{-1})_{11}(\theta_{1c}{}^r)^2+2(\Sigma^{-1})_{12}\theta_{1c}{}^r\theta_{1c'}{}^r+(\Sigma^{-1})_{22}(\theta_{1c'}{}^r)^2\}\right.\right.$$
$$\left.\left.+\frac{1}{\lambda}\{(\Sigma^{-1})_{11}(\theta_{1c}{}^i)^2+2(\Sigma^{-1})_{12}\theta_{1c}{}^i\theta_{1c'}{}^i+(\Sigma^{-1})_{22}(\theta_{1c'}{}^i)^2\}\right]+\frac{1}{\lambda(1+\lambda)N^2|\Sigma|}(\theta_{1c}{}^r\theta_{1c'}{}^i-\theta_{1c'}{}^r\theta_{1c}{}^i)^2\right]^{(1/2)(N-5)}, \quad (13)$$

where N is the dimension of the complex orthogonal matrix, Σ^{-1} is the matrix

$$\Sigma^{-1}=\begin{pmatrix}[(1-\gamma^2)J_c]^{-1} & -\gamma(1-\gamma^2)^{-1}(J_cJ_{c'})^{-1/2}\\ -\gamma(1-\gamma^2)^{-1}(J_cJ_{c'})^{-1/2} & [(1-\gamma^2)J_{c'}]^{-1}\end{pmatrix},$$

$$J_c=\frac{1}{N}\sum_{\alpha=1}^{N}J_{\alpha c}{}^2,\quad \gamma=\frac{\sum J_{\alpha c}J_{\alpha c'}}{[(\sum J_{\alpha c}{}^2)(\sum J_{\alpha c'}{}^2)]^{1/2}}.$$

The dimension N is very large in practice. Assumption (ii) will be satisfied if we take γ to be zero. This leads us to the result that the ensemble average in expression (11) vanishes. Therefore, we see from expressions (8), (10), and (12) that the lower moments of $\mathrm{Re}U_{cc'}{}^{S\ \mathrm{fl.}}$, under assumptions (i) and (ii), are the same as the moments of a Gaussian distribution with mean zero and variance

$$\frac{\pi}{D}\langle|g_{\mu c}|^2|g_{\mu c'}|^2\rangle_\mu/\Gamma.$$

[8] M. G. Kendall, *The Advanced Theory of Statistics* (Charles Griffin and Company Ltd., London, 1945), Vol. I, p. 83.

A similar calculation can also be carried out for the imaginary part of the amplitude $U_{cc'}^{S\,\mathrm{fl.}}$, and it turns out that these moments also approach the moments of the Gaussian distribution with the same variance

$$\frac{\pi}{D}\frac{\langle|g_{\mu c}|^2|g_{\mu c'}|^2\rangle_\mu}{\Gamma}.$$

We note from expression (13) that for the case $N\to\infty$, $\gamma\to0$, correlations of the higher moments of the amplitude θ belonging to channels c and c' depend on the weight function $\rho_N(\lambda)$. The channels c and c' will become completely independent of each other only if $\rho_N(\lambda)[\lambda(1+\lambda)]^{(1/2)(N-5)}$ becomes a δ function. However, in the formulation in Ref. 6, there is nothing which tells us that $\rho_N(\lambda)$ should become a δ function as $N\to\infty$. The most which can be done about $\rho_N(\lambda)$ at present is to guess its form[6] so that it fits the plot of the probability density function of N_μ obtained numerically. The difficulty we are facing here in choosing the form of $\rho_N(\lambda)$ is similar to the one which arises in describing the anamolous radiation widths, where Rosenzweig had to introduce new kinds of ensembles and choose some plausible weight functions.[9] We would like to add further that the weight function $\rho_N(\lambda)$ depends strongly on the boundary condition chosen for the problem. The particular choice of $\rho_N(\lambda)$ in Ref. 6 was almost independent of N, which may not be true for other boundary conditions.

III. FLUCTUATION CROSS SECTION

The partial fluctuation cross section is defined by[5]

$$\sigma_{cc'}^{\mathrm{fl.}}=(\pi/k_c^2)[\langle|U_{cc'}^S|^2\rangle_{\mathrm{av}}-|\langle U_{cc'}^S\rangle_{\mathrm{av}}|^2],\quad(14)$$

where k_c is the wave number in channel c. Using the assumptions (i) and (ii) and expressions (2) and (10), we get for the reaction fluctuation cross section

$$\sigma_{cc'}^{\mathrm{fl.}}=\left(\frac{\pi}{k_c^2}\right)\left(\frac{2\pi}{D}\right)\frac{\langle|g_{\mu c}|^2|g_{\mu c'}|^2\rangle_\mu}{\Gamma}.\quad(15)$$

The ensemble average of the quantity $|\theta_{\mu c}|^2|\theta_{\mu c'}|^2$ is given by[6]

$$\langle|\theta_{\mu c}|^2|\theta_{\mu c'}|^2\rangle_\mu=\tfrac{1}{4}J_cJ_{c'}\frac{N}{N+2}\times\left\{\langle N_\mu^2\rangle_\mu\left[(5\gamma^2+3)+(1-\gamma^2)\frac{N+3}{N-1}\right]+\left[(3\gamma^2+1)-(1-\gamma^2)\frac{N+3}{N-1}\right]\right\},\quad(16)$$

where N_μ is a parameter called the normalization constant.[5] The ensemble average $\langle N_\mu^2\rangle_\mu\geqslant1$. For the case of large N and $\gamma=0$, we have

$$\langle|\theta_{\mu c}|^2|\theta_{\mu c'}|^2\rangle_\mu=J_cJ_{c'}\langle N_\mu^2\rangle_\mu.$$

Using the above value of the ensemble average we can write expression (15) as

$$\sigma_{cc'}^{\mathrm{fl.}}=\left(\frac{\pi}{k_c^2}\right)\left(\frac{2\pi}{D}\right)\frac{\langle\Gamma_{\mu c}\rangle_\mu\langle\Gamma_{\mu c'}\rangle_\mu}{\Gamma}\langle N_\mu^2\rangle_\mu,\quad c\neq c'.\quad(17)$$

The quantities $\Gamma_{\mu c}$ are channel parameters,[5] which add up to the total width Γ_μ when summed over the channel index c.

IV. CONCLUDING REMARKS

In Secs. II and III we have discussed the case of inelastic scattering. We would now like to say a few words about the elastic scattering. Let us consider the elastic fluctuation cross section $\sigma_{cc}^{\mathrm{fl.}}$. The main difficulty which one faces here is to see whether $\sigma_{cc}^{\mathrm{fl.}}$ can be represented as

$$\sigma_{cc}^{\mathrm{fl.}}=\left(\frac{\pi}{k_c^2}\right)\left(\frac{2\pi}{D}\right)\frac{\langle|a_{\mu cc}|^2\rangle_\mu}{\Gamma}.\quad(18)$$

For cases where expression (18) holds we need the ensemble average of $|\theta_{\mu c}|^4$. It is given by[6]

$$\langle|\theta_{\mu c}|^4\rangle_\mu=\frac{N}{N+2}J_c^2[1+2\langle N_\mu^2\rangle_\mu].$$

Expression (18) can then be written as

$$\sigma_{cc}^{\mathrm{fl.}}=\left(\frac{\pi}{k_c^2}\right)\left(\frac{2\pi}{D}\right)2\langle N_\mu^2\rangle_\mu\frac{\langle\Gamma_{\mu c}\rangle_\mu^2}{\Gamma}.\quad(19)$$

The interesting result to note is that expression (19) is similar to expression (3.24) of Ref. 1, except that the factor $2\langle N_\mu^2\rangle_\mu$, which occurs in expression (19) will always be $\geqslant2$, instead of a factor $\leqslant2$ as indicated in Ref. 1.

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[9] N. Rosenzweig, Phys. Letters 6, 123 (1963).