Charge Form Factor of the Alpha Particle

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We calculate analytically the charge form factor $F_E(q^2)$ of the α particle (He^4) with a modified Irving wave function. The parameters of the wave function have been determined from a variational calculation of the binding energy of He^4 using a central velocity-dependent potential. We find good agreement between our values of $F_E(q^2)$ and those obtained from electron-helium elastic-scattering experiments. Also, the agreement with calculations using hard-core potentials is satisfactory.

DURING the past two years a number of groups of workers have measured the charge form factor $F_E(q^2)$ of the α particle (^{4}He) from electron-helium scattering experiments.[1] In the present note we propose to give an analytical calculation of this quantity.

The charge form factor of the α particle is related to the bare form factor $F_B(q^2)$ through the equation

$$F_E(q^2)=F_B(q^2)\times F_{ES}, \quad (1)$$

where

$$F_{ES}=F_{Ep}+F_{En} \quad (2)$$

is the isoscalar form factor. The bare form factor $F_B(q^2)$ is the Fourier transform of the squared α-particle wave function $|\Psi|^2$. Let $\mathbf{r}_1$ denote the radius vector of one of the protons in ^{4}He (we take nucleon 1 in ^{4}He to be a proton) and $\mathbf{R}$, the radius vector of the center-of-mass. Then

$$F_B(q^2)=\int|\psi|^2\exp(i\mathbf{q}\cdot\mathbf{r})d^3r_i, \quad (3)$$

where $\mathbf{r}=\mathbf{r}_1-\mathbf{R}$.

Our calculation uses a four-parameter, modified Irving[2] wave function

$$\psi=N\{\exp[-\alpha(\sum_{i<j}r_{ij}^2)^{1/2}]+A\exp[-\lambda(\sum_{i<j}r_{ij}^2)^{1/2}]\}/(\sum_{i<j}r_{ij}^2)^n;\quad i,j=1,2,3\text{ and }4 \quad (4)$$

in which the normalizing constant is

$$N=\left[\frac{105\times 2^{13-8n}}{\pi^4(8-4n)!}\right]^{1/2}\Big/[\alpha^{4n-9}+2A\{(\alpha+\lambda)/2\}^{4n-9}+A^2\lambda^{4n-9}]^{1/2}. \quad (5)$$

[Actually the normalization constant shown in Eq. (5) is for the transformed wave function obtained by using the transformations given in Eq. (A1) of the Appendix.]

The parameters α, λ, A, and n have been determined by a variational calculation of the binding energy of the α particle with a two-body central velocity-dependent potential.[3] Since the state represented by the wave function ψ is completely symmetric in the relative spatial coordinates of the four nucleons in the α particle, the effective two-body central potential is the average of potentials in ^1S and ^3S states. Therefore, in the variational calculation of the binding energy of the α particle we have used the potentials of Srivastava for the case of the triton.[4]

$$\begin{aligned}V_{\text{eff}}(r_{ij})=&-\tfrac{1}{2}(1+X_{\text{static}})(V_0)_{\text{static}}\exp(-2r_{ij}/\beta_s)\\&+[(V_0)_{\text{vel.dep.}}/2M][p^2\omega_s(r_{ij})+\omega_s(r_{ij})p^2]\\&+[X_{\text{vel}}(V_0)_{\text{vel.dep.}}/2M][p^2\omega_t(r_{ij})+\omega_t(r_{ij})p^2], \quad (6)\end{aligned}$$

where

$$\omega(r)=\exp(-2r/\beta'), \quad (7)$$

and subscripts s and t denote the singlet and triplet states, respectively. The values of the potential pa-

TABLE I. Calculation of the bare form factor $F_B(q^2)$ and charge form factor $F_E(q^2)$ for ^{4}He using the wave function given by Eqs. (4), (5), and (9). $F_B(q^2)$ is the Fourier transform of the squared wave function and $E_E(q^2)$ is given by Eq. (1). $F_{ES}(q^2)$ is calculated from Eq. (11).

q^2 (F^{-2})	$F_B(q^2)$ [a]	$F_B(q^2)$ [b]	$F_{ES}(q^2)$	$E_E(q^2)$ [c] Theoret	$F_E(q^2)$ [d] Exptl
0.5	0.8253	0.837	0.9533	0.7868	0.7959
1.0	0.6836	0.702	0.9101	0.6221	0.6258
1.5	0.5680	0.591	0.8698	0.4941	0.4937
2.0	0.4735	0.499	0.8323	0.3941	0.3910
3.0	0.3317	0.357	0.7645	0.2536	0.2548
4.0	0.2347	0.256	0.7050	0.1655	0.1583
5.0	0.1676	0.184	0.6524	0.1093	0.0964
6.0	0.1205	0.131	0.6058	0.0730	0.0594
7.0	0.0872		0.5643	0.0492	0.0324
8.0	0.0635		0.5270	0.0334	0.0184
9.0	0.0463		0.4935	0.0228	0.0089
10.0	0.0339		0.4632	0.0157	0.0062

[a] Present calculation.
[b] Calculation of Tang and Herndon using hard-core potentials and wave functions, cf. Ref. 7.
[c] Present calculation.
[d] Experimental values of Frosch *et al.*, cf. Ref. 1.

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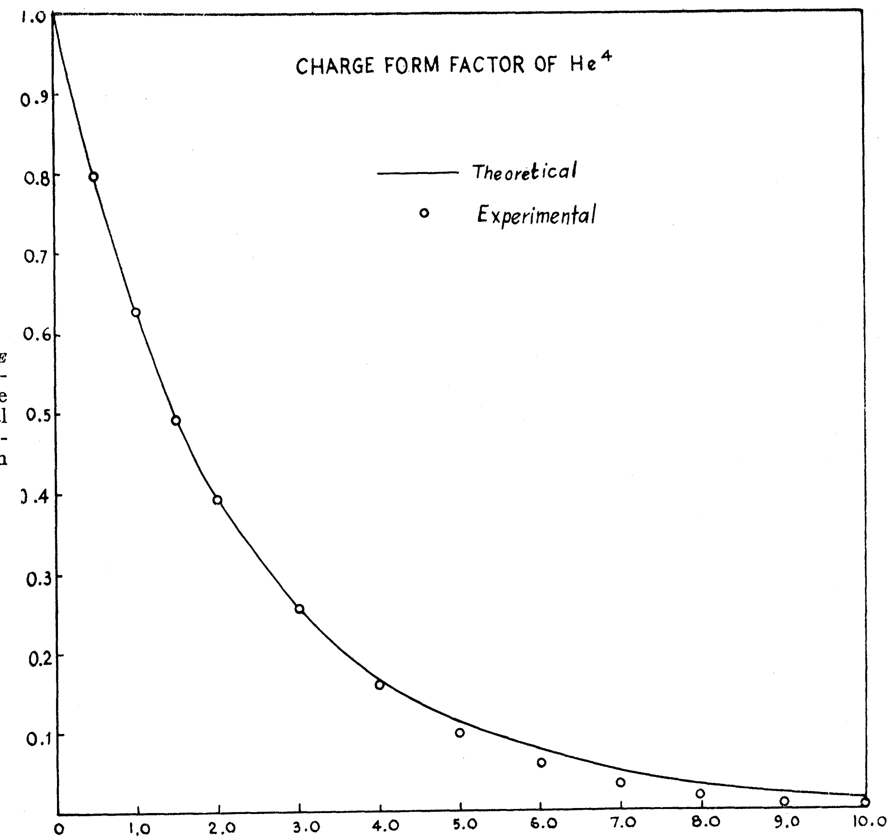

FIG. 1. The charge form factor F_E for the α particle plotted against momentum transfer squared, q^2. The continuous curve shows our theoretical $F_B(q^2)$ while the circles show the experimental $F_E(q^2)$ obtained by Frosch *et al.*, from Table I.

rameters are

$$(V_0)_{\text{static}}=100 \text{ MeV},\quad (V_0)_{\text{vel.dep.}}=2,$$
$$X_{\text{static}}=1.84,\quad X_{\text{vel}}=0.55,$$
$$1/\beta_s=0.625F^{-1},\quad 1/\beta_s'=1.4F^{-1},\quad 1/\beta_t'=1F^{-1}. \quad (8)$$

The ^1S part of the effective potential fits p-p low and high-energy scattering data, while the ^3S parts fits the binding energy of the deuteron and Breit's ^3S phase shifts at $E_{\text{lab}}=147$, 270, and 310 MeV.

For this potential and the trial wave function given by Eqs. (4) and (5), the variational calculation gives the best values of the parameters to be

$$n=0,\quad \alpha=0.90F^{-1},\quad \lambda=1.14F^{-1},\quad \text{and}\quad A=-1.38. \quad (9)$$

The corresponding value for the binding energy E_B of the alpha particle[3] is 30.1 MeV, while[5] the experimental value (E_{exptl}) is 28.29 MeV.

We calculate the bare form factor $F_B(q^2)$ by evaluating the integral in Eq. (3) for the α-particle wave function given by Eqs. (4) and (5). (For the evaluation of the integral see the Appendix.) We get

$$F_B(q^2)=[\alpha^{-9}+2A\{(\alpha+\lambda)/2\}^{-9}+A^2\lambda^{-9}]^{-1}$$
$$\times[f(\alpha,q^2)+2Af((\alpha+\lambda)/2,q^2)+A^2f(\lambda,q^2)],$$

where

$$f(x,q^2)=x^{-9}[1+(3q^2/64x^2)]^{-5}. \quad (10)$$

Equation (10) together with the values of parameters α, λ, and A in Eq. (9) yields the bare form factor in Table I.

We take the isoscalar electric form factor from Dudelzak's thesis[6]:

$$F_{ES}=2[1.14/(1-t/m_\omega^2)]-2[0.64/(1-t/m_\phi^2)],$$

where $m_\omega=783$ MeV, $m_\phi=1020$ MeV, and $t=-q^2$. Expressing t, m_ϕ, and m_ω in units of F^{-2} and substituting $t=-q^2$, we obtain

$$F_{ES}=2[1.14/(1+0.06345q^2)]$$
$$-2[0.64/(1+0.03739q^2)]. \quad (11)$$

These values of F_{ES} for different values of q^2 are tabulated in column four of Table I. Finally, the values of $F_B(q^2)$ and $F_{ES}(q^2)$ are used in Eq. (1) to give the charge form factor $F_E(q^2)$ of the α particle in the fifth column of Table I.

Figure 1 and Table I show that experimental and theoretical values of $F_E(q^2)$ are in close agreement. Also

[5] Thus our calculation gives $E_B>E_{\text{exptl}}$. This is inconsistent with the theory of variational calculation. However, we have not included tensor forces in our two-body potential. The inclusion of tensor forces is very likely to remove this inconsistency by reducing the binding energy obtained with purely central forces. For the effect of tensor forces on the binding energy of the alpha particle, see Blatt and Weisskopf, *Theoretical Nuclear Physics* (John Wiley & Sons, Inc., New York, 1952), Chap. V.

[6] J. S. Levinger (private communication).

we find from Table I that our values of $F_B(q^2)$ agree reasonably well with those of Tang and Herndon,[7] who have used a hard-core potential and wave function in their variational calculation. These comparisons, therefore, lead us to conclude that our wave function for the alpha particle given by Eqs. (4), (5), and (9) and the two-body velocity-dependent potentials given by Eqs. (6)–(8) are quite good. This is contrary to our conclusion in the case of the triton.[8] Perhaps our trial wave function for the triton is not as satisfactory for variational calculation as our modified Irving wave function is for the α particle.

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APPENDIX

Here we discuss the evaluation of integral in Eq. (3) for the bare form factor.

We introduce the following transformations due to Irving[2]

$$\mathbf{u}=\tfrac{1}{2}[(\mathbf{r}_1+\mathbf{r}_2)-(\mathbf{r}_3+\mathbf{r}_4)],\quad \mathbf{v}=(\mathbf{r}_1-\mathbf{r}_2)/\sqrt{2}, \tag{A1}$$
$$\mathbf{w}=(\mathbf{r}_3-\mathbf{r}_4)/\sqrt{2},\quad\text{and}\quad \mathbf{R}=\tfrac{1}{4}(\mathbf{r}_1+\mathbf{r}_2+\mathbf{r}_3+\mathbf{r}_4).$$

Then, the integral in Eq. (3) reduces to the sum of three integrals of the type

$$\int \exp[-4k(u^2+v^2+w^2)^{1/2}] \times\exp[i\mathbf{q}\cdot(\tfrac{1}{2}\mathbf{u}+\mathbf{v}/\sqrt{2})]d^3ud^3vd^3w. \tag{A2}$$

[We have taken $n=0$ in accordance with Eq. (9).]

In order to evaluate (A2), it is instructive to consider a more general type of integral

$$I(\beta,\gamma,\delta;q_1,q_2,q_3)=\int \exp[-(\beta^2u^2+\gamma^2v^2+\delta^2w^2)^{1/2}] \times\exp[i(\mathbf{q}_1\cdot\mathbf{u}+\mathbf{q}_2\cdot\mathbf{v}+\mathbf{q}_3\cdot\mathbf{w})]d^3ud^3vd^3w. \tag{A3}$$

We define two nine-dimensional vectors $\boldsymbol{\varrho}$ and $\mathbf{Q}$ such that

$$\rho_{1,2,3}=\beta(u)_{x,y,z},\quad \rho_{4,5,6}=\gamma(v)_{x,y,z},\quad \rho_{7,8,9}=\delta(w)_{x,y,z}. \tag{A4}$$

$$Q_{1,2,3}=\beta^{-1}(q_1)_{x,y,z},\quad Q_{4,5,6}=\gamma^{-1}(q_2)_{x,y,z},\quad Q_{7,8,9}=\delta^{-1}(q_3)_{x,y,z}. \tag{A5}$$

Then, on account of Eqs. (A4) and (A5), the integral given by Eq. (A3) reduces to

$$I=(\beta\gamma\delta)^{-3}\int \exp(i\mathbf{Q}\cdot\boldsymbol{\varrho}-\rho)d^9\rho. \tag{A6}$$

Next, we follow Sommerfeld[9] and introduce the following transformation equations between Cartesian $(x_1,x_2,\cdots x_i)$ and spherical polar $(\rho,\theta,\phi_1\cdots\phi_{i-2})$ coordinates in the nine-$(i=9)$ dimensional space:

$$\begin{aligned} x_1&=\rho\cos\theta,\\ x_2&=\rho\sin\theta\cos\phi_1,\\ x_3&=\rho\sin\theta\sin\phi_1\cos\phi_2,\\ &\cdots\\ x_8&=\rho\sin\theta\sin\phi_1\sin\phi_2\cdots\sin\phi_6\cos\phi_7,\\ x_9&=\rho\sin\theta\sin\phi_1\sin\phi_2\cdots\sin\phi_6\sin\phi_7. \end{aligned} \tag{A7}$$

Each of the x's ranges from $-\infty$ to $+\infty$ in (A7).

For the whole space $-\infty<x_j<+\infty$ to be covered, the coordinates ρ, θ, $\phi_1,\cdots\phi_7$ must have the limits

$$0<\rho<\infty,\quad 0<\phi_j<\pi,\ j=1,2\cdots6,\quad 0<\theta<\pi,\quad -\pi<\phi_7<+\pi. \tag{A8}$$

Then,

$$\sum_{j=1}^{9} x_j^2=\rho^2 \tag{A9}$$

and the nine-dimensional volume element is given by

$$d^9\rho=\rho^8\sin^7\theta\sin^6\phi_1\sin^5\phi_2\cdots\sin\phi_6 d\rho d\theta d\phi_1\cdots d\phi_7. \tag{A10}$$

Let us choose the x_1 axis along the direction of vector $\mathbf{Q}$ so that $\mathbf{Q}\cdot\boldsymbol{\varrho}=Q\rho\cos\theta$ and (A6) becomes

$$I=(\beta\gamma\delta)^{-3}\int \exp(iQ\rho\cos\theta-\rho)\rho^8\sin^7\theta\sin^6\phi_1\sin^5\phi_2 \times\sin^4\phi_3\sin^3\phi_4\sin^2\phi_5\sin\phi_6 d\rho d\theta d\phi_1 d\phi_2 d\phi_3 d\phi_4 d\phi_5 d\phi_6 d\phi_7.$$

Integrations with respect to the ϕ's are simple and give $\frac{1}{3}\pi^4$. Using Sommerfeld's expansion[9] of the plane wave, $\exp(iQ\rho\cos\theta)$, into products of Bessel functions $J_{n+7/2}(Q\rho)$ and Gegenbauer polynomials $P_n(\cos\theta/7)$ and performing the θ integration we get

$$I=(2\pi)^{9/2}(\beta\gamma\delta)^{-3}Q^{-9}\int x^{9/2}J_{7/2}(x)\exp(-x/Q)dx,$$

where $x=Q\rho$. Integration with respect to x finally leads to

$$I=3\pi^4\times2^{12}(\beta\gamma\delta)^{-3}(1+Q^2)^{-5}, \tag{A11}$$

where $Q^2=(q_1/\beta)^2+(q_2/\gamma)^2+(q_3/\delta)^2$. We use Eqs. (A11) and (A2) to obtain the expression (11) for the bare form factor.

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