Modified Distorted-Wave Born Series

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A method is developed for obtaining a convergent solution for the three-body scattering amplitude in a modified distorted-wave representation. An integral equation for the three-body scattering operator is derived in a similar manner to that of Greider and Dodd. A new analytical form for the transition amplitude of the stripping processes is obtained after some reasonable approximations. This form contains the Butler cutoff and distorted-wave treatments as limiting cases. A curve is drawn to show the significance of this new Born term together with the usual Butler cutoff model for the $Si^{28}(d,p)Si^{29}$ reaction.

I. INTRODUCTION

REARRANGEMENT collisions play a central role in the field of nuclear reactions. In a rearrangement process one is generally faced with three-particle systems, and the problem of finding an exact expression for the scattering or exchange scattering amplitude of such systems. In computing the reaction amplitude one has to deal with the solution of a three-body-model problem,[1] in which the interaction between two of the three particles is assumed to be responsible for the transition. It is usually argued that the lowest term in this interaction (Born term) is quite sufficient for the description of the reaction amplitude. The plane-wave Born approximation and the distorted-wave Born approximation are typical examples which are commonly used. The validity of such approximations was always based on some qualitative arguments, and on the fits with the experimental data. However, the distorted-wave treatment,[2] for example, contains so many ambiguous parameters that it cannot be easily known whether the fitting obtained is really significant.

Recently, the distorted-wave Born series for rearrangement scattering was investigated by Greider and Dodd.[3] These authors have derived an integral equation for the three-body scattering operator, in such a way that the inhomogeneous part is just the distorted-wave Born term. It was found that the iterative solution to this integral equation diverges, because the kernel of the equation obtained is not continuous.[4] Consequently, the distorted-wave Born approximation may not be a correct approximation to the exact amplitude.

In this work, an integral equation for the transition operator of the three-body problem is derived in a manner similar to that obtained by Greider and Dodd.[5] The inhomogeneous part of this equation which reproduces a correct modified distorted-wave Born approximation, may be easily solved in terms of a two-body amplitude with the aid of some reasonable approximations. The case of stripping reactions will be of particular interest, because it will be shown that, in the plane-wave approximation, one obtains the transition amplitude in a new form which contains the original Butler cutoff result as a limiting case. Section II contains general theoretical formulation; Sec. III includes the investigation of the stripping process in the light of the new formalism. In Secs. IV and V numerical calculations for the $Si^{28}(d,p)Si^{29}$ reaction and a discussion of the results obtained are given.

II. THEORY

In this section, we are going to introduce a modified distorted-wave representation for the transition amplitude, which as will be shown is more accurate than the usual representation. For this purpose, let us consider the rearrangement process

$$(a+b)+c \rightarrow b+(a+c)\,,$$

where the parentheses indicate bound states. To describe this process, we split the complete Hamiltonian H of the system into two alternative ways:

$$\begin{aligned} H=H_i+V_i&=(H_0+V_{ab})+(V_{ac}+V_{bc}) \\ &=H_f+V_f=(H_0+V_{ac})+(V_{ab}+V_{bc})\,, \end{aligned}$$

where H_0 is the kinetic energy operator for the relative motion between the particles and V_{ij} is the two body interaction between the particles i and j.

Let ϕ_i and ϕ_f be the wave functions describing the initial and final noninteracting states with plane waves representing the relative motion. They are solutions of

$$(E-H_i)\phi_i=0 \quad \text{and} \quad (E-H_f)\phi_f=0\,.$$

The exact transition matrix element for rearrangement collisions may be given by[6]

$$T_{fi}=\langle\phi_f|V_f|\psi_i^{(+)}\rangle\,, \tag{1}$$

where $\psi_i^{(+)}$ is the outgoing wave solution of the complete Hamiltonian given by

$$\psi_i^{(+)}=\phi_i+(1/E-H+i\epsilon)V_i\phi_i=(1+G^{(+)}V_i)\phi_i\,.$$

Introducing now the distorting potentials $\mathcal{U}_i$ and $\mathcal{U}_f$ in the initial and final channels, respectively, one may write the corresponding distorted waves (DW) in the

[1] W. Tobocman, *Theory of Direct Nuclear Reactions* (Oxford University Press, London, 1961).
[2] C. A. Pearson and M. Coz, Nucl. Phys. **82**, 533 (1961).
[3] K. R. Greider and L. R. Dodd, Phys. Rev. **146**, 671 (1966).
[4] C. Lovelace, Phys. Rev. **135**, B1225 (1964).
[5] L. R. Dodd and K. R. Greider, Phys. Rev. **146**, 675 (1966).
[6] B. A. Lippmann, Phys. Rev. **102**, 264 (1956).

forms

$$\chi_i^{(+)}=\phi_i+\frac{1}{E-H_i+i\epsilon}\mathcal{U}_i\chi_i^{(+)}$$

$$=\left(1+\frac{1}{E-H_i-\mathcal{U}_i+i\epsilon}\mathcal{U}_i\right)\phi_i=\Omega_i^{(+)}\phi_i,$$

$$\chi_f^{(-)}=\phi_f+\frac{1}{E-H_f-i\epsilon}\mathcal{U}_f\chi_f^{(-)}$$

$$=\left(1+\frac{1}{E-H_f-\mathcal{U}_f-i\epsilon}\mathcal{U}_f\right)\phi_f=\Omega_f^{(-)}\phi_f; \quad (2)$$

$\psi_i^{(+)}$ may be now written in terms of $\chi_i^{(+)}$ as follows;

$$\psi_i^{(+)}=\left\{1+\frac{1}{E-H+i\epsilon}(V_i-\mathcal{U}_i)\right\}\chi_i^{(+)}$$

$$=[1+G^{(+)}(V_i-\mathcal{U}_i)]\chi_i^{(+)}. \quad (3)$$

Using Eq. (3), the transition amplitude in the DW representation may, then, be given by

$$T_{fi}^{\mathrm{DW}}=\langle\chi_f^{(-)}|V_f-\mathcal{U}_f^\dagger|\psi_i^{(+)}\rangle$$

$$=\langle\chi_f^{(-)}|(V_f-\mathcal{U}_f^\dagger)+(V_f-\mathcal{U}_f^\dagger)G^{(+)}(V_i-\mathcal{U}_i)|\chi_i^{(+)}\rangle, \quad (4)$$

provided that the following condition holds[3]

$$\lim_{\epsilon\to 0} i\epsilon\langle\chi_f^{(-)}/\phi_i\rangle=0. \quad (5)$$

This will be the case when we choose $\mathcal{U}_f$ to be the optical model potential in the final channel.

Thus, we see that in the DW treatment given above the distorting potential, $\mathcal{U}_f$ must be found from the condition (5) while $\mathcal{U}_i$ is completely arbitrary.

The matrix element (4), may be written alternatively using Eqs. (2) and (3) in the form

$$T_{fi}^{\mathrm{DW}}=\langle\phi_f|U_{fi}^{(+)}|\phi_i\rangle,$$

where

$$U_{fi}^{(+)}=\Omega_f^{(-)\dagger}(V_f-\mathcal{U}_f^\dagger)(1+G^{(+)}V_i).$$

One may now write (after some manipulations) the following integral equation for $U_{fi}^{(+)}$:

$$U_{fi}^{(+)}=\Omega_f^{(-)\dagger}(V_f-\mathcal{U}_f^\dagger)\Omega_i^{(+)}+U_{fi}^{(+)}G_i^{(+)}(V_i-\mathcal{U}_i)\Omega_i^{(+)}, \quad (6)$$

where

$$G_i^{(+)}=\frac{1}{E-H_i+i\epsilon}.$$

This integral equation was investigated by Greider and Dodd,[3] and they have been able to show that the kernel of this equation [i.e., the quantity $G_i^{(+)}(V_i-\mathcal{U}_i)\Omega_i^{(+)}$] is not a continuous one.

The usual method of eliminating the divergence from the kernel of Eq. (6) is to subtract explicitly the dangerous part of that kernel. For this purpose, let us rewrite Eq. (6) in the following abstract form:

$$U_{fi}^{(+)}=I+U_{fi}^{(+)}K, \quad (7)$$

where

$$I=\Omega_f^{(-)\dagger}(V_f-\mathcal{U}_f^\dagger)\Omega_i^{(+)},$$

and

$$K=G_i^{(+)}(V_i-\mathcal{U}_i)\Omega_i^{(+)}.$$

Let us now introduce an arbitrary operator K_0 and rewrite Eq. (7) in the form

$$U_{fi}^{(+)}=I+U_{fi}^{(+)}(K-K_0)+U_{fi}^{(+)}K_0,$$

or

$$U_{fi}^{(+)}=I(1-K_0)^{-1}+U_{fi}^{(+)}(K-K_0)(1-K_0)^{-1}. \quad (8)$$

The choice of K_0 is of course arbitrary except that it must contain at least all the singularities contained in the kernel K.

Now if $(1-K_0)^{-1}$ is a bounded operator,[7] then its product with a continuous operator $(K-K_0)$ will yield another continuous operator $(K-K_0)(1-K_0)^{-1}$ and all the troubles will be overcome. We define another operator g such that

$$\Omega_i^{(+)}Q=g(V_i-\mathcal{U}_i)\Omega_i^{(+)},$$

where $(1-K_0)^{-1}=1+Q$. Consequently, Eq. (7) will take the form

$$U_{fi}^{(+)}=I+\Omega_f^{(-)\dagger}(V_f-\mathcal{U}_f^\dagger)g(V_i-\mathcal{U}_i)\Omega_i^{(+)}+U_{fi}^{(+)}[K-(1-K)Q], \quad (9)$$

and noting that

$$(1-K)=(1-G_i^{(+)}V_i)\Omega_i^{(+)},$$

one may write

$$U_{fi}^{(+)}=\tilde{I}+U_{fi}^{(+)}[G_i^{(+)}-(1-G_i^{(+)}V_i)g]\times(V_i-\mathcal{U}_i)\Omega_i^{(+)}, \quad (10)$$

where

$$\tilde{I}=\Omega_f^{(-)\dagger}\{(V_f-\mathcal{U}_f^\dagger)+(V_f-\mathcal{U}_f^\dagger)g(V_i-\mathcal{U}_i)\}\Omega_i^{(+)}.$$

This equation contains only the arbitrary operator g, and one has to define it in such a way that the kernel in this equation will be a Schmidt one. If we now put

$$g=\frac{1}{E-H+V_x+i\epsilon},$$

where V_x is an arbitrary sum of two-body interactions, we immediately obtain the results derived by Greider and Dodd.[5] However, this form for g is not convenient as V_x is quite arbitrary and leads to a complicated form for the transition amplitude, especially in the case of stripping reactions.

Our aim is to remove from the new kernel of Eq. (10) (which may be denoted by $\mathcal{K}$) all the dangerous terms by the appropriate choice of g, and we proceed as follows:

[7] C. Lovelace, Lecture notes for the Edinburgh Summer School, 1963 (unpublished).

Let us consider the quantity $\mathcal{L}=G_i^{(+)}-(1-G_i^{(+)}V_i)g$, in terms of which $\mathcal{K}=\mathcal{L}(V_i-\mathcal{U}_i)\Omega_i^{(+)}$. Remembering that $V_i=V_{ac}+V_{bc}$, $\mathcal{L}$ may be written down in the two alternative ways

$$\mathcal{L}=G_i^{(+)}-(1-G_i^{(+)}V_{bc})g+G_i^{(+)}V_{ac}g, \quad (11a)$$

$$\mathcal{L}=G_i^{(+)}-(1-G_i^{(+)}V_{ac})g+G_i^{(+)}V_{bc}g. \quad (11b)$$

The following two possible choices for g may be taken: (i). We set [using (11a)] $G_0^{(+)}-(1-G_0^{(+)}V_{bc})g=0$. Hence g will take the form

$$g=g_{bc}=\frac{1}{E-H_0-V_{bc}+i\epsilon} \quad (12)$$

and $\mathcal{K}$ is, then, given by

$$\mathcal{K}=G_i^{(+)}(V_{ab}+V_{ac})g_{bc}(V_i-\mathcal{U}_i)\Omega_i^{(+)}. \quad (13)$$

Fortunately $\mathcal{U}_i$ is quite arbitrary. Choosing $\mathcal{U}_i$ in the form $\mathcal{U}_i=V_{ac}$ we find that the kernel in Eq. (12) is continuous.[4] This result may be convenient for the inelastic-scattering and knockout process in the reaction

$$(a+b)+c\rightarrow b+(a+c)$$

(ii). We set [using (11b)] $G_0^{(+)}-(1-G_0^{(+)}V_{ac})g$. Hence g will take the form

$$g=g_{ac}=\frac{1}{E-H_0-V_{ac}+i\epsilon}. \quad (14)$$

Therefore one may write

$$\mathcal{K}=G_i^{(+)}(V_{ab}+V_{bc})g_{ac}(V_i-\mathcal{U}_i)\Omega_i^{(+)}, \quad (15)$$

and choosing $\mathcal{U}_i=V_{bc}$, we find that $\mathcal{K}$ has no dangerous terms. This result may be applied to the case of stripping reactions in the process $(a+b)+c\rightarrow b+(a+c)$.

III. STRIPPING REACTIONS

Let us consider the stripping reaction

$$(a+b)+T\rightarrow b+(a+T).$$

Using Eqs. (14) and (15), the inhomogeneous term of Eq. (9) will then take the form

$$\tilde{I}=\Omega_f^{(-)\dagger}\{(V_f-\mathcal{U}_f^{\dagger})+(V_f-\mathcal{U}_f^{\dagger})g_{aT}(V_i-\mathcal{U}_i)\}\Omega_i^{(+)}, \quad (16)$$

provided that $\mathcal{U}_i=V_{bT}$. Therefore, the correct distorted-wave Born amplitude (DWB) will have the form

$$T_{fi}^{\mathrm{DWB}}=\langle\chi_f^{(-)}|(V_f-\mathcal{U}_f^{\dagger})\times\left\{1+\frac{1}{E-H_0-V_{aT}+i\epsilon}V_{aT}\right\}|\chi_i^{(+)}\rangle. \quad (17)$$

If we approximate $\mathcal{U}_f$ by V_{bT} (as is usually done) then our matrix element will take the form

$$T_{fi}^{\mathrm{DWB}}=\langle\chi_f^{(-)}|V_{ab}\left\{1+\frac{1}{E-H_0-V_{aT}+i\epsilon}V_{aT}\right\}|\chi_i^{(+)}\rangle. \quad (18)$$

We now try to evaluate this matrix element, first in the plane-wave approximation, and then in the distorted-wave approximation.

Plane-Wave Approximation

In the plane-wave approximation (PWB) our matrix element [Eq. (18)] takes the form

$$T_{fi}^{\mathrm{PWB}}=\langle\phi_f|V_{ab}\left\{1+\frac{1}{E-H_0-V_{aT}+i\epsilon}V_{aT}\right\}|\phi_i\rangle, \quad (19)$$

where we recall that ϕ_i and ϕ_f are defined by

$$(E-H_i)\phi_i=0,\quad (E-H_f)\phi_f=0.$$

Neglecting spin complications one may write (see Fig. 1)

$$\phi_i=\exp(i\mathbf{k}_i\cdot\mathbf{r}_i)\varphi_{lm}(\boldsymbol{\rho})\cdot A_T(\xi_T)A_a(\xi_a)A_b(\xi_b),$$
$$\phi_f=\exp(i\mathbf{k}_f\cdot\mathbf{r}_f)\cdot\Phi_{LM}(\mathbf{r})\cdot A_T(\xi_T)A_a(\xi_a)A_b(\xi_b), \quad (20)$$

where $\mathbf{k}_i,\mathbf{k}_f$ are the wave numbers describing the relative motion in the initial and final states, respectively, φ_{lm} and Φ_{LM} are the wave functions describing the relative motions in the bound states in the initial and final channels, respectively, and $A_i(\xi_i)$ is the internal wave function of the particle i. Let us now transform the variables in the matrix element (2) to the new coordinates $\mathbf{r}$ and $\mathbf{r}_f$ according to

$$\mathbf{r}_i=\frac{m_aM}{m_Am_R}\mathbf{r}+\frac{m_b}{m_A}\mathbf{r}_f,\quad \boldsymbol{\rho}=\frac{m_T}{m_R}\mathbf{r}-\mathbf{r}_f,$$

where $M=m_a+m_b+m_T$, $m_A=m_a+m_b$, $m_R=m_a+m_T$, and m_i is the mass of the particle i. Further, let us make use of the Fourier transformation

$$\varphi_{lm}(\boldsymbol{\rho})=\frac{1}{(2\pi)^3}\int d\mathbf{K}\,G_{lm}(\mathbf{K})e^{i\mathbf{K}\cdot\boldsymbol{\rho}}.$$

Substituting the expressions (20) for ϕ_i and ϕ_f into the matrix element (19) and using the above relations, one may write

$$\tilde{T}_{fi}^{\mathrm{PWB}}=\frac{1}{(2\pi)^3}\int d\mathbf{K}\,G_{lm}(\mathbf{K})\int d\mathbf{r}_f\int d\mathbf{r}\,e^{-i\mathbf{k}_f\cdot\mathbf{r}_f}\times\Phi_{LM}^*(\mathbf{r})V_{ab}\left(\frac{m_T}{m_R}\mathbf{r}-\mathbf{r}_f\right)\times\left\{1+\frac{1}{E-H_0-V_{aT}+i\epsilon}V_{aT}\right\}e^{i\mathbf{q}\cdot\mathbf{r}}e^{i\mathbf{Q}\cdot\mathbf{r}_f}, \quad (21)$$

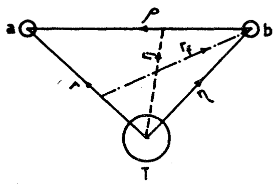

FIG. 1. Schematic diagram for the stripping process $(a+b)T \to b+(a+T)$.

where

$$\mathbf{q}=\frac{m_T}{m_R}\mathbf{K}+\frac{m_a M}{m_A m_R}\mathbf{k}_i \quad \text{and} \quad \mathbf{Q}=\frac{m_b}{m_A}\mathbf{k}_i-\mathbf{K}.$$

Rearranging terms in expression (21) one may find that

$$T_{fi}^{\text{PWB}}=\frac{1}{(2\pi)^3}\int d\mathbf{K}\, G_{lm}(\mathbf{K})\int d\mathbf{r}_f\int d\mathbf{r}\, e^{i(\mathbf{k}-\mathbf{K})\cdot\mathbf{r}_f}$$

$$\times\Phi_{LM}^*(\mathbf{r})V_{ab}\left(\frac{m_T}{m_R}\mathbf{r}-\mathbf{r}_f\right)\psi_{\mathbf{q}}^{(+)}(\mathbf{r}), \quad (22)$$

where

$$\mathbf{k}=\frac{m_b}{m_A}\mathbf{k}_i-\mathbf{k}_f,$$

$$\psi_{\mathbf{q}}^{(+)}(\mathbf{r})=\left\{1+\frac{1}{(\hbar^2/2\mu_{aT})(\nabla_r^2+q^2)-V_{aT}(r)+i\epsilon}V_{aT}(r)\right\}\times e^{i\mathbf{q}\cdot\mathbf{r}},$$

and μ_{ij} is the reduced mass of the particles i and j. Integrating over $\mathbf{r}_f$ in (22) we get

$$\tilde{T}_{fi}^{\text{PWB}}=\frac{1}{(2\pi)^3}\int d\mathbf{K}\, G_{lm}(\mathbf{K})\tilde{V}_{ab}(\mathbf{k}-\mathbf{K})$$

$$\times\langle\Phi_{LM}(\mathbf{r})|e^{i(m_T/m_R)(\mathbf{k}-\mathbf{K})\cdot\mathbf{r}}|\psi_{\mathbf{q}}^{(+)}(\mathbf{r})\rangle, \quad (23)$$

provided that

$$\tilde{V}_{ab}(\mathbf{k}-\mathbf{K})=\int d\mathbf{X}\, e^{-i\mathbf{X}\cdot(\mathbf{k}-\mathbf{K})}V_{ab}(X).$$

To evaluate the integral (23) we adopt the following approximation:

Let us consider the scattering solution $\psi_{\mathbf{q}}^{(+)}(\mathbf{r})$ as a function of the vectors $\mathbf{K}$ and $\mathbf{k}_i$, i.e. let us write

$$\psi_{\mathbf{q}}^{(+)}(\mathbf{r})\equiv\psi^{(+)}\left(\mathbf{r},\frac{m_T}{m_R}\mathbf{K}+\mathbf{p}\right), \quad \mathbf{p}=\frac{m_a M}{m_A m_R}\mathbf{k}_i.$$

Then

$$\psi_{\mathbf{q}}^{(+)}(\mathbf{r})=\exp\left(\frac{m_T}{m_R}\mathbf{K}\cdot\nabla_{\mathbf{p}}\right)\psi^{(+)}(\mathbf{r},\mathbf{p})$$

$$\cong\exp\left(i\frac{m_T}{m_R}\mathbf{K}\cdot\mathbf{r}\right)\psi^{(+)}(\mathbf{r},\mathbf{p})$$

$$=\exp\left(i\frac{m_T}{m_R}\mathbf{K}\cdot\mathbf{r}\right)\psi_{\mathbf{p}}^{(+)}(\mathbf{r}). \quad (24)$$

[This approximation may be accepted insofar as the main contribution to the transition amplitude may come from outside the nuclear surface. Therefore, in our approximation, we let the nabla operator $\nabla_{\mathbf{p}}$ act on the plane-wave limit of $\psi_{\mathbf{p}}^{(+)}(\mathbf{r})$.] Hence, using Eq. (24), one may write

$$\tilde{T}_{fi}^{\text{PWB}}=\left\{\frac{1}{(2\pi)^3}\int d\mathbf{K}\, G_{lm}(\mathbf{K})\tilde{V}_{ab}(\mathbf{k}-\mathbf{K})\right\}$$

$$\times\langle\Phi_{LM}(\mathbf{r})|e^{i(m_T/m_R)\mathbf{k}\cdot\mathbf{r}}|\psi_{\mathbf{p}}^{(+)}(\mathbf{r})\rangle. \quad (25)$$

Note that the integration over $\mathbf{K}$ in Eq. (25) can be easily evaluated using the Schrödinger equation for the relative motion between the particles a and b in the bound state $(a+b)$. Consequently

$$\tilde{T}_{fi}^{\text{PWB}}=-\frac{\hbar^2}{2\mu_{ab}}(k^2+\chi^2)G_{lm}(\mathbf{k})$$

$$\times\langle\Phi_{LM}(\mathbf{r})|e^{i(m_T/m_R)\mathbf{k}\cdot\mathbf{r}}|\psi_{\mathbf{p}}^{(+)}(r)\rangle, \quad (26)$$

where $\chi^2=(2\mu_{ab}/\hbar^2)\epsilon$; ϵ is the binding energy of the particles a and b and μ_{ab} is their reduced mass.

If we approximate $\psi_{\mathbf{p}}^{(+)}(\mathbf{r})$ by its plane-wave solution, we obtain the usual Butler cutoff result.

Distorted-Wave Treatment

In our investigation of the stripping process, the distorting potentials in the initial and final channels are taken as $\mathcal{U}_i=\mathcal{U}_f=V_{bT}$. The corresponding distorted waves [see Eqs. (2) and (3)] are therefore solutions of the following equations (with the appropriate boundary conditions):

$$\chi_i^{(+)}=\left\{1+\frac{1}{E-H_i-V_{bT}+i\epsilon}V_{bT}\right\}\phi_i,$$

$$\chi_f^{(-)}=\left\{1+\frac{1}{E-H_f-V_{bT}-i\epsilon}V_{bT}\right\}\phi_f.$$

To obtain the wave function $\chi_i^{(+)}$, we transform to the variables $\boldsymbol{\varrho}$ and $\boldsymbol{\eta}$ (see Fig. 1), and then express χ_i in the following form[8] (neglecting spin complication):

$$\chi_i=\phi_i\zeta_i(\boldsymbol{\eta})+\Phi(\boldsymbol{\varrho},\boldsymbol{\eta}),$$

where Φ may involve only states orthogonal to ϕ_i. Introducing this expression into the corresponding equation for χ_i (expressed in the new variables $\boldsymbol{\varrho}$ and $\boldsymbol{\eta}$) and then multiplying by ϕ_i^* and integrating over $\boldsymbol{\varrho}$, one gets the following equation for $\zeta_i(\boldsymbol{\eta})$:

$$\left(\nabla_\eta^2+2i\lambda\mathbf{k}_i\cdot\nabla_\eta-\frac{2\mu_{bT}}{\hbar^2}V_{bT}\right)\zeta_i(\boldsymbol{\eta})$$

$$=\frac{2\mu_{bT}}{\hbar^2}\int d\boldsymbol{\varrho}\,\phi_i^*\{\nabla_\rho\cdot\nabla_\eta\}\Phi(\boldsymbol{\varrho},\boldsymbol{\eta}), \quad (27)$$

[8] I. P. Grant, Proc. Phys. Soc. (London) 68A, 244 (1955).

with

$$\lambda = 1 - \frac{m_a \mu_{bT}}{m_b m_A}.$$

An approximation for χ_i may be obtained by neglecting the right-hand side of this equation. This effect means that we neglect the effect of dissociation of the projectile in the field of the target nucleus. Such a procedure will lead to the following expression for $\chi_i^{(+)}$ (neglecting spin complications)

$$\chi_i^{(+)} = \phi_i \zeta_i^{(+)}(\boldsymbol{\eta}), \tag{28}$$

where $\zeta_i^{(+)}$ is the solution of Eq. (27) without the right-hand side, and with outgoing boundary condition. It should be noted that if we multiply $\zeta_i^{(+)}(\boldsymbol{\eta})$ by $\exp(i\lambda\mathbf{k}_i\cdot\boldsymbol{\eta})$ we obtain the wave function describing the scattering of the particle (b) on the target (T) via V_{bT} with outgoing boundary condition. Similarly $\chi_f^{(-)}$ may be written in the form

$$\chi_f^{(-)} \cong \phi_f \zeta_f^{(-)}(\boldsymbol{\eta}), \tag{29}$$

where $\zeta_f^{(-)}$ satisfies the equation

$$\left(\nabla_\eta^2 - 2i\lambda'\mathbf{k}_f\cdot\nabla_\eta - \frac{2\mu_{bT}}{\hbar^2}V_{bT}\right)\zeta_f(\boldsymbol{\eta}) = 0,$$

and where

$$\lambda' = 1 - \frac{m_a m_b}{m_R(m_b + m_T)}.$$

Now, making use of

$$\zeta_i^{(+)}(\boldsymbol{\eta}) = \frac{1}{(2\pi)^3}\int d\mathbf{p}_1\, e^{i\mathbf{p}_1\cdot\boldsymbol{\eta}} a_i^{(+)}(\mathbf{p}_1),$$

and

$$\zeta_f^{(-)}(\boldsymbol{\eta}) = \frac{1}{(2\pi)^3}\int d\mathbf{p}_2\, e^{i\mathbf{p}_2\cdot\boldsymbol{\eta}} a_f^{(-)}(\mathbf{p}_2),$$

and also of the Fourier transform of the bound state wave function of the particles a and b as in Eq. (21), one may write

$$\chi_i^{(+)} = \frac{1}{(2\pi)^6}\int d\mathbf{K}\int d\mathbf{p}_1\, e^{i\mathbf{Q}'\cdot\mathbf{r}_f} e^{i\mathbf{q}'\cdot\mathbf{r}} a_i^{(+)}(\mathbf{p}_1) G_{lm}(\mathbf{K}),$$

with

$$\mathbf{q}' = \frac{m_T}{m_R}\mathbf{K} + \frac{m_a M}{m_A m_R}\mathbf{k}_i + \frac{m_a}{m_R}\mathbf{p}_1,$$

$$\mathbf{Q}' = \frac{m_b}{m_A}\mathbf{k}_i - \mathbf{K} + \mathbf{p}_1.$$

Consequently,

$$\tilde{T}_{fi}^{\mathrm{DWB}} = \frac{1}{(2\pi)^9}\int d\mathbf{K}\int d\mathbf{p}_1\int d\mathbf{p}_2 \times G_{lm}(\mathbf{K}) a_f^{(-)*}(\mathbf{p}_2) a_i^{(+)}(\mathbf{p}_1)\int d\mathbf{r}_f\, e^{i(\mathbf{Q}'-\mathbf{k}_f-\mathbf{p}_2)\cdot\mathbf{r}_f}\mathcal{T}_{fi},$$

where

$$\mathcal{T}_{fi} = \langle e^{i(m_a/m_R)\mathbf{p}_2\cdot\mathbf{r}}\Phi_{LM}(r) | V_{ab} \times\left\{1 + \frac{1}{E - H_0 - V_{aT} + i\epsilon}V_{aT}\right\} | e^{i\mathbf{q}'\cdot\mathbf{r}}\rangle. \tag{30}$$

Since the plane-wave approximation can describe the essential features of this reaction, we assume that in the matrix element $\mathcal{T}_{fi}$ in Eq. (30), most of the contribution will arise from components $\mathbf{p}_{1=0}$ and $\mathbf{p}_{2=0}$; consequently one may find that

$$\tilde{T}_{fi}^{\mathrm{DWB}} \cong \frac{1}{(2\pi)^3}\int d\mathbf{K}\, G_{lm}(\mathbf{K}) \times \langle\Phi_{LM}(\mathbf{r}) | \mathcal{V}(\mathbf{r}) | \psi_{\mathbf{q}}^{(+)}(\mathbf{r})\rangle, \tag{31}$$

where

$$\mathcal{V}(\mathbf{r}) = \langle e^{i\mathbf{k}_f\cdot\mathbf{r}_f}\zeta_f^{(-)}(\mathbf{r}_f) | V_{ab} | e^{i\mathbf{Q}\cdot\mathbf{r}_f}\zeta_i^{(+)}(\mathbf{r}_f)\rangle.$$

In the plane-wave limit, we have $\zeta_i^{(+)} = \zeta_f^{(-)} \to 1$ and we obtain Eq. (27).

IV. NUMERICAL CALCULATIONS

In this section, the transition amplitude for the stripping process in the plane-wave limit of the present model will be considered. Let us consider Eq. (26) and rewrite it in the form

$$T_{fi}^{\mathrm{PWB}} \cong -\frac{\hbar^2}{2\mu_{ab}}(k^2 + \chi^2) G_{lm}(\mathbf{k})\int_{r\geq R_0}\Phi_{LM}^*(\mathbf{r}) \times e^{i(m_T/m_R)\mathbf{k}\cdot\mathbf{r}}\psi_{\mathbf{p}}^{(+)}(\mathbf{r})d\mathbf{r}. \tag{32}$$

For simplicity the case of $L=0$ will be considered. The calculations for other L values are straightforward. Using the partial wave expansions for $\psi_{\mathbf{p}}^{(+)}(\mathbf{r})$ and $\exp(i(m_T/m_R)\mathbf{k}\cdot\mathbf{r})$ and noting that $\Phi_{00} = N_1 F(r)$, where N_1 is a normalization constant, the matrix element (32) takes the form

$$\tilde{T}_{fi}^{\mathrm{PWB}} \cong -\frac{(4\pi)^2 N_1 \hbar^2}{2\mu_{ab}}(k^2 + \chi^2) G_{lm}(\mathbf{k}) \times \sum_{l'm'} e^{i\delta_{l'}} Y_{l'}^{m'*}(\hat{p}) Y_{l'}^{m'}(\hat{q})\int_{r\geq R_0}^{\infty} r^2 dr \times F(r) j_{l'}(qr)\psi_{l'}^{(+)}(p,r), \tag{33}$$

where

$$\psi_{\mathbf{p}}^{(+)}(\mathbf{r}) = 4\pi\sum_{l'm'} e^{i\delta_{l'}}\psi_{l'}^{(+)}(pr) Y_{l'}^{m'*}(\hat{p}) Y_{l'}^{m'}(\hat{p});$$

$$\mathbf{q} = -\frac{m_T}{m_R}\mathbf{k}.$$

A crude value for the integral in Eq. (33) may be now obtained by taking the value of the integrand at radius R_0 near the nuclear surface. Hence, Eq. (23) becomes,

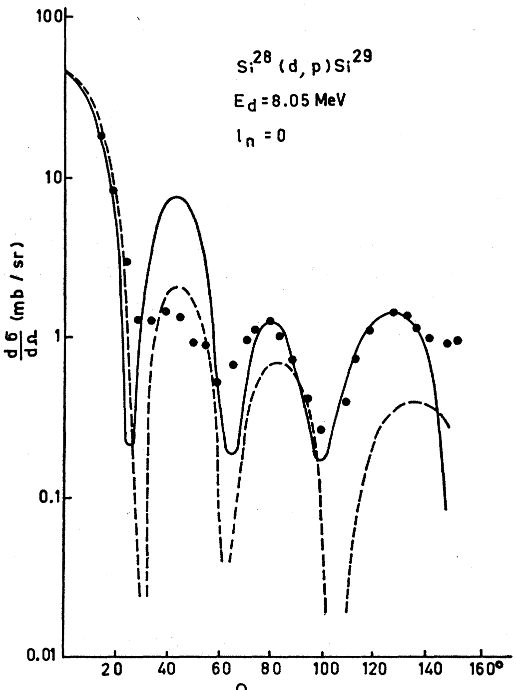

FIG. 2. Angular distribution of the $Si^{28}(d,p)Si^{29}$ reaction. Solid line—present work; dashed line—Butler cutoff model.

after summing over the projections,

$$\tilde{T}_{fi}^{\text{PWB}} \cong -\frac{(4\pi)N_1R_0^2F(R_0)}{2\mu_{ab}\alpha}(k^2+\chi^2) \times G_{lm}(\mathbf{k})\sum_{l'=0}^{\infty} e^{i\delta_{l'}}(2l'+1)P_{l'}(\cos(\mathbf{p},\mathbf{q})) \times \psi_{l'}^{(+)}(PR_0)j_{l'}(qR_0)\,, \quad (34)$$

where α is the wavenumber of the captured particle.

Now let us consider the reaction $Si^{28}(d,p)Si^{29}$. Assuming the internal deuteron wave function to be of the form $\varphi_d = N_2(e^{-\chi r}/r)$ and calculating the phase shifts from a square-well potential U_0 which reproduces the binding energy of the captured neutron in the $2S$-state in Si^{29}, we get

$$\tilde{T}_{fi} \cong -\frac{(4\pi)R_0^2F(R_0)N_1N_2\hbar^2}{\alpha}\sum_{l'}(2l'+1) \times C_{l'}P_{l'}(\cos(\mathbf{p},\mathbf{q}))j_{l'}(qR_0)\,, \quad (35)$$

where

$$C_{l'} = e^{i\delta_{l'}}\cos\delta_{l'}[j_{l'}(k_0R_0) - \tan\delta_{l'}n_{l'}(k_0R_0)]\,,$$

and

$$k_0^2 = p^2 + \frac{2\mu_{nT}}{\hbar^2}U_0\,.$$

Now, in the usual Butler cutoff representation, the matrix element of the above-mentioned reaction, using the same approximations, will take the form

$$T_{fi}^{\text{Butler}} = -\frac{(4\pi)R_0^2F(R_0)N_1N_2}{\alpha}j_0(PR_0)\,, \quad (36)$$

where

$$\mathbf{P} = \mathbf{k}_i - \frac{m_T}{m_R}\mathbf{k}_f\,.$$

Figure 2 shows the comparison between our results and the experimental data taken from Ref. 9, where we have used the following parameters

$U_0 = 47.5$ MeV with a range $d = 4.1$ F

$R_0 = 7.4$ F.

V. DISCUSSION

In the present work, an approach is suggested for obtaining the three-body amplitude of stripping and other direct reactions. It has been shown that in the case of stripping reactions a new form is obtained for the transition amplitude in the plane and distorted-wave treatments. The usual Butler cutoff theory and distorted-wave theory appear as limiting cases of the present matrix element.

In the case of knockout and inelastic scattering processes, one obtains the same results of Greider and Dodd,[5] which reproduce the impulse approximation.

Numerical comparison between our results and the Butler cutoff theory is presented in Fig. 2. From this comparison it seems that the present theory shows some improvement over the usual plane-wave Born approximations and gives good agreement with the experimental data, especially at large angles. Moreover, the order of magnitude of the differential cross section will be larger than that obtained from the usual theory [see Eqs. (35) and (36)]; thus the extracted values of the reduced widths will be greater, which is supported by the shell model calculations.

It is hoped that the present results may help to clarify some of the difficulties which are associated with the three-body problems.

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